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Matrix-Spacetimes and a $2D$ Lorentz-Covariant Calculus in Any Even Dimension.

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Abstract

A manifestly Lorentz-covariant calculus based on two matrix-coordinates and their associated derivatives is introduced. It allows formulating relativistic field theories in any even-dimensional spacetime. The construction extends a single-coordinate matrix formalism based on coupling spacetime coordinates with the corresponding Γ-matrices.

A $2D$ matrix-calculus can be introduced for each one of the structures, adjoint, complex and transposed acting on Γ-matrices. The adjoint structure works for spacetimes with (n,n) signature only. The complex structure requires an even number of timelike directions. The transposed structure is always defined. A further structure which can be referred as "spacetime-splitting" is based on a fractal property of the Γ-matrices. It is present in spacetimes with dimension $D = 4n+2$.

The conformal invariance in the matrix-approach is analyzed. A complex conjugation is present for the complex structure, therefore in euclidean spaces, or spacetimes with $(2,2)$, $(2,4)$ signature and so on.

As a byproduct it is here introduced an index which labels the classes of inequivalent Γ-structures under conjugation performed by real and orthogonal matrices. At least two timelike directions are necessary to get more than one classes of equivalence. Furthermore an algorithm is presented for iteratively computing D-dimensional Γ matrices from the p and q-dimensional ones where $D = p+q+2$.

Possible applications of the $2D$ matrix calculus concern the investigation of higher-dimensional field theories with techniques borrowed from $2D$-physics.

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1 Introduction.

In the last few years physicists started getting accustomed with the likely possibility that the ultimate theory would be non-commutative. Many investigations on the role of non-commutative geometry took place [1]. In a somewhat different context, attempts to penetrate the mysteries of M-theory have been made invoking the so-called M(atrix) theory [2] and matrix string theory [3]. The latter in particular is a non-perturbative formulation which allows a non-trivial dynamics for strings by assuming the target-space coordinates being of matrix type.

The above-mentioned approaches are not immediately related with the topics discussed in the present paper but they constitute their natural premises and background. Moreover the results here discussed point to further investigations in that direction.

Our present work deals with the issue of finding a manifestly Lorentz-covariant description of relativistic field theories in any even spacetime dimension in terms of a formalism which involves matrix-type coordinates. Let us postpone for a while answering the question why should we bother about such a formulation and let us first discuss the main ideas involved.

It is somewhat a trivial remark, found in standard textbooks [4], that the Lorentz-group can be recovered and interpreted in terms of matrix-type coordinates. On the other hand it is clear, following the original ideas of Dirac, i.e. expressing the d'Alembertian $\Box$ box operator through its squared root $\not\partial$, that to such a derivative can be associated, as in any case involving derivatives, a space expressed through a coordinate which is now matrix-valued. A Lorentz-covariant calculus, endorsed with matrix-type integrals is immediately at disposal. The above considerations are perhaps not very deep. In any case they did not find applications especially because they lead to feasible descriptions, but nothing is gained and much is lost with respect to the standard case. The reason is clear, the lack of extra-structures. For instance, if we work in a single coordinate matrix-type formalism, then we have no room left to introduce in our theories antisymmetric tensors like curvatures $F_{\mu\nu}$ which require antisymmetry properties among indices and therefore at least two coordinates. The restriction is so strong that we are not even allowed to formulate QED or Yang-Mills theories. Therefore, if we wish to play the "matrix game" in a purposeful way we need at least two coordinates.

In reality "two" is quite sufficient for our scopes. More than that, it is precisely what we need. Indeed two dimensions are just enough to formulate all kind of theories we could be possibly interested in. Besides, an impressive list of methods and techniques have been elaborated to deal precisely with field theories in $2D$. Let us just mention one issue for all, integrability. Integrable field theories are well understood in $2D$ [5] due to the possibility of representing equations of motion as zero-curvature equations in the form $[\partial_Z - L_Z, \partial_{\overline{Z}} - L_{\overline{Z}}] = 0$ where L_Z, $L_{\overline{Z}}$ are Lax pairs. In higher dimensions analyzing integrability is much more problematic [6]. We have reasons to believe that our approach could shed light on this subject. Indeed the point of view we are advocating here is that we can, formally, deal even-dimensional spacetimes as a matrix-valued $2D$ space. With a pictorial image, we can say that we boost dimensions to the Flatland.

It is clear that non-commutative features are present with respect to theories formulated on the plane. These extra structures however, far from being undesired, are welcome and natural. They are the expected price we must pay for living in a higher-dimensional world.

Even if as a consequence we are not automatically guaranteed that working methods in the standard $2D$-plane continue to work in high-D, nevertheless our approach helps attacking problems with techniques which, so to speak, are "driven by the $2D$ formalism itself". As an example and with respect to the above mentioned integrability issue, this would imply investigating the matrix-analogs of the ordinary $2D$ Lax pairs. A (partial) list of other topics and areas which could benefit from this approach will be discussed in the conclusions.

The key ingredient we demand is acting with even-dimensional Poincaré generators on two matrix-valued coordinates $Z, \overline{Z}$ which, in order to leave the construction as simple as possible, we require being mutually commuting

$$i) \qquad [Z, \overline{Z}] = 0 \tag{1}$$

The differential calculus involves two derivatives ∂_Z, $\partial_{\overline{Z}}$ which should satisfy a factorization (Lorentz-covariant) property as follows

$$ii) \qquad \partial_Z \partial_{\overline{Z}} \propto \Box \cdot \mathbb{1} \tag{2}$$

This property can also be rephrased in more geometrical terms by requiring the (pseudo)-euclidean quadratic form $ds^2 = dx_\mu dx_\nu \eta^{\mu\nu}$ being expressed through $d\overline{Z} \cdot dZ = ds^2 \cdot \mathbb{1}$, where dZ, $d\overline{Z}$ are matrix-valued differentials. The commutativity of $Z, \overline{Z}$ implies the commutativity of the derivatives, therefore $\partial_Z \partial_{\overline{Z}} = \partial_{\overline{Z}} \partial_Z$.

The disentangling of $Z, \overline{Z}$ further requires that

$$iii \quad a) \qquad \partial_Z \overline{Z} = \partial_{\overline{Z}} Z = 0 \tag{3}$$

while the normalization condition

$$iii \quad b) \qquad \partial_Z Z = \partial_{\overline{Z}} \overline{Z} = \mathbb{1} \tag{4}$$

can be imposed. Please notice that in the above formulas the action of derivatives is a left action (not a free one).

The three listed properties are non-trivial ones. In order to make them work two different schemes can be adopted. The first one is based on non-trivial identities satisfied by the Clifford Γ-matrices and involving vector-indices contractions (from time to time we refer to such identities as "vector-traces", being understood they are not the standard traces taken w.r.t the spinorial indices). The second one uses a fractal property of the same Γ-matrices, i.e. an algorithm which allows computing higher-dimensional Γ-matrices from lower-dimensional ones. As we discuss later in the text, this fractal property encodes the information that the Lorentz-algebra has the structure of a homogeneous space.

The "vector trace"-case should be analyzed for each one of the three structures, adjoint, transposed or complex, which act on Γ matrices. While the transposed structure allows to satisfy the three properties above for any even space-time, the adjoint action restricts the $D = 2n$ spacetime to have (n, n) signature, and the complex structure restricts the signature to have an even number of time-coordinates. In the case of the complex structure

$Z, \overline{Z}$ are mutually complex conjugated ($\overline{Z} = Z^*$), while no conjugation is present in all the remaining cases.

The second scheme, which for reasons that will become clear later will be referred as the "splitting case", works only when the dimensionality of the spacetime is restricted to the values $D = 4n+2$ (n is a non-negative integer). In this case the signature is arbitrary.

Contrary to the standard calculus, the matrix-calculus here discussed naturally encodes the mentioned trace or fractal properties.

It is worth mentioning that when the formulas here reported (for the whole set of constructions mentioned above) are specialized to the $D = 2$ case, we trivially recover the ordinary $2D$ formalism in either the euclidean or Minkowski spacetime.

As we discuss at length in the text our calculus is manifestly Poincaré covariant and a $2D$ matrix-integration can be easily constructed. We explicitly apply it to bosonic theories and QED fields to show how to recover the results of the standard formulation.

It is worth mentioning that as a byproduct of the matrix-construction here discussed some other results are found. In particular, motivated by finding the consistency conditions under which the complex structure gives rise to a $2D$-matrix coordinate calculus, we are able to introduce an index which labels the classes of inequivalent Γ-structures under conjugation realized by matrices both real and orthogonal. This index is shown to classify the Wick rotations mapping the euclidean D-dimensional space to a spacetime with $(k, D-k)$ signature.

The algorithm mentioned before is here furnished. It is a realization of D-dimensional Γ-matrices in terms of p and q-dimensional ones, where D, p, q are even integer numbers satisfying the relation $D = p + q + 2$.

The scheme of the paper is the following.

In section 2 we introduce and discuss at first the covariant calculus for a single matrix coordinate. In section 3 the conformal invariance is analyzed in the light of the matrix-approach. In section 4 the basic properties concerning Γ matrices, as well as the conventions used, are reported. The algorithm expressing higher-dimensional Γ-matrices from the lower-dimensional ones is presented in section 5. Section 6 is devoted to discuss the consistency conditions for a $2D$ matrix-calculus in the "vector-trace" approach. It is shown that the vanishing of $\Gamma^\mu {\Gamma_\mu}^\dagger$, $\Gamma^\mu {\Gamma_\mu}^*$ or $\Gamma^\mu {\Gamma_\mu}^T$ is required. In section 7 the complete solution is furnished. The already mentioned restrictions to the $2D$ matrix-calculus with the adjoint or complex structure arise as a consequence. In section 8 the index discussed before is introduced and computed. It is shown how to relate it to Wick rotations from euclidean spaces to pseudoeuclidean spacetimes. The $2D$ matrix formalism is revisited and compact formulas are given in section 9. In section 10 a relativistic separation of the matrix-variables is explained. In section 11 the formula realizing higher-dimensional Γ-matrices from lower-dimensional ones is used to present a different (inequivalent) way of introducing the $2D$ matrix-coordinate calculus. It applies for $p = q$, that is when the spacetime is $D = 4n + 2$-dimensional. Section 12 is devoted to explain how to apply the matrix-calculus to forms. In the conclusions we make some comments about the $2D$ matrix calculus and discuss its possible applications.

2 Matrix coordinates.

Originally the $\partial\!\!\!/$ derivative was introduced by Dirac to be applied on spinors in order to define the dynamics of spinorial fields. However, as mentioned in the introduction, $\partial\!\!\!/$ admits another interpretation. Indeed it can be regarded as acting on a matrix-valued coordinate space. It turns out that e.g. bosonic fields can be described within a Lorentz-covariant framework in such a manner.

