CBPF-NF-063/96

θ-Vacua in the Light-Front Quantized Schwinger Model †

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Abstract

The light-front (LF) quantization of the bosonized Schwinger model is discussed in the *continuum formulation.* The proposal, successfully used earlier for describing the spontaneous symmetry breaking (SSB) on the LF, of separating first the scalar field into the dynamical condensate and the fluctuation fields before employing the standard Dirac method works here as well. The condensate variable, however, is now shown to be a q-number operator in contrast to the case of SSB where it was shown to be a c-number or a background field. The condensate or θ-vacua emerge straightforwardly together with their continuum normalization which avoids the violation of the cluster decomposition property in the theory. Some topics on the *front form* theory are summarized in the Appendices and attention is drawn to the fact that the theory quantized, say, at equal x^+ seems already to carry information on equal x^- commutators as well.

Key-words: Light-front; θ-Vacua; Spontaneous symmetry breaking; Gauge theory.
IF-UERJ 034/96; hep-th@xxx.lanl.gov/9610149
September 1996

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† *Invited talk given at the First US-Brasilian Workshop on Topology, Geometry, and Physics, UNICAMP, Campinas, SP, Brasil, July '96* and at the Intl. *Workshop on Light-Front Quantization and Non-Perturbative QCD*, Iowa State University, Ames, Iowa, USA, *March '96.* To be published in the Proceedings of the US-Brasilian Workshop.

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1. Introduction

Dirac [1] in 1949 pointed out the advantages of studying the relativistic quantum dynamics of physical system on the hyperplanes of the light-front: $x^0 + x^3 = const.$, the *front form*. The LF or light-cone coordinates, in place of (x^0, x^1, x^2, x^3), are then convenient to use; they are defined by $(x^+, x^-, x^\perp)$ where $x^\pm = (x^0 \pm x^3)/\sqrt{2} = x_\mp$ and $x^\perp = (x^1, x^2)$. The metric tensor for the indices $\mu = +, -, 1, 2$ may be read from the Lorentz invariant expression $A^\mu B_\mu = g_{\mu\nu} A^\mu B^\nu = g^{\mu\nu} A_\mu B_\nu = A^+ B^- + A^- B^+ - A^\perp . B^\perp$. We make the *convention* to regard ‡ $x^+ \equiv \tau$ as the LF *time* coordinate while $x^- \equiv x$ is the longitudinal spatial coordinate and we study the evolution in τ of the dynamical system. The LF components of any four-vector or tensor are similarly defined. We note that the separation of two points x and y on equal-τ plane is also spacelike. It becomes lightlike when $x^\perp = y^\perp$ but, unlike in the equal-time case, the points need not be coincident since $(x^- - y^-)$ may take any value. The *microcausality principle* leads to locality requirement only in $x^\perp$ and the appearence of any nonlocality in the longitudinal coordinate in the theory would not be unexpected [2].

The transformation from the conventional to LF components is, however, *not a Lorentz transformation* and the structure of the *LF phase space* is notably different when compared with the conventional phase space. For example, the momentum four-vector is $(k^-, k^+, k^\perp)$ where k^- is the LF energy while $k^\perp$ and k^+ indicates the transverse and the longitudinal components of the momentum. A massive particle on the mass shell, $k^- = (m^2 + {k^\perp}^2)/(2k^+)$, has positive definite values for $k^\pm$ in contrast to $-\infty \le k^{1,2,3} \le \infty$ for the usual components. An immediate consequence is that the *vacuum* in the LF *quantized theory* may become simpler than the one in the conventional (equal-time) theory and in many cases the interacting theory vacuum on the LF may be the same as the perturbation theory vacuum. For example, the conservation of the total longitudinal momentum would not permit the excitations of particle-antiparticle pairs by the LF vacuum (having $\sum k^+ = 0$). The SSB on the LF, for example, is described [2] in a way different from the conventional one even though the physical outcome, as expected, is the same (Apppendix C).

An important advantge pointed out by Dirac is that *seven* out of the ten Poincaré

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‡ We can of course make the convention with the role of x^+ and x^- interchanged. In Appendix D we illustrate by an example how the equal-x^+ quantized theory does seem to contain information on the equal-x^- commutators as well.

generators are *kinematical* on the LF while in the conventional theory constructed on the hyperplanes $x^0 = const.$, the *instant form*, only *six* have this property. Also the notions of spin on the LF for massive and massless particles seem to get unified (Appendices A and B).

Another notable feature of a relativistic theory in the *front form* is that it gives rise to a singular Lagrangian, e.g., a constrained dynamical system [3]. It leads in general to a reduction in the number of independent field operators on the corresponding phase space. The vacuum structure may then become more tractable and the computation of the physical observables may become simpler. It is illustrated below by the detailed study of the Schwinger model [4]. In the conventional framework, for example, the QCD vacuum is quite complex due to the *infrared slavery* and it contains also the gluonic and fermionic condensates. There seems also to exist a contradiction between the Standard Quark Model and the QCD containing a sea of partons (quarks, anti-quarks and gluons); the *front form* theory may throw light on such problems.

The LF field theory was rediscovered in 1966 by Weinberg [5] in his Feynman rules adapted for infinite momentum frame. Latter it was demonstrated [6] that these rules correspond to the quantization of the theory on the LF. The recent revival [7-9] of the interest in LF quantization owes, say, to the difficulties encountered in the computation of nonperturbative effects in the *instant form* QCD or in the study of the relativistic bound states of light fermions [8,7] which can not be handled, say, by the Lattice gauge theory. The LF coordinates have proved also very useful in the study of the theories of (super-) strings and membranes as well.

The purpose of the present work is to show how the quantization of the massless Schwinger model on the LF leads in a straightforward way to the θ-vacua known to emerge [10,11] in the *instant form* theory. The important feature of the continuum normalization of these states arises naturally on the LF. This is in contrast to the discussions in the conventional Lorentz or the Coulomb gauge framework where we have to invoke arguments to impose it, so as to avoid [10-12] the violation of the the cluster decomposition property. The discussion on the LF is more transparent due to a reduced number of independent fields. It is obtained here by making use of the method proposed earlier in connection with the *front form* description of the SSB (and the tree level Higgs mechanism) [13,2]. The scalar field (of the equivalent bosonized Schwinger model) is separated, based on physical considerations, into the dynamical bosonic condensate and the quantum fluctuation fields.