Since the idea of using matrix coordinates is at the very core of our further developments let us introduce and discuss in some detail the theory of a single-matrix coordinate at first.

We consider the following matrix-valued objects:
i) the matrix coordinate $Z = x_\mu \Gamma^\mu$
ii) the matrix derivative [1] $\partial_Z = \frac{1}{D}\partial_\mu \Gamma^\mu$
iii) the matrix differential $dZ = dx_\mu \Gamma^\mu$.

The above objects are all Γ-valued, where the Γ^μ denote any set of D-dimensional Γ-matrices (the signature of the space-time does not play any role for the moment and can be left arbitrary).

Matrix-valued functions are Γ-valued functions (Φ) of the single Z matrix-variable (i.e. $\Phi \equiv \Phi(Z)$). Both Lorentz and Poincaré invariances are automatically encoded in the above formalism. Indeed not only $\partial_Z^2 = \frac{1}{D^2}\Box \cdot \mathbb{1}$, but also the quadratic form dZ^2 satisfies

$$dZ^2 \;=\; ds^2 \cdot \mathbb{1} \tag{5}$$

(here, as in the introduction, $ds^2 = dx_\mu dx_\nu \eta^{\mu\nu}$).

It turns out that linear transformations which include the Poincaré group as a subgroup leave invariant this quadratic form. Indeed the differential $d = dx_\mu \partial^\mu$ can be reexpressed in matrix-coordinate form as

$$d \cdot \mathbb{1} \;=\; \frac{D}{2}(dZ \cdot \frac{\partial}{\partial Z} + \frac{\partial}{\partial Z} \cdot dZ) \tag{6}$$

so that for a generic $f(Z)$ function of Z we can write $df = \frac{D}{2}(dZ \cdot \frac{\partial}{\partial Z} + \frac{\partial}{\partial Z} \cdot dZ) \cdot f$. Notice that when f is the identity ($f(Z) \equiv Z$) we recover, as it should be, the above definition for dZ. We wish to point out that, since we are dealing with matrix-valued objects, some care has to be taken when performing computations with respect to the ordinary case. Non-commutative issues imply for instance that $dZ \cdot Z \neq Z \cdot dZ$.

If we specialize the f-transformation to be given by

$$Z' \;=\; f(Z) = S \cdot Z \cdot S^{-1} + K \tag{7}$$

where S is an element of the D-dimensional Lorentz group (i.e. $S\Gamma^\mu S^{-1} = \Lambda^\mu{}_\nu \Gamma^\nu$) and K is a constant matrix which for what we need is sufficent to take of the form $K = k_\mu \cdot \Gamma^\mu$, we therefore obtain $dZ' = S \cdot dZ \cdot S^{-1}$ which further implies $dZ'^2 = dZ^2$ since the latter is proportional to the identity.

The calculus can be further enlarged to accomodate a formal definition of a matrix-valued volume integration form and a matrix-valued delta-function. They both coincide

[1] the $\frac{1}{D}$ normalization is introduced for convenience in order to normalize $\partial_Z \cdot Z = \mathbb{1}$.

with the standard manifestly relativistic covariant definitions. To express them in matrix form is sufficent to recall the definition of Γ^{D+1}, the D-dimensional analog of γ^5, as the Lorentz-invariant product of the D-dimensional Γ^μ

$$\Gamma^{D+1} = \epsilon \Gamma^0 \cdot \Gamma^1 \cdot \ldots \cdot \Gamma^{D-1} \tag{8}$$

with $\epsilon = (-1)^{\frac{(s-t)}{4}}$. Here t denotes the number of timelike coordinates with + signature and $s = D - t$ the number of spacelike coordinates with $-$ signature. Therefore we can write

$$dV = dx_0 \cdot \ldots \cdot dx_{D-1} \cdot \mathbb{1} = \epsilon d\Gamma(0) \cdot \ldots d\Gamma(D-1) \cdot \Gamma^{D+1}$$

(here $d\Gamma(i) = dx_i \Gamma^i$) and

$$\delta(Z, W) = \delta(x_0 - y_0) \cdot \ldots \cdot \delta(x_{D-1} - y_{D-1}) \cdot \mathbb{1} = \epsilon \delta_\Gamma(0) \cdot \ldots \cdot \delta_\Gamma(D-1)$$

(where $\delta_\Gamma(i) = \delta(x_i - y_i)\Gamma^i$).

Let $K = k_\mu \cdot \Gamma^\mu$. The identity $\frac{1}{2}(K \cdot Z + Z \cdot K) = k_\mu x^\mu \cdot \mathbb{1}$ allows us to express the solutions to the free equations of motion for the bosonic massive field Φ in terms of the matrix-coordinate Z-representation. Indeed, if $K \cdot K = m^2 \cdot \mathbb{1}$, the equation

$$(D^2 {\partial_Z}^2 + m^2)\Phi = 0 \tag{9}$$

admits solutions which can be written as

$$\Phi(Z) = \int dV_K [a(K) e^{\frac{i}{2}(K\cdot Z + Z\cdot K)} + a^*(K) e^{-\frac{i}{2}(K\cdot Z+Z\cdot K)}] \tag{10}$$

where the modes $a(K)$ can be expanded in Laurent expansion as $a(K) = \sum_{n\in} \; a_n K^n$ and the a_n coefficients for our scopes can be assumed to be c-numbers.

At least for this particular case within the single-coordinate matrix formalism we are able to recover the results obtained in the standard framework. The mentioned feature that the $\not\partial$ derivative need not be associated with only spinorial fields arises as a byproduct.

3 The conformal invariance in the matrix-approach.

The matrix nature of the coordinate in the matrix calculus introduces noncommutative features. In this section we discuss this topic and show how the conformal invariance can be recovered within such a formalism.

At first it should be noticed that even and odd powers of Z behave differently. Due to the previous section results we get that $Z^{2n} = (x^2)^n \cdot \mathbb{1}$ is proportional to the identity, while $Z^{2n+1} = (x^2)^n \cdot Z$. As a consequence only the subclass of "odd" transformations of the kind $Z \mapsto Z^{2n+1}$ admits a realization in the ordinary spacetime coordinates x_μ as $x_\mu \mapsto x_\mu (x^2)^n$, for any integer-valued n. "Even" transformations (i.e. mappings $Z \mapsto Z^{2n}$) cannot be realized on the x_μ coordinates, while they are acceptable transformations in the Z-coordinate realization.

Simple algebraic manipulations show that the *left* action of the ∂_Z derivative on powers of Z leads to

$$\begin{aligned} \partial_Z Z^{2n} &= \tfrac{2n}{D} Z^{2n-1} \\ \partial_Z Z^{2n+1} &= (\tfrac{2n+D}{D}) Z^{2n} \end{aligned} \quad (11)$$

The commutation relation between Z and ∂_Z is given by

$$[\partial_Z, Z] \;=\; \mathbb{1} - \tfrac{2}{D} \cdot l_{\mu\nu}\Sigma^{\mu\nu} \quad (12)$$

where $l_{\mu\nu}$ and $\Sigma^{\mu\nu}$ are respectively the spacetime and spinorial generators of the Lorentz algebra:

$$\begin{aligned} l_{\mu\nu} &= x_\mu\partial_\nu - x_\nu\partial_\mu \\ \Sigma^{\mu\nu} &= \tfrac{1}{4}[\Gamma^\mu, \Gamma^\nu] \end{aligned} \quad (13)$$

The extra-term on the r.h.s. of (12) is clearly absent in $D = 1$ dimension. All the informations that we are dealing with a higher dimensional spacetime are therefore encoded in this extra operator [2].

The D-dimensional conformal group is defined as the set of transformations leaving invariant the relation $ds^2 = 0$. It is well known that for $D > 2$ the number of generators n_C in the conformal group is given by $n_C = n_L + 2D + 1$, where $n_L = \frac{1}{2}(D^2 - D)$ is the number of generators in the Lorentz group. The extra generators are given by the D translations, the D special conformal transformations plus the dilatation. In the matrix-coordinate realization this result is recovered as follows. While the Poincaré generators have been discussed in the previous section and the dilatation is simply given by $Z \mapsto \lambda Z$, the only crucial points concerns how to obtain the D special conformal transformations. They are given by the composition of the Poincaré transformations with the conformal inversion, which in our case is expressed through the transformation

$$Z \;\mapsto\; Z' = \frac{1}{Z} \quad (14)$$

(i.e. $x_\mu \mapsto \frac{x_\mu}{(x^2)}$). It is a simple algebraic check to prove that $dZ^2 = 0$ is preserved by (14). No other power transformation of Z for a different value of the exponent shares this feature. For instance $dZ^{2n+1} \cdot dZ^{2n+1}$ is not proportional to ds^2 because an extra contribution of the kind

$$4n(n+1)(x^2)^{2n-1} dx_\alpha x^\alpha dx_\beta x^\beta,$$

which vanishes only for $n = -1$, is present. This one and similar other consistency checks make ourselves comfortable with the intrinsic coherency of the matrix-coordinate formalism.

[2]It is tempting to regard (12) as a deformation (depending on a $\kappa = \frac{1}{D}$ parameter) of the standard commutator. Perhaps D-dimensional relativistic theories could therefore be analyzed in the light of the deformation theory which, in a different context, has been employed to recover quantization from classical structures (see e.g. [7]). However we will not elaborate more on such aspects in the present paper.

In the $D = 1$ dimension the conformal group coincides classically with the 1-dimensional diffeomorphisms group whose algebraic structure is given by the infinite-dimensional Witt algebra (centerless Virasoro algebra), spanned by the l_n generators

$$l_n = -z^{(n+1)}\frac{\partial}{\partial z} \tag{15}$$

which satisfy the commutation relations

$$[l_n, l_m] = (n-m)l_{n+m} \tag{16}$$

We expect that this algebra should be recovered in higher dimensions as well. This is the case indeed. If we define for any given D

$$L_n = -\frac{D}{2}Z^{2n+1}\partial_Z \tag{17}$$

then the L_n generators satisfy (16), i.e. $[L_n, L_m] = (n-m)L_{n+m}$.

In the special $D = 1$ limit this algebra coincides with the Witt subalgebra spanned by the "even" generators $\frac{1}{2}l_{2n}$; such a subalgebra coincides with the Witt algebra itself [3].