The Dirac procedure [3] is then followed in order to construct the Hamiltonian formulation and the quantized theory.

After a brief discussion in Sec. 2 of the Schwinger model its bosonized version is quantized on the LF. The *condensate* or θ-vacua are described in Sec. 3 while the conclusions are summarized in Sec. 4. Appendices contain some material related to the *front form* field theory.

2. LF Quantization of the Schwinger Model

The field theory model is simply the two dimensional massless quantum electrodynamics with the Lagrangian

$$\mathcal{L} = \bar{\psi}\, i\gamma^\mu \partial_\mu \psi - \frac{1}{4} F^{\mu\nu} F_{\mu\nu} - e\bar{\psi}\, \gamma^\mu \psi A_\mu, \tag{1}$$

where ψ is a two-component spinor field† and A_μ is the $U(1)$ gauge field. The Lagrangian is invariant under the global $U(1)_5$ chiral transformations $\psi \to exp(i\gamma_5 \alpha)\, \psi$ apart from under the usual $U(1)$ gauge transformations. The model is exactly solvable and its physical spectrum consists solely of massive vector field [4]. This was made explicit by Lowenstein and Swieca [10,11] through their operator solution of the model in the *instant form* framework in the Lorentz gauge and where the complex structure of the ground state was also studied. It has also been studied in the Coulomb gauge [12] and on the light-cone in the discretized formulation [14] where there are many subtleties in the fermionic version (1). We will work here directly in *the continuum formulation* (Appendix C) on the LF so that the spurious finite volume effects get automatically suppressed. For the purposes of studying the vacuum structure it is also convenient to study the *equivalent* bosnized version.

The LF coordinates are now (x^+, x^-) where $x^\pm = (x^0 \pm x^1)/\sqrt{2} = x_\mp$. The conserved Noether currents defined by $j_\mu = \bar{\psi}\gamma_\mu \psi$ and $j_{5\mu} = \bar{\psi}\gamma_5\gamma_\mu\psi$ satisfy the relation $j_5^\mu = \epsilon^{\mu\nu} j_\nu$ which is particular to the two dimensional theory. Written explicitly, $j_5^+ = -j^+ = -\sqrt{2}{\psi_2}^\star \psi_2$, $j_5^- = +j^- = \sqrt{2}{\psi_1}^\star \psi_1$. The classical conservation of both the current leads to $j^+ = j^+(x^-)$ and $j^- = j^-(x^+)$ like in the case of the free Dirac field. In the quantized theory, however, the currents must be defined, say, by the introduction of the of the point-splitting of operators. When the gauge coupling is also present we are required to construct gauge invariant currents together with the point-splitting. The divergence of the

† $\gamma^0 = \sigma_1, \gamma^1 = i\sigma_2, \gamma_5 = -\sigma_3, \eta^{00} = -\eta^{11} = 1, \epsilon^{-+} = \epsilon^{01} = +1$, and $\gamma_5\gamma^\mu = \epsilon^{\mu\nu}\gamma_\nu$ etc.

axial current is then found anomalous: $\partial_\mu j_5^\mu = (e/(2\pi))\epsilon_{\mu\nu}F^{\mu\nu}$ and the eqs. of motion lead to $(\partial_\mu\partial^\mu + e^2/\pi)\epsilon^{\mu\nu}F_{\mu\nu} = 0$ implying dynamical mass generation [4] for the gauge field in the quantized theory. The same results may also be obtained by employing functional integal methods [11].

The abelian gauge theory under consideration has been extensively studied and its exact solvability [4,10] derives from the remarkable property of one-dimensional fermion systems, viz, that they can equivalently be described in terms of canonical one-dimensional boson fields [15]. The operator solution [10] mentioned above in a sense amounts to bosonization. Some of the relevant correspondences in abelian bosonization are $\bar\psi\psi = K : \cos 2\sqrt{\pi}\,\phi :$, $\bar\psi\gamma_5\psi = K : \sin 2\sqrt{\pi}\,\phi :$, $\bar\psi\gamma_5\gamma_\mu\psi = \partial_\mu\phi/\sqrt{\pi}$, $\bar\psi\gamma_\mu\psi = \epsilon_{\mu\nu}\partial^\nu\phi/\sqrt{\pi}$, $\bar\psi\, i\gamma.\partial\psi = \frac{1}{2}\partial_\mu\phi\partial^\mu\phi$ where ϕ is a bosonic scalar field and K is a constant. The fermionic condensate $< \bar\psi\psi >_0$, for example, may then be expressed in terms of the value of the bosonic condensate. The bosonized theory can also be constructed with the use of the functional integral method [11]. The original fermionic and the bosonized theories are *equivalent* in the sense that they have the same current commutation relations and the energy-momentum tensor is the same when expressed in terms of the currents.

The Lagrangian density of the *bosonized* massless Schwinger Model takes the following form

$$\mathcal{L} = \frac{1}{2}\partial_\mu\phi\partial^\mu\phi - gA_\mu\epsilon^{\mu\nu}\partial_\nu\phi - \frac{1}{4}F^{\mu\nu}F_{\mu\nu}, \tag{2}$$

where $g = e/\sqrt{\pi}$. It carries in it all the symmetries of the original fermionic model including the information on the mass generation. Under the $U(1)$ gauge field transforamtion the scalar field is left invariant while under the $U_5(1)$ chiral transformations, in view of the correspondences above, it suffers a translation by a constant. This is crucial in obtaining the so called θ- or *condensate* vacua. The quantization of the bosonized theory would allow us to describe the vacuum structure of the original theory and to compute, say, the fermionic condensate.