If we do not limit ourselves to consider "even" generators, but we enlarge the structure to accomodate "odd" generators of the kind $M_n = -\frac{D}{2}Z^{2n}\partial_Z$ in $D > 1$, then we no longer find a closed algebraic structure since the commutator between M_n, L_m involves extra operators

$$[M_n, L_m] = (\tfrac{2m-2n+D}{4}) \cdot Z^{2n+2m} \cdot \partial_Z - \tfrac{3}{4}Z^{2n+2m} \cdot l_{\mu\nu}\Sigma^{\mu\nu} \cdot \partial_Z \tag{18}$$

A closed linear algebraic structure should therefore include the extra operators (the second term in the r.h.s.) and any other new operator arising from the commutation of the previous ones, a procedure which has been encountered for instance when dealing with W_∞-algebra structures, see [8] and references therein.

Let us conclude this section by pointing out that no contradiction is present with the previous result that the conformal algebra in higher dimension is finite-dimensional. Indeed only in $D = 1$ the Witt algebra admits a geometrical interpretation as a conformal algebra. We have seen that for $D > 2$ the conformal relation $dZ^2 = 0$ is preserved by a group of transformations with a finite number of generators only (while the $D = 2$ case can be treated within the standard conformal calculus).

4 Γ-matrices and basic notations.

In the two previous sections we have investigated the single matrix-coordinate formalism and explained in some detail how it works. To be able to go a step further and analyze the $2D$ matrix-coordinates approach we need at first to check whether is it possible to solve the three conditions (from i) to iii)) formulated in the introduction. This can be

[3]It is a property of the Virasoro algebra that any subalgebra spanned by $\tilde{l}_n = \frac{1}{k}l_{kn}$ generators for any given positive integer k, is still equivalent to the full Virasoro algebra. If the non-trivial cocycle for the central extension is chosen to be of the form $cn^3\delta_{n+m,0}$, then the central charge $\tilde{c}$ present in the $\tilde{l}_n$ subalgebra is rescaled to be $\tilde{c} = kc$.

done only when properties of Γ-matrices for any spacetime are taken into account. For that reason this section is devoted to analyze Γ matrices and establish our notations and conventions. Concerning this material, we have used [9] as basic references.

A Γ-structure associated to a given spacetime, is a matrix representation of the Clifford algebra generators Γ^μ, which satisfy the anticommutation relations

$$\Gamma^\mu\Gamma^\nu + \Gamma^\nu\Gamma^\mu = 2\eta^{\mu\nu}\mathbb{1}_\Gamma \tag{19}$$

(here $\eta^{\mu\nu}$ is any (pseudo)-euclidean metric in D dimension). The representation is realized by $2^{\frac{D}{2}} \times 2^{\frac{D}{2}}$ matrices which further satisfy the unitarity requirement

$$\Gamma^{\mu\dagger} \;=\; \Gamma^{\mu-1} \tag{20}$$

as well as the tracelessness condition

$$tr\Gamma^\mu \;=\; 0 \tag{21}$$

for any μ.

The commutator is

$$\Gamma^\mu\Gamma^\nu - \Gamma^\nu\Gamma^\mu = 4\Sigma^{\mu\nu}\mathbb{1}_\Gamma \tag{22}$$

$\Sigma^{\mu\nu}$, already introduced in (13), is the generator of the Lorentz (pseudo-rotations) group.

For a matter of convenience and without loss of generality we can work in the so-called Weyl representation for Γ^μ, which occours when the dimensionality D of the spacetime is even; the Γ^μ are block-diagonal

$$\Gamma^\mu \;=\; \begin{pmatrix} 0 & \sigma^\mu \\ \tilde{\sigma}^\mu & 0 \end{pmatrix} \tag{23}$$

The dimensionality of the σ, $\tilde{\sigma}$ matrices is $dim_\sigma = dim_{\tilde{\sigma}} = 2^{\frac{D}{2}-1}$.
It is worth mentioning that all the results found in the present paper are representation-independent and not specific of the above presentation.

Any generic Y matrix, constructed with Γ-matrices and their products, have spinorial transformation properties (dotted and undotted indices) of the following kind

$$Y \;=\; \begin{pmatrix} \star_\alpha{}^\beta & \star_{\alpha\dot{\beta}} \\ \star^{\dot{\alpha}\beta} & \star^{\dot{\alpha}}{}_{\dot{\beta}} \end{pmatrix} \tag{24}$$

The extra matrix Γ^{D+1}, introduced with the correct normalization in (9), together with the D Γ^μ satisfy the (19) and (22) algebra in $(D+1)$-dimensions and is block-diagonal (the blocks have equal size)

$$\Gamma^{D+1} = \begin{pmatrix} \mathbb{1}_\sigma & 0 \\ 0 & -\mathbb{1}_{\tilde{\sigma}} \end{pmatrix}$$

In terms of σ^μ and $\tilde{\sigma}^\mu$ the (19) and (22) algebra reads as follows

$$\begin{aligned} \sigma^\mu\tilde{\sigma}^\nu + \sigma^\nu\tilde{\sigma}^\mu &= 2\eta^{\mu\nu}\mathbb{1}_\sigma \\ \tilde{\sigma}^\mu\sigma^\nu + \tilde{\sigma}^\nu\sigma^\mu &= 2\eta^{\mu\nu}\mathbb{1}_{\tilde{\sigma}} \end{aligned} \tag{25}$$

and respectively

$$
\begin{array}{rcl}
\sigma^\mu\tilde{\sigma}^\nu - \sigma^\nu\tilde{\sigma}^\mu &=& 4\sigma^{\mu\nu}{\bf 1\!\!1}_\sigma \\
\tilde{\sigma}^\mu\sigma^\nu - \tilde{\sigma}^\nu\sigma^\mu &=& 4\tilde{\sigma}^{\mu\nu}{\bf 1\!\!1}_{\tilde{\sigma}}
\end{array}
\qquad (26)
$$

while

$$
\Sigma^{\mu\nu} = \begin{pmatrix} \sigma^{\mu\nu} & 0 \\ 0 & \tilde{\sigma}^{\mu\nu} \end{pmatrix}
$$

From the even-dimensional euclidean Γ-matrices we can reconstruct the Γ-matrices for any other signature by applying a Wick rotation, realized as follows: let $\overline{\mu}$ be a direction with $-$ signature. The correponding $\Gamma^{\overline{\mu}}$ is obtained from the euclidean ${\Gamma_E}^{\overline{\mu}}$ through ${\Gamma_E}^{\overline{\mu}} \mapsto \Gamma^{\overline{\mu}} = i{\Gamma_E}^{\overline{\mu}}$, i.e.

$$
\begin{array}{rcl}
{\sigma_E}^{\overline{\mu}} &\mapsto& \sigma^{\overline{\mu}} = i{\sigma_E}^{\overline{\mu}} \\
\tilde{\sigma}_E^{\overline{\mu}} &\mapsto& \tilde{\sigma}^{\overline{\mu}} = i\tilde{\sigma}_E^{\overline{\mu}}
\end{array}
\qquad (27)
$$

The Γ^ν matrices along the timelike ν directions ($\eta^{\nu\nu} = +1$) are left unchanged. The absolute sign in the (27) transformations is just a matter of choice.

The adjoint, complex and transposed structures can be introduced in terms of three unitary matrices, conventionally denoted as A, B, C in the literature [9], satisfying

$$
\Gamma^{\mu\dagger} = (-1)^{t+1}A\Gamma^\mu A^\dagger \qquad (28)
$$

$$
\Gamma^{\mu *} = \eta B\Gamma^\mu B^\dagger \qquad (29)
$$

$$
\Gamma^{\mu T} = \eta(-1)^{t+1}C\Gamma^\mu C^\dagger \qquad (30)
$$

η is here a sign ($\eta = \pm 1$) which in principle can be evaluated but need not be specified for our purposes.

In the euclidean (positive signature $+...+$) and only in the euclidean case the Γ^μ matrices can be assumed all hermitians ($\Gamma^{\mu\dagger} = \Gamma^\mu$ for any μ).

For simplicity in the following the three above structures, adjoint, complex and transposed, will also be referred as A, B, C-structures. For completeness let us report here the following properties satisfied by A, B, C:

$$
A = \Gamma^0 \cdot ... \cdot \Gamma^{t-1} \qquad (31)
$$

$$
B^T = \varepsilon B, \qquad (32)
$$

$$
C = B^T A \qquad (33)
$$

ε is a sign ($\varepsilon = \pm 1$) which is expressed [9] through $\varepsilon = \cos\frac{\pi}{4}(s-t) - \eta\sin\frac{\pi}{4}(s-t)$ (as before t and $s = D - t$ denote respectively the number of timelike and spacelike coordinates). In the formula for A the product of Γ is restricted to timelike coordinates only.

We have furthermore

$$
A^{-1} = (-1)^{\frac{t(t-1)}{2}}A \qquad (34)
$$

$$
A^* = \eta^t BAB^{-1} \qquad (35)
$$

$$
A^T = \eta^t CAC^{-1} \qquad (36)
$$

$$
C^T = \varepsilon\eta^t(-1)^{\frac{t(t-1)}{2}}C \qquad (37)
$$

5 An algorithm to iteratively compute Γ matrices.

In this section we present an algorithm which encodes fractal properties of the Γ-matrices and allows to iteratively compute Γ-matrices in any dimension and for any signature of the space-time by the knowledge of lower-dimensional Γ-matrices. As a consequence the computation of any set of D-dimensional Γ-matrices satisfying the (19) algebra is recovered from the sole knowledge of the three Pauli matrices.

The algorithm here presented is central for our analysis of the $2D$ matrix-coordinates formalism in the "splitting" case and is also quite useful in proving the vector-contraction identities we introduce and discuss in the next section.

The Γ-matrices in even D spacetime dimension can be represented from the γ-matrices in $(p+1)$ and $(q+1)$ spacetime dimensions (we will use capital and lower letters for reasons of typographical clarity) where the even integers p, q satisfy the condition

$$D \;=\; p+q+2 \tag{38}$$

Since, as recalled in the previous section, Γ-matrices for any signature are obtained from the euclidean Γ-matrices through a Wick rotation, it is sufficient to present our formulas in the case when all the Γ-matrices involved (in $(p+1)$, $(q+1)$ and D dimensions) are euclidean.

The capital index $M = 0, 1, ..., D-1$ is used to span the D-dimensional space, while $m = 0, 1, ..., p$ and $\overline{m} = 0, 1, ..., q$ are respectively employed for the $(p+1)$ and $(q+1)$-dimensional spaces. The corresponding Γ-matrices will be denoted as ${\Gamma_D}^M$, ${\gamma_p}^m$, ${\gamma_q}^{\overline{m}}$.

The symbol $\mathbb{1}_n$ will denote the $2^{\frac{n}{2}} \times 2^{\frac{n}{2}}$ identity matrix.

For concision of notations the symbols $\mathbb{1}_0$ and ${\gamma_0}^0$ (i.e. the "1-dimensional Γ-matrix") will both denote the constant number 1.