We first make the *separation*, proposed in Ref. [13], in the scalar field (a generalized function) : $\phi(\tau, x^-) = \omega(\tau) + \varphi(\tau, x^-)$, where $\omega(\tau)$ is the (dynamical) bosonic *condensate* variable and the field φ represents the quantum fluctuations. This enabled us to give [13] a description on the LF of the SSB and (tree level) Higgs mechanism and where the variable ω was shown to be a c-number, e.g., a background field. In the Schwinger model, on the contrary, it will be shown below to turn out as a q-number operator and its eigenvalues would label the vacuum states. We will set $\int dx^- \varphi(x^+, x^-) = 0$ so that the entire zero-momentum mode of ϕ is represented by the condensate variable. The *chiral transformations* would be defined [12] as: $\omega \to \omega + const.$, $\varphi \to \varphi$, and $A_\mu \to A_\mu$ so that the boundary conditions at infinity on φ are left unaltered (see Sec. 3). The bosonized Lagrangian then becomes

$$L = \int_{-R/2}^{R/2} dx^- \Big[\dot{\varphi}\varphi' + g(A_+\varphi' - A_-\dot{\varphi}) + \frac{1}{2}(\dot{A}_- - {A_+}')^2\Big] - g\dot{\omega}\int_{-R/2}^{R/2} dx^- A_-(\tau, x^-) \quad (3)$$

Here $R \to \infty$, an overdot (a prime) indicates the partial derivative with respect to $x^+ \equiv \tau$ ($x^- \equiv x$), and φ is assumed to satisfy the conditions required for the existence of the Fourier transform in the spatial variable x^-.

The last term of (3) shows that the light-cone gauge $A_- = 0$ is not convenient to adopt in the present case. It is suggested and in fact will be shown below that the zero-momentum mode of A_-, viz, $h(\tau) \equiv \int dx^- A_-$ and ω form a canonically conjugate pair. We would instead adopt the gauge [16] $\partial_- A_- = 0$ which is shown to be accessible on the phase space. The canonical momenta defined from (3) are

$$\begin{aligned}
\pi &= \frac{\delta\mathcal{L}}{\delta\dot{\varphi}} = \varphi' - gA_- \\
E^+ &= \frac{\delta\mathcal{L}}{\delta\dot{A}_+} = 0 \\
E^- &= \frac{\delta\mathcal{L}}{\delta\dot{A}_-} = (\dot{A}_- - {A_+}') \\
p &= \frac{\partial L}{\partial\dot{\omega}} = -gh(\tau) \equiv -\pi_\omega
\end{aligned} \quad . \quad (4)$$

The primary constraints [3] are thus $\chi \equiv \pi - \varphi' + gA_- \approx 0$, $E^+ \approx 0$, and $T(\tau) \equiv (-\pi_\omega + gh) \approx 0$, where $\approx$ indicates the weak equality [3]. The canonical Hamiltonian is found to be

$$H_c = \int dx^- \Big[\frac{1}{2}{E^-}^2 + E^- {A_+}' - gA_+\varphi'\Big]. \quad (5)$$

and we take the preliminary [3] Hamiltonian as

$$H' = H_c + \int dx^- \Big[u_+ E^+ + u\chi\Big] + \lambda(\tau)T(\tau), \quad (6)$$

where u_+, u, and λ are the Lagrange multiplier fields. We assume initially the standard equal-τ Poisson brackets: $\{\pi_\omega(\tau), \omega(\tau)\} = +1$, $\{E^\mu(\tau, x^-), A_\nu(\tau, y^-)\} = -\delta^\mu_\nu \delta(x^- - y^-)$, $\{\pi(\tau, x^-), \varphi(\tau, y^-)\} = -\delta(x^- - y^-)$ etc. from which it follows, for example, that $\{E^\mu(\tau, x^-), h(\tau)\} = -\delta^\mu_-$. On requiring the persistency of the constraints in τ employing $df/d\tau = \{f, H'\} + \partial f/\partial\tau$ we derive the secondary constraints $K(\tau) \equiv \int dx^-[E^- + A'_+] \approx 0$ and $\Omega \equiv \partial_-(E^- + g\varphi) \approx 0$. We go over now to an extended Hamiltonian including these constraints as well and repeat the procedure. No new constraints are shown to be generated since only the equations which would determine the Lagrange multiplier fields are left. The constraints Ω and E^+ are easily shown to be first class [3] while T, K, and χ are second class [3]. From the eqs. of motion we show that we may determine the multiplier fields such that $\partial_- A_- \approx 0$ and $(E^- + A'_+) \approx 0$ along with their persistency conditions are satisfied. We may thus add to the theory these ones as the (external) gauge-fixing constraints, corresponding to the two first class constraints found above. The whole set of constraints then becomes second class and we may proceed to construct Dirac brackets [3], which would replace the Poisson ones, such that the weak equalities may be replaced by the strong equalities (even) inside them.

It is straightforward to construct the Dirac bracket iteratively. We first handle the pair $T \approx 0$, $K \approx 0$ with $\{T, K\} \approx gR$, $\{T, T\} \approx 0$, and $\{K, K\} \approx 0$. The corresponding modified equal-τ bracket is easily constructed[‡]

$$\{f, k\}_1 = \{f, k\} + \frac{1}{gR}\Big[\{f, T\}\{K, k\} - \{f, K\}\{T, k\}\Big], \tag{7}$$

We verify, for example, $\{f, T\}_1 \equiv 0$, $\{K, f\}_1 \equiv 0$ etc.. The modified bracket differs from the Poisson one only when the variables ω, E^-, E^+, or A_- are to be found in the functionals f and k. For example, $\{\omega, h\}_1 = -(1/g)$, $\{\omega, E^+\}_1 = 0$, $\{E^-, A_-\}_1 = \{E^-, A_-\} + (1/R)$, $\{f, \chi\}_1 = \{f, \chi\} - (1/R)\{f, T\}$, $\{\chi, \chi\}_1 = \{\chi, \chi\} = -2\partial^x_- \delta(x^- - y^-)$.

A further modification

$$\{f, k\}_2 = \{f, k\}_1 + \frac{1}{4}\int\int du dv\ \{f, \chi\}_1 \epsilon(u - v)\{\chi, k\}_1, \tag{8}$$

allows us to set $\chi = 0$ also as strong equality. We find, for example, $\{\omega, \omega\}_2 = 0$,

[‡] We make the convention that the first variable in an equal-τ bracket $\{f, k\}$ refers to longitudinal coordinate x^- while the second one to y^-. We remind that we are working *in the continuum formulation* and that $R \equiv \int_{-R/2}^{R/2} dx \to \infty$ at the end of the computation.

$\{E^-, E^-\}_2 = (-g^2/4)[\epsilon(x-y) + (1/R)\int du \ \{\epsilon(u-x) - \epsilon(u-y)\}]$, $\{\omega, E^-\}_2 = -(g/(4R))\int du\, \epsilon(u-y)$, $\{\varphi, \pi\}_2 = (1/2)\delta(x-y)$, $\{\varphi, \varphi\}_2 = (-1/4)\epsilon(x-y)$, $\{\varphi, E^-\}_2 = (g/4)\epsilon(x-y)$, $\{E^+, \Omega\}_2 = 0$, $\{\Omega, \Omega\}_2 = 0$ etc., and that E^+ and Ω continue to remain first class even with respect to $\{,\}_2$.

Next the *gauge-fixing* constraint $\chi_3 \equiv \partial_- A_- \approx 0$ along with $\Omega \approx 0$ are implemented. The constraint matrix has elements $\{\chi_3, \chi_3\}_2 = 0$, $\{\chi_3, \Omega\}_2 = -\partial_x^2 \delta(x-y)$, and $\{\Omega, \Omega\}_2 = 0$ and its inverse is easily found to be $i\sigma_2\, \partial_x^{-2}\delta(x-y)$. Hence we define

$$\{f, k\}_3 = \{f, k\}_2 + \int du \left[\{f, A_-\}_2 \{E^- + g\varphi, k\}_2 - \{f, E^- + g\varphi\}_2 \{A_-, k\}_2\right] \qquad (9).$$

We find $\{A_-, \omega\}_3 = 1/(gR)$ and $\{A_-, f\}_3 \to 0$ when $R \to \infty$ for $f \neq \omega$. Some of the others are $\{\omega, E^-\}_3 = (-g/(4R))\int du\, \epsilon(u-y)$, $\{h, E^-\}_3 = 0$, $\{\omega, h\}_3 = -1/g$, $\{E^-, E^-\}_3 = (-g^2/4)\epsilon(x-y)$.

The remaining first class constraint $E^+ \approx 0$ may be taken care of by adding still another *gauge-fixing* constraint $\Phi \equiv E^- + A'_+ \approx 0$. The two dimensional constraint matrix $C(x,y)$ here has the elements: $C_{11} = \{E^+, E^+\}_3 = 0$, $C_{12} = C_{21} = \{E^+, \Phi\} = \{E^+, A'_+\} = \partial_x \delta(x-y)$, $C_{22} = \{\Phi, \Phi\}_3 = -g^2\epsilon(x-y)/4$. An inverse matrix C^{-1} is shown to have the elements $C^{-1}_{12} = C^{-1}_{21} = (1/2)\epsilon(x-y)$ and $C^{-1}_{22} = 0$ while C^{-1}_{11} satisfies $\partial_x^2 C^{-1}_{11}(x,y) = (g^2/4)\epsilon(x-y)$. The final Dirac bracket which implements all the constraints is thus constructed to be

$$\begin{aligned}\{f, k\}_D = \{f, k\}_3 &\\ &- \frac{1}{2}\int\int du dv \Big[\{f, \Phi\}_3 \epsilon(u-v)\{E^+, k\}_3 - \{f, E^+\}_3 \epsilon(u-v)\{\Phi, k\}_3 \\ &+ 2\{f, E^+\}_3\, C^{-1}_{11}(u,v)\, \{E^+, k\}_3\Big]\end{aligned} \qquad (10)$$

We find $\{E^-, E^-\}_D = -g^2\epsilon(x-y)/4$, $\{\varphi, \varphi\}_D = -\epsilon(x-y)/4$, $\{A_-, k\}_D = 0$ for $k \neq \omega$, $\{A_-, \omega\}_D = 1/(gR)$, $\{\pi_\omega, \omega\}_D = 1$, $\{A_+, A_+\}_D = C^{-1}_{11}(x,y)$, and $\{\omega, E^-\}_D = -g\int du\, \epsilon(u-y)/(4R)$ among the others. All the constraints can now be written as strong equalities and we are left behind with the independant variables φ, ω, π_ω and $E^- = -\partial_- A_+ = -g\varphi$. The Hamiltonian density is effectively given by $\mathcal{H}_D = {E^-}^2/2 + \partial_-(E^- A_+)$ and $H_D = \int dx\, {E^-}^2/2 = g^2\int dx\, \varphi^2/2$ corresponding to a free scalar field φ of mass e^2/π. It may be checked that for the self-consistency with the Lagrangian eqs. we require that A_+ should satisfy the periodic boundary conditions at infinity in x^-.

3. θ-Vacua

We discuss now the vacuum state in the LF quantized theory. The *ultimate physical conclusions* would of course coincide with those following from the *instant form* theory. The *quantized theory* is obtained through the correspondence [3] $i\{f,k\}_D \to [f,k]$ with the commutators of the field operators. We find $[\omega,\omega]=0$, $[\pi_\omega,\omega]=i$, $[\omega,E^-]=[\omega,\varphi]=[\pi_\omega,\varphi]=[\omega,H_D]=[\pi_\omega,H_D]=0$, and the well known LF commutator $[\varphi,\varphi]=-i\epsilon(x-y)/4$ which leads to $2\pi[j^+(\tau,x),j^+(\tau,y)] = i\partial_-\delta(x-y)$, where the right hand side is the Schwinger term. The condensate field ω turns out to be a q-number (operator) in the present model in contrast to the case of the LF quantized scalar theory where it is shown (Appendix C) to be a c-number. This is expected since the chiral symmetry in the model is realized as translation of the value of the condensate. The latter must then be represnted by an operator ω in order that its eigenvalues may be shifted by another operator. The $\omega(\tau)$ in our discussion has been treated as a dynamical variable and we let the Dirac procedure to determine if it is a c- or q-number in the quantized theory.