It is a simple algebraic exercise to prove that the set of ${\Gamma_D}^M$ matrices can be realized through the position

$${\Gamma_D}^M \;=\; \begin{pmatrix} 0 & \mathbb{1}_q \otimes {\gamma_p}^m ; \quad -i{\gamma_q}^{\overline{m}} \otimes \mathbb{1}_p \\ \mathbb{1}_q \otimes {\gamma_p}^m ; \quad i{\gamma_q}^{\overline{m}} \otimes \mathbb{1}_p & 0 \end{pmatrix} \tag{39}$$

with $M \equiv (m, p+1+\overline{m})$.

The condition (38) is necessary in order to match the dimensionality of the Γ-matrices in the left and right side (due to (38) the dimension of the r.h.s. matrix is $2 \cdot 2^{\frac{p}{2}} \cdot 2^{\frac{q}{2}} = 2^{\frac{D}{2}}$ if (38) is taken into account). A further consequence of the (38) condition is that the "generalized γ^5-matrices" (9) of the kind ${\gamma_p}^p$ and ${\gamma_q}^q$ are necessarily present, which implies a decomposition of the even dimensional D spacetime into two *odd*-dimensional $p+1$ and $q+1$-spacetimes.

The decomposition realized by (39) works for any couple of even integers p, q satisfying the (38) condition. This implies that for any given even integer D the number n_D of inequivalent decompositions (factoring out the ones trivially obtained by exchanging $p \leftrightarrow q$) is given by $n_D = \frac{1}{4}(D+r)$, where either $r = 0$ or $r = 2$ according respectively if D is a multiple of 4 or not. At the lowest dimensions we have the following list of allowed decompositions:

$$D = 2 \quad \leftarrow \quad \{(p = 0, q = 0)\}$$

$$
\begin{array}{rcl}
D=4 & \leftarrow & \{(p=0,q=2)\} \\
D=6 & \leftarrow & \{(p=0,q=4), \quad (p=2,q=2)\} \\
D=8 & \leftarrow & \{(p=0,q=6), \quad (p=2,q=4)\} \\
D=10 & \leftarrow & \{(p=0,q=8), \quad (p=2,q=6), \quad (p=4,q=4)\}
\end{array}
\tag{40}
$$

and so on. It is worth mentioning here that in issues involving Kaluza-Klein compactifications to lower-dimensional spacetimes the above result can find useful applications in suggesting which one of the allowed decompositions is the most convenient to choose.

Since the formula (39) is quite important for our purposes it is convenient to furnish it in two other presentation. We can write in the Weyl realization

$$
\begin{array}{rcl}
{\sigma_D}^M & = & (\mathbb{1}_q \otimes {\gamma_p}^m; \quad -i{\gamma_q}^{\overline{m}} \otimes \mathbb{1}_p) \\
\tilde{\sigma}_D^M & = & (\mathbb{1}_q \otimes {\gamma_p}^m; \quad i{\gamma_q}^{\overline{m}} \otimes \mathbb{1}_p)
\end{array}
\tag{41}
$$

${\Gamma_D}^M$ can also be expressed through

$$
\begin{array}{rcl}
{\Gamma_D}^m & = & \tau_x \otimes \mathbb{1}_q \otimes {\gamma_p}^m \\
{\Gamma_D}^{p+1+\overline{m}} & = & \tau_y \otimes {\gamma_q}^{\overline{m}} \otimes \mathbb{1}_p
\end{array}
\tag{42}
$$

with the help of the off-diagonal Pauli matrices τ_x, τ_y.

The three Pauli matrices given by

$$
\tau_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \tau_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \tau_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}
\tag{43}
$$

can be regarded as the Γ-matrices for the euclidean three-dimensional space. It is evident that any D-dimensional Γ-matrix can be constructed, with repeated applications of the (39) formula, by tensoring the (43) Pauli matrices. The statement made at the beginning of this section is therefore proven.

Let us make some comments about the algebraic meaning of the formula (39). The generators Σ^{MN} of the Lorentz transformations are expressed through the commutators of Γ matrices, see (13). By using the (39) decomposition the index M is splitted into m, $\overline{m}$ indices. The Lorentz algebra $\mathcal{G}$ admits a decomposition in three subspaces $\mathcal{M}_+$, $\mathcal{M}_-$ and $\mathcal{K}$, spanned respectively by the generators $\Sigma^{mn} \in \mathcal{M}_+$, $\Sigma^{\overline{mn}} \in \mathcal{M}_-$ and $\Sigma^{m\overline{n}} \in \mathcal{K}$.

$\mathcal{M}_+$ and $\mathcal{M}_-$ are the $\mathcal{G}$ subalgebras corresponding to the Lorentz algebras for the $(p+1)$-dimensional and respectively the $(q+1)$-dimensional subspaces entering the (39) decomposition. By setting $\mathcal{M} =_{def} \mathcal{M}_+ \oplus \mathcal{M}_-$, the full Lorentz algebra is expressed as

$$
\mathcal{G} = \mathcal{M} \oplus \mathcal{K}
\tag{44}
$$

The Lorentz commutators in $\mathcal{G}$ satisfy the following set of symbolic relations

$$
\begin{array}{rcl}
[\mathcal{M}, \mathcal{M}] & = & \mathcal{M} \\
[\mathcal{M}, \mathcal{K}] & = & \mathcal{K} \\
[\mathcal{K}, \mathcal{K}] & = & \mathcal{M}
\end{array}
\tag{45}
$$

The existence of such relations gives to the Lorentz algebra $\mathcal{G}$ the structure of a homogeneous space w.r.t. its (44) decomposition. The existence of the (39) representation for the Γ-matrices is just a reflection of such a homogeneity property.

6 The $2D$ matrix formalism in the "vector-trace" approach.

In this section we start discussing how to implement our program which prescribes the introduction of two distinct Z, $\overline{Z}$ matrix-coordinates. We recall that the basic properties required (from i) to $iiib$)) have already been presented in the introduction.

It is quite evident that we have no longer the possibility to identify one of the coordinates (let's say Z) with the position $Z = x_\mu \Gamma^\mu$ as in the single matrix-coordinate formalism, since in this case no room is left to introduce the second coordinate $\overline{Z}$, commuting with the previous one and satisfying $d\overline{Z} \cdot dZ = ds^2 \cdot \mathbb{1}$. A different strategy has to be employed. In this section we present one, which we conventionally call the "trace" approach since, as we will see, it involves some identities concerning contractions of vector indices of Γ-matrices ("vector traces"). Another approach based on a different construction will be discussed in the next sections. An important feature which should be stressed here is the fact that the requirements put by the $2D$ matrix-coordinates formalism lead to some non-trivial constraints concerning the structure of space-times. Different matrix-solutions can be found to our program depending on the dimensionality and the signature of the spacetimes.

We gain much more freedom to investigate our problem if we take as starting building blocks to construct matrix-valued objects not just the Γ^μ-matrices themselves, but instead the σ^μ, $\tilde{\sigma}^\mu$ blocks (together with their conjugated matrices under adjoint, transposed or complex action) entering the Weyl realization (23).

Let us introduce in order to simplify notations

$$\omega = x_\mu \sigma^\mu, \qquad \tilde{\omega} = x_\mu \tilde{\sigma}^\mu \tag{46}$$

The spinorial (dotted and undotted indices) transformation properties for ω, $\tilde{\omega}$ and their A, B, C-transformed quantities are as follows

$$\{\omega, \tilde{\omega}^\dagger, \omega^*, \tilde{\omega}^T\} \equiv \star_{\alpha\dot{\beta}}, \qquad \{\tilde{\omega}, \omega^\dagger, \tilde{\omega}^*, \omega^T\} \equiv \star^{\dot{\alpha}\beta} \tag{47}$$

In accordance with the above transformation properties the first and the second set of matrix-valued objects have to be inserted in matrices of the kind of (24) respectively in the upper right (lower left) corner.

The σ's and $\tilde{\sigma}$'s matrices satisfy the anticommutation and commutation relations given by (25) and (26). Analogous relations are immediately obtained by applying on them the A, B, C-transformations (30). The requirement of commutativity ($[Z, \overline{Z}] = 0$), as well as the disentangling of the coordinates under the *left* action of derivatives (i.e. $\partial_Z \overline{Z} = \partial_{\overline{Z}} Z = 0$) can be solved with the help of the (25) relations. They apply however only if at most a single matrix of the kind of ω, $\tilde{\omega}$ (or their conjugated quantities) is inserted in the upper right or lower left diagonal block of a bigger matrix (24) to build up Z, $\overline{Z}$. For that reason we do not consider here the possibility that mixed terms could be present. The investigation about the possibility to solve the above relations in this context is much more involved and does not seem to use general arguments as the case we are analyzing here. It is therefore left as an open problem for further investigations. On the other hand the construction involving single blocks is here fully analyzed and the complete solution is furnished.

We ask for matrix-valued Z, $\overline{Z}$ of the kind

$$Z = \begin{pmatrix} 0 & \omega \\ \star & 0 \end{pmatrix}, \quad \overline{Z} = \begin{pmatrix} 0 & \star \\ \tilde{\omega} & 0 \end{pmatrix} \tag{48}$$

To keep covariance the $\star$ in the above formulas should be replaced either by the 0-matrix or by the matrices in (47) with the right covariance properties. The commutation requirement $[Z, \overline{Z}] = 0$ rules out the possibility to use the 0-matrix so that the only left possibilities are either

$$Z = \begin{pmatrix} 0 & \omega \\ \omega^{\#} & 0 \end{pmatrix}, \quad \overline{Z} = \begin{pmatrix} 0 & \tilde{\omega}^{\#} \\ \tilde{\omega} & 0 \end{pmatrix} \tag{49}$$

or

$$Z = \begin{pmatrix} 0 & \omega \\ \tilde{\omega}^{*} & 0 \end{pmatrix}, \quad \overline{Z} = \begin{pmatrix} 0 & \omega^{*} \\ \tilde{\omega} & 0 \end{pmatrix} \tag{50}$$

(since the transposed and the adjoint case are formally similar it is convenient to introduce a unique symbol $\#$ to denote both of them, i.e. $\# \equiv T, \dagger$).

In both the above cases the identification of $Z, \overline{Z}$ through either (49) or (50) implies that the commutativity property is satisfied in consequence of (25). In the two $\#$-cases above the commutativity requires for instance the vanishing of the expression

$$x_\mu x_\nu (\sigma^\mu \tilde{\sigma}^\nu - \tilde{\sigma}^{\nu\,\#} \sigma^{\mu \#}) \tag{51}$$

This is indeed so as it can be realized by expanding the term inside the parenthesis in its symmetric and antisymmetric component under the $\mu \leftrightarrow \nu$ exchange. Notice the role of the $-$ sign and the fact that Z, $\overline{Z}$ in (49) are correctly "fine-tuned" in order to guarantee the commutativity. A similar analysis works for the complex $*$-case as well.

The (25) identities imply the following relations

$$\sigma^\mu \tilde{\sigma}_\mu = D \cdot \mathbb{1}_\sigma \qquad \tilde{\sigma}^\mu \sigma_\mu = D \cdot \mathbb{1}_{\tilde{\sigma}} \tag{52}$$

(where from now on the Einstein convention over repeated indices is understood).

Such identities allow us to introduce the derivative ∂_Z, $\partial_{\overline{Z}}$ which satisfy the *ii*) condition and the normalization requirement *iiib*). They are given in the $\#$-cases by

$$\partial_Z = \frac{1}{D} \begin{pmatrix} 0 & \partial_\mu \tilde{\sigma}^{\mu\,\#} \\ \partial_\mu \tilde{\sigma}^\mu & 0 \end{pmatrix}, \quad \partial_{\overline{Z}} = \frac{1}{D} \begin{pmatrix} 0 & \partial_\mu \sigma^\mu \\ \partial_\mu \sigma^{\mu\,\#} & 0 \end{pmatrix} \tag{53}$$

and in the $*$-case by

$$\partial_Z = \frac{1}{D} \begin{pmatrix} 0 & \partial_\mu \sigma^{\mu\,*} \\ \partial_\mu \tilde{\sigma}^\mu & 0 \end{pmatrix}, \quad \partial_{\overline{Z}} = \frac{1}{D} \begin{pmatrix} 0 & \partial_\mu \sigma^\mu \\ \partial_\mu \tilde{\sigma}^{\mu\,*} & 0 \end{pmatrix} \tag{54}$$

In both the # and $*$-cases we have the relation

$$\partial_Z \partial_{\overline{Z}} = \partial_{\overline{Z}} \partial_Z = \frac{1}{D^2} \Box \cdot \mathbb{1} \tag{55}$$

Up to now all the properties required by the $2D$ matrix-coordinates formalism have been satisfied. The last property which should be implemented, but a fundamental one, is the "disentangling condition" *iiia*).

One can immediately check that $\partial_Z \overline{Z} = \partial_{\overline{Z}} Z = 0$ is satisfied whether, according to the different cases, the following contractions of the vector indices give vanishing results:

$$\mathcal{A} \equiv \sigma^\mu {\sigma_\mu}^\dagger, \quad \mathcal{B} \equiv \sigma^\mu \tilde{\sigma}_\mu^*, \quad \mathcal{C} \equiv \sigma^\mu {\sigma_\mu}^T \tag{56}$$

(and similarly $\tilde{\mathcal{A}} = \tilde{\sigma}^\mu \tilde{\sigma}_\mu^\dagger$, $\tilde{\mathcal{B}} = \tilde{\sigma}^{\mu\, *} \sigma_\mu$ and $\tilde{\mathcal{C}} = \tilde{\sigma}^\mu \tilde{\sigma}_\mu^T$ should vanish as well). $\mathcal{A}$, $\mathcal{B}$, $\mathcal{C}$ are all proportional to $\mathbb{1}_\sigma$ with a proportionality factor a, b, c respectively (one can easily realize that $\tilde{\mathcal{A}}$, $\tilde{\mathcal{B}}$, $\tilde{\mathcal{C}}$ are proportional to $\mathbb{1}_{\tilde{\sigma}}$ with the same a, b, c constant factors).

An equivalent way of rephrasing the above properties reads as follows

$$\begin{aligned} \Gamma^\mu {\Gamma_\mu}^\dagger &= a \cdot \mathbb{1}_\Gamma \\ \Gamma^\mu {\Gamma_\mu}^* &= b \cdot \mathbb{1}_\Gamma \\ \Gamma^\mu {\Gamma_\mu}^T &= c \cdot \mathbb{1}_\Gamma \end{aligned} \tag{57}$$

We are therefore left to determine under which conditions the above a, b, c constants are vanishing.

Before going ahead let us however point out that while a is always representation-independent, c in principle could not be representation-independent (in effect it is so and is always vanishing) and b is representation-independent only in the euclidean case (for generic signatures its value depend on the way the Wick rotation (27) has been performed). The algebraic meaning of b as an index labelling classes of equivalence of Γ-structures under conjugations determined by *both* real and orthogonal matrices will be discussed in section 8.

The remark concerning the representation-independence can be immediately understood when realizing that a different Γ-structure satisfying the Weyl condition is recovered from the Γ^μ by by simultaneously rescaling all σ's and $\tilde{\sigma}$'s through

$$\begin{aligned} \sigma^\mu &\mapsto -i\sigma^\mu \\ \tilde{\sigma}^\mu &\mapsto i\tilde{\sigma}^\mu \end{aligned} \tag{58}$$

Under such a transformation a, b, and c are mapped as follows: $a \mapsto a$, $b \mapsto -b$, $c \mapsto -c$.

The above transformation can also be reexpressed with the help of the Pauli matrices as

$$\Gamma^\mu \mapsto (i\tau_z \otimes \mathbb{1}) \cdot \Gamma^\mu \tag{59}$$

In the next section we compute the coefficients a, b, c for any even-dimensional spacetime.

7 The vector-contraction identities.

In the previous section we have furnished the motivations why we are interested in computing the "vector-contractions" expressed by the formula (57), i.e. the coefficients a, b, c. Here we furnish the results together with their proofs.

The following properties hold:

$$i) \quad a \;=\; t-s \tag{60}$$

where, as usual, t (s) denotes the number of timelike (spacelike) directions in $D = t+s$ dimensions;

$$ii) \quad b \;=\; 2(t_+ - t_-) \tag{61}$$

where t_+ (respectively t_-) are non-negative integers denoting the number of time-directions (whose total number is $t = t_+ + t_-$) associated to Γ-matrices which are symmetric (respectively antisymmetric) under transposed conjugation in the Weyl representation;

$$iii) \quad c \;=\; 0 \tag{62}$$

identically in any spacetime.

As a result the $2D$ matrix-formalism defined in terms of the A-structure works only in $(t=n, s=n)$ spacetimes, while in terms of the C-structure it is always defined for any even-dimensional spacetime. For what concerns b, it can assume among other possible values, the 0-value only when the spacetime admits an even number of time directions, under the assumption $t_+ = t_- = m$, $t = 2m$.

The B-structure turns out to be defined only for spacetimes with even number of timelike (+ signature) and even number of spacelike (- signature) directions.

The strategy to prove the above statements is the following.

For what concerns the computation of a we can start with the euclidean case. In this case we can consistently assume

$$\Gamma^{\mu\dagger} \;=\; \Gamma^{\mu} \tag{63}$$

By using (25) we get that a in the euclidean is $a = D$. The (27) "Wick rotation" applied to the $\overline{\mu}$ direction leads to a change of sign in the contribution of $\Gamma^{\overline{\mu}}\Gamma_{\overline{\mu}}$ (indeed $+1 \mapsto -1$), which proves the above result.

For what concerns c we proceed as follows. At first we notice that any Wick rotation leaves unchanged the contribution of the corresponding direction so that c does not depend on the signature of the spacetime. It is therefore enough to compute c in the euclidean case. We can do it iteratively by determining the value c_{D+2} of c in $(D+2)$-dimensions from its D-dimensional value c_D. It is convenient to do so with the help of the (39) formula, taken with the "extremal" decomposition $q=0$, $p=D$. We get for ${\sigma_{D+2}}^{\mu}$:

$${\sigma_{D+2}}^{\mu} \;=\; ({\Gamma_D}^{\tilde{\mu}}; \quad \Gamma^{D+1}; \quad -i\cdot \mathbb{1}_D) \tag{64}$$

where $\tilde{\mu} = 0,1,...,D-1$. The two last terms on the right hand side give opposite contributions which cancel each other to the computation of c_{D+2}. Therefore $c_{D+2} = c_D$.

On the other hand an immediate computation shows that in $D = 2$, $c_2 = 0$. The (62) formula is therefore proven.

The above result is just one way of proving the well-known property that in the Weyl representation the Γ-matrices can all be chosen simultaneously either symmetric or antisymmetric under transposition

$$\Gamma^{\mu T} = {\epsilon_\mu}^T \Gamma^\mu; \qquad {\epsilon_\mu}^T = \pm 1 \tag{65}$$

and that the number of "symmetric" μ_+ directions (${\epsilon_{\mu_+}}^T = +1$) is equal to the number of "antisymmetric" μ_- directions (${\epsilon_{\mu_-}}^T = -1$).

Due to the transformation (58) the (anti-)symmetric character of the $\Gamma^{\overline{\mu}}$ along the $\overline{\mu}$ direction (and therefore the sign in (65)) is arbitrary and conventional since (58) reverts the symmetry properties under transposition. The relative sign between two arbitrary directions however is left unchanged and acquires an absolute meaning.

For later purposes it is convenient to introduce the sign ${\epsilon_\mu}^*$ as

$$\Gamma^{\mu\,*} \;=\; {\epsilon_\mu}^* \Gamma^\mu \tag{66}$$

We remark that ${\epsilon_\mu}^*$ changes sign when a Wick rotation is performed along μ.

It should be noticed that in an even $D = t+s$ spacetime with (t,s)-signature the choice of which Γ matrices should be assumed T-symmetric (T-antisymmetric) can be made in different ways. Let us denote with t_+ (t_-) the number of time-directions associated to T-symmetric (T-antisymmetric) Γ matrices; s_+ (s_-) will denote the number of T-symmetric (T-antisymmetric) spatial directions. Clearly, from the previously stated results

$$\begin{aligned} t &= t_+ + t_- \\ s &= s_+ + s_- \\ t_+ + s_+ &= t_- + s_- = \frac{1}{2}(t+s) \end{aligned} \tag{67}$$

It turns out that b can be recovered from the computations already performed by setting

$$\begin{aligned} &\Gamma^\mu \cdot {\Gamma_\mu}^* = \Gamma^\mu \cdot ({\Gamma_\mu}^T)^\dagger = \\ &= \Gamma^{\mu_+} \cdot {\Gamma_{\mu_+}}^\dagger - \Gamma^{\mu_-} \cdot {\Gamma_{\mu_-}}^\dagger = (t_+ - s_+ - t_- + s_-) \cdot \mathbb{1}_\Gamma, \end{aligned}$$

that is, due to (67), $b = 2(t_+ - t_-)$. QED.

8 An index labelling the inequivalent Γ-structures under real orthogonal conjugation and their associated Wick rotations.