The original fermionic Schwinger model is invariant under *global* $U(1)_5$ chiral transformations $\psi \to exp(i\alpha\gamma_5)\,\psi$, $\bar\psi\psi \to \bar\psi\, exp(i2\alpha\gamma_5)\,\psi$. From $exp(i\alpha\gamma_5) = (\cos\alpha + i\gamma_5\sin\alpha)$ it is clear that the $U(1)_5$ like $U(1)$ is a compact group with $\alpha=\alpha_0+2\pi n$, $n=0,\pm1,\pm2,..$, and, for example, $n=0$ with $0\le\alpha_0\le 2\pi$ would enumerate all the distinct elements of the group. For $\alpha=n\pi$, $n=0,\pm1,\pm2,..$, $\bar\psi\psi$ and $\bar\psi\gamma_5\psi$ are clearly left invariant under chiral transformations. From the correspondence $\bar\psi\,\psi \leftrightarrow K\,:\,\cos(2\sqrt{\pi}\phi)\,:$ implied in the construction of the bosonized theory it follows that the chiral transformation on the bosonic field ϕ is realized by $\omega\to\omega+\beta/\sqrt{\pi}$, $\varphi\to\varphi$ with $\beta=\beta_0+n\pi$, $n=0,\pm1,\pm2,..$, $0\le\beta_0\le\pi$. The ω is therefore an angular (coordinate or) variable with the period $\sqrt{\pi}$. In the quantized field theory the chiral transformations are generated by the unitary operator $Q_5(\beta)$ such that‡ $Q_5^\dagger(\beta)\,\omega\,Q_5(\beta) = \omega+\beta/\sqrt{\pi}$, $Q_5^\dagger(\beta)\,\varphi\,Q_5(\beta)=\varphi$ and $Q_5(\pi)=Q_5(0)$. The commutation relations given above result in $exp(-ia\pi_\omega)\,\varphi\,exp(ia\pi_\omega)=\varphi$, $exp(-ia\pi_\omega)\,\omega\,exp(ia\pi_\omega)=\omega+a$. The unitary field theory operator which generates the chirality transformation may then be defined as $Q_5(\beta)\equiv exp(i\pi_\omega\,\beta/\sqrt{\pi})$.

In view of the commutation relations above the space of states may be built as a tensor product, written as $|\varphi)\otimes|n\}$, of a conventional Fock space for the massive field φ and a space spanned by the eigenvalues of π_ω. From $Q_5(\pi)=exp(i\pi_\omega\sqrt{\pi})=Q_5(0)=I$, where I

‡ Here ω,φ, and π_ω are field operators while β, a, and n are c-no. constants.

is identity operator, it follows that π_ω has discrete eigenvalues : $\pi_\omega|n\} = 2n\sqrt{\pi}|n\}$ where $n = 0, \pm 1, \pm 2, ..$ and $\{m|n\} = \delta_{mn}$. With the help of $exp(ia\omega)\,\pi_\omega\, exp(-ia\omega) = \pi_\omega + a$ we may construct the ladder operators $d^\pm = exp(\mp 2i\sqrt{\pi}\omega)$ such that $d^\pm|n\} = |(n\pm 1)\}$. The Hamiltonian contains only the field φ and it commutes with the chirality operator $Q_5(\beta)$ and $d^\pm$. We have infinite degeneracy corresponding to the *chiral* vacuum states $|0) \otimes |n\}$ with $n = 0, \pm 1, \pm 2, ..$ since ω and π_ω are absent from the LF Hamiltonian. There are also no transitions between these vacua characterized by the differnt values of n. In the original Schwinger solution the vacuum state was chosen to be the chirally symmetric state $|0) \otimes |n = 0\}$ which leads to the violation of the cluster decomposition property. This may be avoided by choosing instead the (alternative) vacuum state for the model to be an eigenstate of the condensate operator. The state vectors which are degenerate with respect to the eigenvalue ω' of the condensate operator are easily constructed§

$$|\omega'\} \otimes |\varphi) = \frac{1}{\pi^{1/4}} \sum_{n=-\infty}^{\infty} e^{2i\sqrt{\pi}\, n\, \omega'}\, |n\} \otimes |\varphi) = e^{i\pi_\omega \omega'}|\omega' = 0\} \otimes |\varphi). \qquad (11)$$

where $0 \leq \omega' \leq \sqrt{\pi}$ or $0 \leq \theta' \equiv 2\sqrt{\pi}\omega' \leq 2\pi$ specify the physical values of the condensate. These states have the continuum normalization $\{\omega''|\omega'\} = \delta(\omega'' - \omega')$ and it comes out naturally in our discussion of the LF quantized theory. This is in contrast to the arguments required in the equal-time formulation to impose it so as to avoid the violation of the cluster decomposition property and other inconsistencies in the theory [10-12]. The *condensate-* or *θ-vacuum* in the theory under consideration is the state $|\omega'\} \otimes |0)$ with a fixed given value for θ' and we note that $Q_5(\beta)|\omega'\} = |\omega' + \beta/\sqrt{\pi}\}$. The vacuum state is infinitely degenerate and there are no transitions among the states with different values of θ'. The chiral symmetry is thus spontaneously broken. The corresponding generator π_ω also does not annihilate the vacuum state. This feature here is different from the one found in connection with the description on the LF of the SSB (and Higgs mechanism). There the LF vacuum is annihilated by all of the symmetry generators (in contrast to the case of the equal-time formulation) and the broken symmetry is manifested in the quantized theory Hamiltonian (Appendix C). The fermionic condensate in the Schwinger model, for example, may also be computed

§ In the coordinate representation $\omega|\omega'\} = \omega'|\omega'\}$ and π_ω is represented by $i\partial/\partial\omega'$. We recall also the *Poisson summation formula* of the distribution theory: $\delta(x) = \sum_{-\infty}^{\infty} e^{i2\pi nx}$ for $-1 \leq x \leq 1$ and note that $\{\omega'|\pi_\omega|\omega''\} = i\partial\delta(\omega' - \omega'')/\partial\omega'$ etc.

$$
\begin{aligned}
(0| \otimes \{\omega''| : \bar{\psi}\psi : |\omega'\} \otimes |0) &= K(0| \otimes \{\omega''| : \cos 2\sqrt{\pi}(\omega+\varphi) : |\omega'\} \otimes |0) \\
&= K(0| \otimes \{\omega''| : \cos(2\sqrt{\pi}\varphi) : |\omega'\} \otimes |0) \cos(2\sqrt{\pi}\omega') \qquad (12) \\
&= K \cos\theta' \ \delta(\omega'' - \omega')
\end{aligned}
$$

5. Conclusions

On the LF the θ-vacua in the massless Schwinger model are obtained in straightforward fashion on quantizing the equivalent bosonized theory. Self-consistent Hamiltonian formulation on the LF may be built by first seperating the scalar field into the dynamical condensate and quantum fluctuation fields and then following the Dirac procedure. The same procedure allowed earlier to describe also the SSB and (tree level) Higgs mechanism. The physical results may get a different description on the LF compared to the conventional one [13,19]. The integrability of QCD_2 has been conjectured [17] recently from the studies in the conventional framework employing the equivalent bosonic description. It would be interesting to study the vacuua here in the *front form* theory and also to find an alternative (simpler) proof [18] of the conjecture.