At this point of our analysis it is convenient to make a little digression in order to explain the algebraic significance of the coefficient b which can be reintroduced through the position ($b = 2I$):

$$I \;=\; \frac{1}{2^{\frac{D}{2}+1}} \cdot tr(\Gamma^\mu \cdot {\Gamma_\mu}^*) = (t_+ - t_-) \tag{68}$$

I is an index with a deep algebraic meaning. We recall at first a fundamental property of the Γ-structures (defined in section 4), known as the "fundamental Pauli theorem" (see [10]), stating that they are all unitarily equivalent. This implies that given two Γ-structures, denoted as ${\Gamma_I}^\mu$, ${\Gamma_{II}}^\mu$, a unitary matrix S ($S^{-1} = S^\dagger$) can always be found such that ${\Gamma_{II}}^\mu = S{\Gamma_I}^\mu S^\dagger$ for any μ. Moreover, up to a normalization factor, S is uniquely determined. As a consequence the representation-independence of the Dirac equation is guaranteed.

The index I, as shown by the previous section computations, is not defined on the above equivalence class. However one can easily realize that I is well-defined on the class of equivalence of Γ-structures which are conjugated under a real and orthogonal transformation, i.e. such that ${\Gamma_{II}}^\mu = O{\Gamma_I}^\mu O^T$ for any μ, with O a real-valued matrix belonging to the orthogonal group $O(2^{\frac{D}{2}})$, $2^{\frac{D}{2}}$ being the dimensionality of Γ^μ. The index I is therefore mathematically meaningful and can find useful applications in issues where reality conditions, not just unitary equivalence, have to be imposed. We already know that there exists spacetimes for which I assumes different values. Such spacetimes support inequivalent Γ-structures under real and orthogonal conjugation. The fact that inequivalent real structures arise out of a single "complex" structure is of course not at all surprising. In a related area we have the example of the real forms which are associated to a given complex Lie algebra.

The index I admits another interpretation. It classifies the inequivalent ways a Wick rotation can be performed from the euclidean D-dimensional space to a given $(t, s = D-t)$ pseudoeuclidean space. We will briefly discuss this topic in the following. Our considerations will be simplified if we analyze not just the index I itself, but its modulus $|I|$. By taking into account the (58) transformation, $|I|$ classifies the equivalence-classes of Γ-structures under conjugation for the tensor group $O(2^{\frac{D}{2}}) \otimes \mathbf{Z}_2$.

Without loss of generality (to recover the condition below it is sufficient to perform a $t \leftrightarrow s$ exchange), we can further restrict t to be $t \leq \frac{D}{2}$. Under this restriction the index $|I|$ for an odd number of time-directions ($t = 2k+1$) assumes all the possible $k+1$ different odd-values $|I| = 1, 3, ..., 2k+1$ (i.e. for $t_+ = 0, 1, ..., k$ in the reverse order), while for an even number of time directions ($t = 2k$) it assumes all the possible $k+1$ even values $|I| = 0, 2, ..., 2k$ (here again for $t_+ = 0, 1, ..., k$ in the reverse order).

Please notice that not only in the euclidean, but even in the generalized Minkowski case ($t = 1, s = D-1$), $|I|$ detects just one class of equivalence.

In practice Wick rotations corresponding to a given value of $|I|$ can be quite easily constructed. Let us start with the euclidean $D = 2n$ space. The directions are splitted into two classes on n elements each, according to the (anti-)symmetry property under transposition for their associated Γ-matrices (or equivalently, their ${\epsilon_\mu}^*$ (66) sign). We can list them as $[SS...S|AA...A]$ or as $[++...+|--...-]$. We recall that the Wick rotation flips the ${\epsilon_\mu}^*$ sign so that

$$([SS...|AA...] \equiv [++...|--...]) \ \mapsto ([(iS)S...|A...] \equiv [-+...|--...]) \qquad (69)$$

with a clear use of the symbols.

It is evident that for any n the passage from the euclidean $(2n, 0)$ space to the $(1, 2n-1)$-Minkowski spacetime can be done unambiguously when (58) is taken into account, indeed $|I|$ can only be $|I| = 1$. However, starting from the $t = 2$ case, the Wick rotation can be done in inequivalent ways. For instance the passage from the euclidean $(4,0)$ space to the $(2,2)$ spacetime can be done through either

$$i) \quad [++|--] \;\mapsto\; [++|++] \tag{70}$$

(i.e. $t_+ = 0$, $t_- = 2$) so that $|I| = 2$, or

$$ii) \quad [++|--] \;\mapsto\; [+-|-+] \tag{71}$$

(i.e. $t_+ = t_- = 1$) with $|I| = 0$.

Similarly, the passage $(6,0) \rightarrow (2,4)$ can be done through either

$$i) \quad [+++|---] \;\mapsto\; [+++|-++] \tag{72}$$

($t_+ = 0$, $t_- = 2$, $|I| = 2$) or

$$ii) \quad [+++|---] \;\mapsto\; [++-|--+] \tag{73}$$

($t_+ = t_- = 1$, $I = 0$).

As from the Wick rotations $(6,0) \rightarrow (3,3)$, we can have either

$$i) \quad [+++|---] \;\mapsto\; [+++|+++] \tag{74}$$

($t_+ = 0$, $t_- = 3$, $|I| = 3$), or

$$ii) \quad [+++|---] \;\mapsto\; [++-|-++] \tag{75}$$

($t_+ = 1$, $t_- = 2$, $|I| = 1$).

The iteration of the procedure to more general cases is now evident.

In all the above transformations we have picked up a Wick rotation which is representative of its class of equivalence. The fact that inequivalent Γ-structures, labelled by the index $|I|$, can be associated to a given space-time has immediate consequences to our problem of finding a $2D$ matrix-valued complex structure. Indeed, as discussed in section 6, the only structure which endorses a complex structure for the Z, $\overline{Z}$ matrix-coordinates is the B-structure. Formula (50) applies and we get

$$\overline{Z} \;=\; Z^* \tag{76}$$

As remarked in the previous section the only spacetimes which allow a $2D$-matrix valued complex calculus are those for which $b \equiv I = 0$. We already noticed that this implies an even number of time coordinates (and an even number of space coordinates due to the assumption that D is even). The discussion of this section shows however that in order to get a $2D$-matrix valued complex calculus, it is not sufficient just to pick up a $(2k, 2n-2k)$ spacetime. A "correct" Wick rotation from the euclidean (one of those leading to $t_+ = t_- = k$) has to be performed. For even times there is a Γ structure which satisfies $|I| = 0$. Such a Γ structure (with its associated Wick rotations) has to be carefully determined. In the $(2,2)$ case for instance it corresponds to the formula (71), while the Wick-rotation (70), belonging to a different Γ-structure, must be discarded.

We conclude this section by remarkig that issues involving two-times physics are at present quite investigated, see e.g. [11].

9 The $2D$-matrix formalism revisited.

In this section we collect all the results previously obtained concerning the $2D$-matrix calculus and present them in a single unifying framework which makes formally similar the analysis of the three A, B, C cases discussed so far. The "splitting case" S, whose discussion is postponed to a later section, also fits the following formulas.

Let us introduce at the first the matrices ${\Omega^\mu}_{(\star)} \equiv \Omega^\mu$, ${\overline{\Omega}^\mu}_{(\star)} \equiv \overline{\Omega}^\mu$, where the $(\star)$ index denotes one of the constructions (A, B, C or S) which proves to work. In the following the $(\star)$ index will be omitted in order not to burden the notation, but it should be understood.

The Ω's and $\overline{\Omega}$'s matrices, with $\overline{\Omega}^\mu \neq \Omega^\mu$, are constructed to satisfy the anticommutation relations

$$\begin{aligned} \Omega^\mu \overline{\Omega}^\nu + \Omega^\nu \overline{\Omega}^\mu &= 2\eta^{\mu\nu}\mathbb{1} \\ \overline{\Omega}^\mu \Omega^\nu + \overline{\Omega}^\nu \Omega^\mu &= 2\eta^{\mu\nu}\mathbb{1} \end{aligned} \tag{77}$$

An useful identity which immediately follows is

$$\overline{\Omega}^\mu \Omega_\mu = \Omega^\mu \overline{\Omega}_\mu = D \cdot \mathbb{1} \tag{78}$$

A further requirement which has been imposed is expressed by the formula

$$\Omega^\mu \Omega_\mu = \overline{\Omega}^\mu \overline{\Omega}_\mu = 0 \tag{79}$$

(here and above the Einstein convention is understood). The latter relation, in the A, B, C cases, is a consequence of the vector-contractions properties of Γ-matrices, and the conditions when is satisfied have been discussed section 7.

We can introduce the matrix coordinates Z, $\overline{Z}$, and their relative matrix derivatives ∂_Z, $\partial_{\overline{Z}}$ through

$$\begin{aligned} Z &= x_\mu \Omega^\mu \\ \overline{Z} &= x_\mu \overline{\Omega}^\mu \end{aligned} \tag{80}$$

$$\begin{aligned} \partial_Z &= \frac{1}{D}\partial_\mu \overline{\Omega}^\mu \\ \partial_{\overline{Z}} &= \frac{1}{D}\partial_\mu \Omega^\mu \end{aligned} \tag{81}$$

It is convenient to formally define the following (anti)-commutators

$$\begin{aligned} \Omega^\mu \Omega^\nu \pm \Omega^\nu \Omega^\mu &= {\Xi_\pm}^{\mu\nu} \\ \overline{\Omega}^\mu \overline{\Omega}^\nu \pm \overline{\Omega}^\nu \overline{\Omega}^\mu &= {\overline{\Xi}_\pm}^{\mu\nu} \\ \Omega^\mu \overline{\Omega}^\nu - \Omega^\nu \overline{\Omega}^\mu &= \Omega^{\mu\nu} \\ \overline{\Omega}^\mu \Omega^\nu - \overline{\Omega}^\nu \Omega^\mu &= \overline{\Omega}^{\mu\nu} \end{aligned} \tag{82}$$

For our purposes we do not need to compute them esplicitly, however the formulas for $\Omega^{\mu\nu}$, $\overline{\Omega}^{\mu\nu}$ will be presented at the end.

The following commutation relations hold

$$
\begin{aligned}
[Z,\overline{Z}] &= 0 \\
[\partial_Z, Z] &= [\partial_{\overline{Z}},\overline{Z}] = \mathbb{1} - \frac{1}{4D} l_{\mu\nu}(\overline{\Omega}^{\mu\nu} + \Omega^{\mu\nu}) \\
[\partial_Z,\overline{Z}] &= -\frac{1}{2D} l_{\mu\nu} \Xi_-{}^{\mu\nu} \\
[\partial_{\overline{Z}}, Z] &= -\frac{1}{2D} l_{\mu\nu} \Xi_-{}^{\mu\nu}
\end{aligned}
\tag{83}
$$

where $l_{\mu\nu} = x_\mu\partial_\nu - x_\nu\partial_\mu$.