Acknowledgements

Acknowledgements with thanks are due to James Vary, Stan Brodsky, Heinrich Leutwyler, Maria Cristina Abdalla, Elcio Abdalla, and Luiz Alvarez-Gaumé for constructive suggestions. Thanks are due also to Waldyr Rodrigues Jr. and the Organizers of the Workshop at the UNICAMP for the invitation at the very stimulating interdisciplinary meeting. The kind invitation of James Vary and the hospitality at the Workshop at the Iowa state University, the hospitality at the Theory Division at CERN, where the present work started, and the facilities offered at LAFEX are gratefully acknowledged.

Appendix A: Poincare Generators on the LF

The Poincaré generators in coordinate system (x^0, x^1, x^2, x^3), satisfy $[M_{\mu\nu}, P_\sigma] = -i(P_\mu g_{\nu\sigma} - P_\nu g_{\mu\sigma})$ and $[M_{\mu\nu}, M_{\rho\sigma}] = i(M_{\mu\rho}g_{\nu\sigma} + M_{\nu\sigma}g_{\mu\rho} - M_{\nu\rho}g_{\mu\sigma} - M_{\mu\sigma}g_{\nu\rho})$ where the metric is $g_{\mu\nu} = diag\,(1,-1,-1,-1)$, $\mu = (0,1,2,3)$ and we take $\epsilon_{0123} = \epsilon_{-+12} = 1$. If we define $J_i = -(1/2)\epsilon_{ikl}M^{kl}$ and $K_i = M_{0i}$, where $i,j,k,l = 1,2,3$, we find $[J_i, F_j] = i\epsilon_{ijk}F_k$

for $F_l = J_l, P_l$ or K_l while $[K_i, K_j] = -i\epsilon_{ijk}J_k$, $[K_i, P_l] = -iP_0 g_{il}$, $[K_i, P_0] = iP_i$, and $[J_i, P_0] = 0$.

The LF generators are P_+, P_-, P_1, P_2, $M_{12} = -J_3$, $M_{+-} = -K_3$, $M_{1-} = -(K_1 + J_2)/\sqrt{2} \equiv -B_1$, $M_{2-} = -(K_2 - J_1)/\sqrt{2} \equiv -B_2$, $M_{1+} = -(K_1 - J_2)/\sqrt{2} \equiv -S_1$, and $M_{2+} = -(K_2 + J_1)/\sqrt{2} \equiv -S_2$. We find $[B_1, B_2] = 0, [B_a, J_3] = -i\epsilon_{ab}B_b, [B_a, K_3] = iB_a, [J_3, K_3] = 0, [S_1, S_2] = 0, [S_a, J_3] = -i\epsilon_{ab}S_b, [S_a, K_3] = -iS_a$ where $a, b = 1, 2$ and $\epsilon_{12} = -\epsilon_{21} = 1$. Also $[B_1, P_1] = [B_2, P_2] = iP^+, [B_1, P_2] = [B_2, P_1] = 0, [B_a, P^-] = iP_a, [B_a, P^+] = 0, [S_1, P_1] = [S_2, P_2] = iP^-, [S_1, P_2] = [S_2, P_1] = 0, [S_a, P^+] = iP_a, [S_a, P^-] = 0, [B_1, S_2] = -[B_2, S_2] = -iJ_3, [B_1, S_1] = [B_2, S_2] = -iK_3$. For $P_\mu = i\partial_\mu$, and $M_{\mu\nu} \to L_{\mu\nu} = i(x_\mu\partial_\nu - x_\nu\partial_\mu)$ we find $B_a = (x^+P^a - x^aP^+), S_a = (x^-P^a - x^aP^-), K_3 = (x^-P^+ - x^+P^-)$ and $J_3 = (x^1P^2 - x^2P^1)$. Under the conventional *parity* operation $\mathcal{P}$: ($x^\pm \leftrightarrow x^\mp, x^{1,2} \to -x^{1,2}$) and ($p^\pm \leftrightarrow p^\mp, p^{1,2} \to -p^{1,2}$), we find $\vec{J} \to \vec{J}$, $\vec{K} \to -\vec{K}$, $B_a \to -S_a$ etc.. The six generators P_l, M_{kl} leave $x^0 = 0$ hyperplane invariant and are called [1] *kinematical* while the remaining P_0, M_{0k} the *dynamical* ones. On the LF there are *seven* kinematical generators : $P^+, P^1, P^2, B_1, B_2, J_3$ and K_3 which leave the LF hyperplane, $x^0 + x^3 = 0$, invariant and the three *dynamical* ones S_1, S_2 and P^- form a mutually commuting set. We note that each of the set $\{B_1, B_2, J_3\}$ and $\{S_1, S_2, J_3\}$ generates an $E_2 \simeq SO(2) \otimes T_2$ algebra; this will be shown below to be relevant for defining the *spin* for massless particle. Including K_3 in each set we find two subalgebras each with four elements. Some useful identities are $e^{i\omega K_3}\, P^\pm\, e^{-i\omega K_3} = e^{\pm\omega}\, P^\pm$, $e^{i\omega K_3}\, P^\perp\, e^{-i\omega K_3} = P^\perp, e^{i\bar{v}.\bar{B}}\, P^-\, e^{-i\bar{v}.\bar{B}} = P^- + \bar{v}.\bar{P} + \frac{1}{2}\bar{v}^2P^+, e^{i\bar{v}.\bar{B}}\, P^+\, e^{-i\bar{v}.\bar{B}} = P^+, e^{i\bar{v}.\bar{B}}\, P^\perp\, e^{-i\bar{v}.\bar{B}} = P^\perp + v^\perp P^+, e^{i\bar{u}.\bar{S}}\, P^+\, e^{-i\bar{u}.\bar{S}} = P^+ + \bar{u}.\bar{P} + \frac{1}{2}\bar{u}^2P^-, e^{i\bar{u}.\bar{S}}\, P^-\, e^{-i\bar{u}.\bar{S}} = P^-, e^{i\bar{u}.\bar{S}}\, P^\perp\, e^{-i\bar{u}.\bar{S}} = P^\perp + u^\perp P^-$ where $P^\perp \equiv \bar{P} = (P^1, P^2)$, $v^\perp \equiv \bar{v} = (v_1, v_2)$ and $(v^\perp.P^\perp) \equiv (\bar{v}.\bar{P}) = v_1P^1 + v_2P^2$ etc. Analogous expressions with P^μ replaced by X^μ can be obtained if we use $[P^\mu, X_\nu] \equiv [i\partial^\mu, x_\nu] = i\delta^\mu_\nu$.

Appendix B: LF Spin Operator. Hadrons in LF Fock Basis

The Casimir generators of the Poincaré group are : $P^2 \equiv P^\mu P_\mu$ and W^2, where $W_\mu = (-1/2)\epsilon_{\lambda\rho\nu\mu}M^{\lambda\rho}P^\nu$ defines the Pauli-Lubanski pseudovector. It follows from $[W_\mu, W_\nu] = i\epsilon_{\mu\nu\lambda\rho}W^\lambda P^\rho$, $[W_\mu, P_\rho] = 0$ and $W.P = 0$ that in a representation charactarized by particualr eigenvalues of the two Casimir operators we may simultaneously diagonalize P^μ along with just one component of W^μ. We have $W^+ = -[J_3P^+ + B_1P^2 - B_2P^1], W^- = J_3P^- + S_1P^2 - S_2P^1, W^1 = K_3P^2 + B_2P^- - S_2P^+$, and $W^2 = -[K_3P^1 + B_1P^- - S_1P^+]$ and it shows that W^+ *has a special place* since it contains only the kinematical generators [19]. On the LF we define $\mathcal{J}_3 = -W^+/P^+$ as the *spin operator*[17]. It may be shown to commute with P_μ, B_1, B_2, J_3, and K_3. For $m \neq 0$ we may use the parametrizations $p^\mu : (p^- = (m^2 + {p^\perp}^2)/(2p^+), p^+ = (m/\sqrt{2})e^\omega, p^1 = -v_1 p^+, p^2 = -v_2 p^+)$ and $\tilde{p}^\mu : (1,1,0,0)(m/\sqrt{2})$ in the rest frame. We have $P^2(p) = m^2 I$ and $W(p)^2 = W(\tilde{p})^2 = -m^2[J_1^2 + J_2^2 + J_3^2] = -m^2 s(s+1)I$ where s assumes half-integer values. Starting from the rest state $|\tilde{p}; m, s, \lambda, ..\rangle$ with $J_3\,|\tilde{p}; m, s, \lambda, ..\rangle = \lambda\,|\tilde{p}; m, s, \lambda, ..\rangle$ we may build an arbitrary eigenstate of $P^+, P^\perp, \mathcal{J}_3$ (and P^-) on the LF by

$$|p^+, p^\perp; m, s, \lambda, ..\rangle = e^{i(\bar{v}.\bar{B})} e^{-i\omega K_3} |\tilde{p}; m, s, \lambda, ..\rangle$$

If we make use of the following *identity* [19] for the spin operator

$$\mathcal{J}_3(p) = \; J_3 + v_1 B_2 - v_2 B_1 = \quad e^{i(\bar{v}.\bar{B})}\; J_3\; e^{-i(\bar{v}.\bar{B})}$$

we find $\mathcal{J}_3\,|p^+, p^\perp; m, s, \lambda, ..\rangle = \lambda\,|p^+, p^\perp; m, s, \lambda, ..\rangle$. Introducing also $\mathcal{J}_a = -(\mathcal{J}_3 P^a + W^a)/\sqrt{P^\mu P_\mu}$, $a = 1, 2$, which do, however, contain dynamical generators, we verify that $[\mathcal{J}_i, \mathcal{J}_j] = i\epsilon_{ijk}\mathcal{J}_k$.

For $m = 0$ case when $p^+ \neq 0$ a convenient parametrization is $p^\mu : (p^- = p^+ {v^\perp}^2/2,\, p^+, p^1 = -v_1 p^+, p^2 = -v_2 p^+)$ and $\tilde{p} : (0, p^+, 0^\perp)$. We have $W^2(\tilde{p}) = -(S_1^2 + S_2^2){p^+}^2$ and $[W_1, W_2](\tilde{p}) = 0$, $[W^+, W_1](\tilde{p}) = -ip^+ W_2(\tilde{p})$, $[W^+, W_2](\tilde{p}) = ip^+ W_1(\tilde{p})$ showing that W_1, W_2 and W^+ generate the algebra $SO(2) \otimes T_2$. The eigenvalues of W^2 are hence not quantized and they vary continuously. This is contrary to the experience so we impose that the physical states satisfy in addition $W_{1,2}|\,\tilde{p};\, m = 0, ..\rangle = 0$. Hence $W_\mu = -\lambda P_\mu$ and the invariant parameter λ is taken to define as the *spin* of the massless particle. From $-W^+(\tilde{p})/\tilde{p}^+ = J_3$ we conclude that λ assumes half-integer values as well. We note that $W^\mu W_\mu = \lambda^2 P^\mu P_\mu = 0$ and that the definition of the *LF spin operator appears unified for massless and massive particles.* A parallel discussion based on $p^- \neq 0$ may also be given.

As an illustration consider the three particle state on the LF with the total eigenvalues p^+, λ and $p^\perp$. In the *standard frame* with $p^\perp = 0$ it may be written as $(|x_1 p^+, k_1^\perp; \lambda_1\rangle |x_2 p^+, k_2^\perp; \lambda_2\rangle |x_3 p^+, k_3^\perp; \lambda_3\rangle\ )$ with $\sum_{i=1}^3 x_i = 1$, $\sum_{i=1}^3 k_i^\perp = 0$, and $\lambda = \sum_{i=1}^3 \lambda_i$. Aplying $e^{-i(\bar{p}.\bar{B})/p^+}$ on it we obtain $(|x_1 p^+, k_1^\perp + x_1 p^\perp; \lambda_1\rangle |x_2 p^+, k_2^\perp + x_2 p^\perp; \lambda_2\rangle |x_3 p^+, k_3^\perp + x_3 p^\perp; \lambda_3\rangle\ )$ now with $p^\perp \neq 0$. The x_i and $k_i^\perp$ indicate relative (invariant) parameters and do not depend upon the reference frame. The x_i is the fraction of the total longitudinal momentum carried by the i^{th} particle while $k_i^\perp$ its transverse momentum. The state of a pion with momentum $(p^+, p^\perp)$, for example, may be expressed as an expansion over the LF Fock states constituted by the different number of partons [7]