When either the A ($\equiv \dagger$) or the C structure ($\equiv T$) are employed we have (as before $\# \equiv \dagger, T$)

$$
{\Omega_{(A,C)}}^{\mu} = \begin{pmatrix} 0 & \sigma^{\mu} \\ \sigma^{\mu\,\#} & 0 \end{pmatrix}, \quad {\overline{\Omega}_{(A,C)}}^{\mu} = \begin{pmatrix} 0 & \tilde{\sigma}^{\mu\,\#} \\ \tilde{\sigma}^{\mu} & 0 \end{pmatrix}
\tag{84}
$$

When the B ($\equiv *$) structure is employed we have

$$
{\Omega_{(B)}}^{\mu} = \begin{pmatrix} 0 & \sigma^{\mu} \\ \tilde{\sigma}^{\mu\,*} & 0 \end{pmatrix}, \quad {\overline{\Omega}_{(B)}}^{\mu} = \begin{pmatrix} 0 & \sigma^{\mu\,*} \\ \tilde{\sigma}^{\mu} & 0 \end{pmatrix}
\tag{85}
$$

In this case

$$
{\overline{\Omega}_{(B)}}^{\mu} = {\Omega_{(B)}}^{\mu\,*}
\tag{86}
$$

Due to the hermiticity property of the euclidean Γ-matrices in the euclidean space the B-structure and the C-structure coincide.

Let us furnish here for completeness the expression for $\Omega^{\mu\nu}$, $\overline{\Omega}^{\mu\nu}$ in the three A, B, C cases. We get

$$
{\Omega_{(A,C)}}^{\mu\nu} = \begin{pmatrix} \sigma^{\mu\nu} & 0 \\ 0 & -\tilde{\sigma}^{\mu\nu\,\#} \end{pmatrix}, \quad {\overline{\Omega}_{(A,C)}}^{\mu\nu} = \begin{pmatrix} -\sigma^{\mu\nu\,\#} & 0 \\ 0 & \tilde{\sigma}^{\mu\nu} \end{pmatrix}
\tag{87}
$$

and respectively

$$
{\Omega_{(B)}}^{\mu\nu} = \begin{pmatrix} \sigma^{\mu\nu} & 0 \\ 0 & \tilde{\sigma}^{\mu\nu\,*} \end{pmatrix}, \quad {\overline{\Omega}_{(B)}}^{\mu\nu} = \begin{pmatrix} \sigma^{\mu\nu\,*} & 0 \\ 0 & \tilde{\sigma}^{\mu\nu} \end{pmatrix}
\tag{88}
$$

The solutionsof the free equations of motion in the $2D$- matrix formalism (confront discussion at the end of section 2) are expressed with the help of $K = k_\mu\Omega^\mu$, $\overline{K} = k_\mu\Omega^\mu$ through

$$
k_\mu x^\mu \cdot \mathbb{1} = \frac{1}{2}(K \cdot Z + \overline{ZK})
\tag{89}
$$

so that

$$
\begin{aligned}
\partial_Z e^{ik_\mu x^\mu \cdot \mathbb{1}} &= \frac{i}{D} K e^{ik_\mu x^\mu \cdot \mathbb{1}} \\
\partial_{\overline{Z}} e^{ik_\mu x^\mu \cdot \mathbb{1}} &= \frac{i}{D} \overline{K} e^{ik_\mu x^\mu \cdot \mathbb{1}}
\end{aligned}
\tag{90}
$$

10 A relativistic separation of variables.

Instead of using Z, $\overline{Z}$ we can make a change of variables and introduce the $2D$-matrix coordinates $Z_\pm$ defined as follows:

$$Z_\pm = Z \pm \overline{Z} \tag{91}$$

The commutativity property clearly still holds

$$[Z_+, Z_-] = 0 \tag{92}$$

while the $\partial_\pm$ matrix-derivatives can be introduced

$$\partial_\pm = \frac{1}{2}(\partial_Z \pm \partial_{\overline{Z}}) \tag{93}$$

in order to satisfy, as a left action on $Z_\pm$,

$$\partial_\pm Z_\pm = \mathbb{1}, \quad \partial_\pm Z_\mp = 0 \tag{94}$$

The (pseudo)-euclidean quadratic form $ds^2 \cdot \mathbb{1}$ can now be read as follows

$$d\overline{Z} \cdot dZ = \frac{1}{4}({dZ_+}^2 - {dZ_-}^2) \tag{95}$$

Therefore Z_+ (Z_-) can be regarded as single-matrix coordinates, as those introduced in section 2, for the (pseudo)-euclidean spaces associated (up to a global sign) to the quadratic forms ${dZ_+}^2$ and ${dZ_-}^2$ respectively. In this context (91) can be seen as a separation of variables which preserves the relativistic structure of the theory.

It is not difficult to prove that, while the Z, $\overline{Z}$ matrix-coordinates are constructed with the full set of x_μ coordinates, no matter which structure has been used to define them, Z_+ involves only half of the x_μ coordinates. The remaining "half-sector" of the x_μ's appears in Z_-. The even $D = (2n)$-dimensional spacetime is therefore splitted in two n-dimensional relativistic spacetimes.

Such a result is a consequence of the following easy-to-prove equalities

$$\begin{aligned} \Omega^\mu + \overline{\Omega}^\mu &= \Gamma^\mu + \overline{\Gamma}^\mu \\ \Omega^\mu - \overline{\Omega}^\mu &= \Gamma^{D+1} \cdot (\Gamma^\mu - \overline{\Gamma}^\mu) \end{aligned} \tag{96}$$

where Γ^{D+1} has been introduced in (9). $\overline{\Gamma}^\mu$ denotes $\Gamma^{\mu\,\dagger}$, $\Gamma^{\mu\,T}$ or $\Gamma^{\mu\,*}$ according to the context.

Let us analyze in detail the situation for each one of the three A, B, C structures so far investigated.

The A structure (the adjoint case) works only when the $D = 2n$ spacetime admits n space directions and n time directions. We recall that the $\dagger$-conjugation property of Γ^μ depends on its signature. It turns out as a consequence that both Z_+, Z_- describe an euclidean n-dimensional space (we apply on the space described by Z_- an overall change of the signature).

For what concerns the C-structure we recall the results presented in the previous sections. We can denote as t_+ and s_+ the number of respectively timelike and spacelike directions which are associated to symmetric Γ-matrices. Similarly t_- and s_- are the number of timelike and spacelike directions whose Γ-matrices are antisymmetric. The relations (67) among $t_\pm$, $s_\pm$ hold. As a consequence the Z_+ (Z_-) coordinate describes a relativistic spacetime with signature (t_+, s_+) (and respectively (t_-, s_-)).

The same result applies also when the B (complex) structure is considered. The vanishing of the index I as introduced in (68) now requires $t_+ = t_-$ and $s_+ = s_-$. Let us $(t, s) \equiv (2k, 2n - 2k)$ be the signature of the original spacetime. The spacetime described by Z_+ results having the same signature as the spacetime furnished by the Z_- matrix coordinate i.e.

$$(t_+, s_+) = (t_-, s_-) = (k, n - k).$$

This is the last result which completes our analysis concerning the relativistic separation of variables.

11 The splitting case.

In this section we present a different way, alternative to the construction so far employed, of solving the set of relations (77). It is based on the Γ-matrices realization expressed by the formula (39). Due to the presence in (39) of tensor products of lower-dimensional spacetimes Γ matrices, the construction based on (39) will be referred as the "splitting case". It proceeds as follows. At first we introduce two matrix-valued coordinates X_+ and X_- through the positions

$$X_+ = x_m \cdot \begin{pmatrix} 0 & \mathbb{1}_q \otimes {\gamma_p}^m \\ \mathbb{1}_q \otimes {\gamma_p}^m & 0 \end{pmatrix} \tag{97}$$

and

$$X_- = x_{\overline{m}} \cdot \begin{pmatrix} 0 & {\gamma_q}^{\overline{m}} \otimes \mathbb{1}_p \\ {\gamma_q}^{\overline{m}} \otimes \mathbb{1}_p & 0 \end{pmatrix} \tag{98}$$

The conventions introduced in section 5 are employed. In particular m takes value in a $(p+1)$-dimensional space and $\overline{m}$ in a $(q+1)$-dimensional one. The total spacetime is $D = p + q + 2$ (38).

Clearly $X_\pm$ commute

$$[X_+, X_-] = 0 \tag{99}$$

The matrix coordinates $X_\pm$ realize a relativistic separation of variables since the quadratic pseudoeuclidean form ds^2 can be written as

$$ds^2 \cdot \mathbb{1} = d{X_+}^2 + d{X_-}^2 \tag{100}$$

The matrix derivatives $\partial_\pm$ can be introduced through

$$\partial_+ = \frac{1}{(p+1)} \cdot \partial_m \begin{pmatrix} 0 & \mathbb{1}_q \otimes {\gamma_p}^m \\ \mathbb{1}_q \otimes {\gamma_p}^m & 0 \end{pmatrix} \tag{101}$$

and

$$\partial_- = \frac{1}{(q+1)} \cdot \partial_{\overline{m}} \begin{pmatrix} 0 & {\gamma_q}^{\overline{m}} \otimes 1\!\!1_p \\ {\gamma_q}^{\overline{m}} \otimes 1\!\!1_p & 0 \end{pmatrix} \tag{102}$$

$\partial_\pm$ are correctly normalized so that their left action on $X_\pm$ produce

$$\partial_+ X_+ = \partial_- X_- = 1\!\!1 \tag{103}$$

Moreover the disentangling condition

$$\partial_+ X_- = \partial_- X_+ = 0 \tag{104}$$

is verified.

With the help of $X_\pm$ we can construct the matrix-valued Z, $\overline{Z}$ which allow to decompose the quadratic form ds^2 through

$$ds^2 \cdot 1\!\!1 = d\overline{Z} \cdot dZ$$

This can be done by setting

$$\begin{aligned} Z &= X_+ + iX_- \\ \overline{Z} &= X_+ - iX_- \end{aligned} \tag{105}$$

Z, $\overline{Z}$ commute. They can be regarded as $2D$ matrix-valued coordinates as discussed in the introduction. In order to define a calculus we need the introduction of the matrix derivatives ∂_Z, $\partial_{\overline{Z}}$. In the light of the "splitting approach" here discussed, this can be done unambiguously by setting

$$\begin{aligned} \partial_Z &= \frac{1}{2}(\partial_+ - i\partial_-) \\ \partial_{\overline{Z}} &= \frac{1}{2}(\partial_+ + i\partial_-) \end{aligned} \tag{106}$$

The above ∂_Z, $\partial_{\overline{Z}}$ derivatives satisfy all the required properties; they commute and moreover

$$\begin{aligned} \partial_Z Z = \partial_{\overline{Z}} \overline{Z} = 1\!\!1 \\ \partial_Z \overline{Z} = \partial_{\overline{Z}} Z = 0 \end{aligned} \tag{107}$$

as left action.

The only crucial point left is whether ∂_Z, $\partial_{\overline{Z}}$ realize a factorization of the d'Alembertian $\Box$ operator. It follows that

$$\partial_{\overline{Z}} \partial_Z = \frac{1}{4}({\partial_+}^2 + {\partial_-}^2) \tag{108}$$

On the other hand ${\partial_\pm}^2$ satisfy

$$\begin{aligned} {\partial_+}^2 &= \frac{1}{(p+1)^2} 1\!\!1_q \otimes 1\!\!1_p \Box_+ \\ {\partial_-}^2 &= \frac{1}{(q+1)^2} 1\!\!1_q \otimes 1\!\!1_p \Box_- \end{aligned} \tag{109}$$

where $\Box_+$ and $\Box_-$ are the d'Alembertian for respectively the $(p+1)$ and the $(q+1)$ dimensional subspaces. It turns out that the property

$$\partial_{\overline{Z}}\partial_Z \;\propto\; \Box$$

is verified only in the case

$$p \;=\; q \tag{110}$$

that is, the subspaces associated to X_+, X_- have equal dimensions.

We recall that p, q entering (39) are even dimensional, so that we can set $p = q = 2n$. From the (38) condition $D = p + q + 2$, it follows that the $2D$-matrix coordinate calculus can be introduced with the splitting condition only for spacetimes whose dimensionality D is restricted to be an even integer of the kind

$$D \;=\; 4n+2 \tag{111}$$

for some integral n.

This conclusion furnishes also the proof that the "splitting case" here considered is different from the previously analyzed A, B, C cases. In such cases only restrictions to the signature of the spacetimes (for the A and B structures) were found, while the dimensionality itself of the spacetimes is no further restricted (besides the initial even-dimesionality requirement).

Let us conclude this section by pointing out that Z, $\overline{Z}$ can be represented as

$$\begin{aligned} Z &= x_\mu\Omega^\mu = x_m\Omega^m + x_{\overline{m}}\Omega^{\overline{m}} \\ \overline{Z} &= x_\mu\overline{\Omega}^\mu = x_m\Omega^m + x_{\overline{m}}\Omega^{\overline{m}} \end{aligned} \tag{112}$$

(μ is an index which spans the D dimensional spacetime), where Ω^μ, $\overline{\Omega}^\mu$ can be immediately read from (97,98) and (105).

One can easily check that the derivatives $\partial_\pm$ of formulas (101,102) can be represented in the form

$$\begin{aligned} \partial_Z &= \frac{1}{D}\partial_\mu\overline{\Omega}^\mu \\ \partial_{\overline{Z}} &= \frac{1}{D}\partial_\mu\Omega^\mu \end{aligned} \tag{113}$$

only when the equality $p = q$ is satisfied. The algebra which has been analyzed in section 9 can be formally recovered in the splitting (S) case. The "splitting" is another construction which allows satisfying the relations (77).

12 An application to forms.

In the introduction we mentioned that one possible application for the $2D$-matrix formalism consists in investigating abelian and Yang-Mills gauge theories. In this respect it is convenient to outline how differential forms can be introduced in the light of the $2D$ matrix formalism. We sketch it here. Notations and conventions are those reported in section 9.

With the help of the differentials $dZ = dx_\mu \Omega^\mu$, $d\overline{Z} = dx_\mu \overline{\Omega}^\mu$ we can construct the wedge products

$$\begin{aligned} dZ \wedge dZ &= \frac{1}{2} dx_\mu \wedge dx_\nu \cdot \Xi_-{}^{\mu\nu} \\ d\overline{Z} \wedge dZ &= \frac{1}{2} dx_\mu \wedge dx_\nu \cdot \overline{\Omega}_-{}^{\mu\nu} \\ dZ \wedge d\overline{Z} &= \frac{1}{2} dx_\mu \wedge dx_\nu \cdot \Omega_-{}^{\mu\nu} \\ d\overline{Z} \wedge d\overline{Z} &= \frac{1}{2} dx_\mu \wedge dx_\nu \cdot \overline{\Xi}_-{}^{\mu\nu} \end{aligned} \tag{114}$$

Notice that, due to the matrix character of the coordinates, $dZ \wedge dZ \neq 0$ and similarly $d\overline{Z} \wedge d\overline{Z} \neq 0$, while $dZ \wedge d\overline{Z} \neq -d\overline{Z} \wedge dZ$.

The differential operator d which satisfies the nilpotency condition

$$d^2 = 0 \tag{115}$$

can be decomposed through

$$d \cdot \mathbb{1} = dx^\mu \partial_\mu \cdot \mathbb{1} = \partial + \overline{\partial} \tag{116}$$

where ∂, $\overline{\partial}$ are given by

$$\begin{aligned} \partial &= \frac{D}{2} \partial_Z \cdot dZ \\ \overline{\partial} &= \frac{D}{2} d\overline{Z} \cdot \partial_{\overline{Z}} \end{aligned} \tag{117}$$

The equality (116) is a consequence of the (77) relation.

Please notice that the order in which derivatives and differentials are taken is important because they are no longer commuting in the matrix case.

The wedge products between the differential operators ∂, $\overline{\partial}$ is in general complicated. The simplest expression, the only one which deserves being here reported, is for $\overline{\partial} \wedge \partial$:

$$\overline{\partial} \wedge \partial = \frac{1}{32} \Box dx_\mu \wedge dx_\nu \overline{\Omega}^{\mu\nu} \tag{118}$$

A one-form A can be introduced with the positions

$$\begin{aligned} A_Z &= A_\mu \overline{\Omega}^\mu \\ A_{\overline{Z}} &= A_\mu \Omega^\mu \end{aligned} \tag{119}$$

Indeed we have for A

$$A = A_\mu dx^\mu \cdot \mathbb{1} = \frac{1}{2}(A_Z dZ + d\overline{Z} A_{\overline{Z}}) \tag{120}$$

In the abelian case a gauge transformation is simply realized by the mapping

$$A \mapsto A + d\Lambda = A + (\partial + \overline{\partial})\Lambda$$

where Λ is a matrix-valued 0-form.

The stress-energy tensors $F_{\mu\nu}$ are introduced as two-forms with the standard procedure

$$
\begin{aligned}
F &= dx_\mu \wedge dx_\nu F^{\mu\nu} \cdot \mathbb{1} = \frac{1}{2} dx_\mu \wedge dx_\nu (\partial^\mu A^\nu - \partial^\nu A^\mu) \cdot \mathbb{1} = \\
&= dA = (\partial + \overline{\partial})A
\end{aligned} \tag{121}
$$

13 Conclusions.

In this paper we have introduced a matrix-calculus to describe relativistic field theories in higher-dimensional spacetimes. We discussed the single-matrix approach, which can be applied for instance to scalar bosonic theories, and the $2D$ matrix calculus, by far more general, which employes matrix-valued Z, $\overline{Z}$ coordinates. We pointed out the manifest Lorentz-covariance of our approach; furthermore we investigated the consistency conditions which made it possible.

In order to solve this problem we produced some other results as byproducts. The recursive formula (39) to construct Γ-matrices is an example. The computation of the coefficients in the "vector-trace" formulas (57) is another one. This computation has also lead us to introduce an index labelling inequivalent Γ-structures under conjugation realized by real orthogonal matrices. Such an index describes as well the equivalence classes of Wick rotations from the euclidean into the pseudoeuclidean spacetimes.

Since a short summary of the main results here presented has already been furnished in the introduction, we do not repeat it now. Rather, we prefer to give some commentaries concerning the potentialities of the formalism we have constructed.

It surely deserves being stressed the fact that the existence of a $2D$ matrix-calculus relies on non-trivial properties concerning dimension and signature of spacetimes. These properties are described by nice mathematical formulas. At a purely formal level we dispose of a very attractive mathematical construction. Spacetimes of different dimensions and signature can be formally treated on equal footing. The different properties they share are automatically encoded in the calculus. This feature could be even more relevant for its supersymmetric extension (presently under construction). It is expected to put even more restrictions on the allowed spacetimes. It seems more than a mere possibility that the spacetimes which can be consistently defined would be those obtained from superstring. The question concerning the nature of the spacetime and its signature [12] can in principle be raised for the $2D$ matrix calculus.

At a less formal level and more down-to-earth point of view, we have of course to ask ourselves the question about the usefulness and applicability of the whole construction. So, let us state it clearly. We dispose of a formalism which can be jokingly named as "fat flat space" (where "fat" stands for matrix). In the present paper we have just unveiled the basic roots of such a formalism. Of course more work is required to introduce e.g. lagrangians, Poisson brackets, hamiltonians and so on, or to deal with curved spacetimes, but in fact there is no obstacle in performing such extensions. Indeed they can be carried out quite straightforwardly. The main point here is that our construction can in principle lead to investigate higher-dimensional relativistic field theories by borrowing the techniques employed for standard $2D$ physics.

In the introduction we already mentioned the issue of integrability. In fact we have a lot more. In standard $2D$ physics hamiltonian methods are widely used. They are more powerful than lagrangian methods and, due to the fact that the $2D$ Poincaré invariance admits only three generators, in just $2D$ the loss of the manifest Lorentz-covariance implicit in the hamiltonian approach is not a such a big loss. Our Z, $\overline{Z}$ coordinates can in principle be used for such a hamiltonian description. Moreover, issues like current algebras can be investigated in the light of the $2D$ matrix approach. This could mean the extension of WZNW theories to higher-dimensional spacetimes (see [13]), as well as their possible hamiltonian reductions ([14]) to higher-dimensional relativistic Toda field theories ([15]).

Another topic in mathematical physics which can profit of the present formalism concerns issues of index theorem and computation of the index for elliptic operators in higher dimension. The recursive formula (39) provides the basis for factorizing elliptic operators (just repeating the steps done for the standard d'Alembertian). Heat-kernel computations can be made in terms of the $2D$-matrix calculus.

Let us finally mention that the "splitting of variables" described in section 11 admits an useful application in analyzing reductions from higher-dimensional spacetime to lower-dimensional ones, with a procedure which can be regarded as a "folding" of spacetimes (allowing to express e.g. de Sitter or anti-de Sitter spacetimes from an underlying 10-dimensional theory). This is the content of a work currently at an advanced stage of preparation.

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