$$|\pi : p^+, p^\perp\rangle = \sum_{n,\lambda} \int \bar{\Pi}_i \frac{dx_i d^2 {k^\perp}_i}{\sqrt{x_i}\, 16\pi^3} |n :\, x_i p^+, x_i p^\perp + {k^\perp}_i, \lambda_i\rangle\; \psi_{n/\pi}(x_1, {k^\perp}_1, \lambda_1; x_2, ...)$$

where the summation is over all the Fock states n and spin projections λ_i, with $\bar{\Pi}_i dx_i = \Pi_i dx_i\;\; \delta(\sum x_i - 1)$, and $\bar{\Pi}_i d^2 k_i^\perp = \Pi_i dk_i^\perp\;\; \delta^2(\sum k_i^\perp)$. The wave function of the parton $\psi_{n/\pi}(x, k^\perp)$ indicates the probability amplitue for finding inside the pion the partons in the Fock state n carrying the 3-momenta $(x_i p^+, x_i p^\perp + k_i^\perp)$. The Fock state of the pion is also off the energy shell : $\sum k_i^- > p^-$.

The *discrete symmetry* transformations may also be defined on the LF Fock states [19]. For example, under the conventional parity $\mathcal{P}$ the spin operator $\mathcal{J}_3$ is not left invariant. We may rectify this by defining *LF Parity operation* by $\mathcal{P}^{lf} = e^{-i\pi J_1}\mathcal{P}$. We find then $B_1 \to -B_1, B_2 \to B_2, P^\pm \to P^\pm, P^1 \to -P^1, P^2 \to P^2$ etc. such that $\mathcal{P}^{lf}|p^+, p^\perp; m, s, \lambda, ..\rangle \simeq |p^+, -p^1, p^2; m, s, -\lambda, ..\rangle$. Similar considerations apply for charge conjugation and time inversion. For example, it is straightforward to construct the free *LF Dirac spinor* $\chi(p) = [\sqrt{2}p^+\Lambda^+ + (m - \gamma^a p^a)\,\Lambda^-]\tilde{\chi}/\sqrt{\sqrt{2}p^+ m}$ which is also an eigenstate os $\mathcal{J}_3$ with eigenvalues $\pm 1/2$. Here $\Lambda^\pm = \gamma^0\gamma^\pm/\sqrt{2} = \gamma^\mp\gamma^\pm/2 = (\Lambda^\pm)^\dagger$, $(\Lambda^\pm)^2 = \Lambda^\pm$, and $\chi(\tilde{p}) \equiv \tilde{\chi}$ with $\gamma^0\tilde{\chi} = \tilde{\chi}$. The conventional (equal-time) spinor can also be constructed by the procedure analogous to that followed for the LF spinor and it has the well known form $\chi_{con}(p) = (m + \gamma.p)\tilde{\chi}/\sqrt{2m(p^0 + m)}$. Under the conventional parity operation $\mathcal{P} : \chi'(p') = c\gamma^0\chi(p)$ (since we must require $\gamma^\mu = L^\mu{}_\nu\, S(L)\gamma^\nu S^{-1}(L)$ etc.). We find $\chi'(p) = c[\sqrt{2}p^-\Lambda^- + (m - \gamma^a p^a)\,\Lambda^+]\,\tilde{\chi}/\sqrt{\sqrt{2}p^- m}$. For $p \neq \tilde{p}$ it is not proportional to $\chi(p)$ in contrast to the result in the case of the usual spinor where $\gamma^0\chi_{con}(p^0, -\vec{p}) = \chi_{con}(p)$ for $E > 0$ (and $\gamma^0\eta_{con}(p^0, -\vec{p}) = -\eta_{con}(p)$ for $E < 0$). However, applying parity operator twice we do show $\chi''(p) = c^2\chi(p)$ hence leading to the usual result $c^2 = \pm 1$. The LF parity operator over spin 1/2 Dirac spinor is $\mathcal{P}^{lf} = c\,(2J_1)\,\gamma^0$ and the corresponding transform of χ is shown to be an eigenstate of $\mathcal{J}_3$.

Appendix C: SSB Mechanism. Continuum Limit of Discretized LF Quantized Theory. Nonlocality of LF Hamiltonian.

The existence of the continuum limit [20,2] of the Discretized Light Cone Quantized (DLCQ) [21] theory, the nonlocal nature of the LF Hamiltonian, and the description of the SSB on the LF were clarified [13] only recently.

Consider first the two dimensional case with $\mathcal{L} = [\dot{\phi}\phi' - V(\phi)]$. The eq. of motion, $\dot{\phi}' = (-1/2)\delta V(\phi)/\delta\phi$, shows that $\phi = const.$ is a possible solution. We write $\phi(x,\tau) = \omega(\tau) + \varphi(x,\tau)$ The general case of $\omega = \omega(\tau)$ can be made [2] but to make the discussion here short we assume that ω is a constant so that $\mathcal{L} = \dot{\varphi}\varphi' - V(\phi)$. Dirac procedure is applied now to construct Hamiltonian field theory which may be quantized. We may avoid using distribuitions if we restrict x to a finite interval from $-R/2$ to $R/2$. The *physical limit to the continuum ($R \to \infty$)*, however, must be taken latter to remove the spurious finite volume effects. Expanding φ by Fourier series we obtain $\phi(\tau,x) \equiv \omega + \varphi(\tau,x) = \omega + \frac{1}{\sqrt{R}}q_0(\tau) + \frac{1}{\sqrt{R}}\sum'_{n\neq 0}\; q_n(\tau)\; e^{-ik_n x}$ where $k_n = n(2\pi/R)$, $n = 0, \pm 1, \pm 2, ...$ and the *discretized theory* Lagrangian becomes $i\sum_n k_n\, q_{-n}\, \dot{q}_n - \int dx\; V(\phi)$. The momenta conjugate to q_n are $p_n = ik_n q_{-n}$ and the canonical LF Hamiltonian is found to be $\int dx\, V(\omega + \varphi(\tau,x))$. The primary constraints are thus $p_0 \approx 0$ and $\Phi_n \equiv p_n - ik_n q_{-n} \approx 0$ for $n \neq 0$. We follow the *standard* Dirac procedure [3] and find three weak constraints $p_0 \approx 0$, $\beta \equiv \int dx\, V'(\phi) \approx 0$, and $\Phi_n \approx 0$ for $n \neq 0$ on the phase space and they are shown to be second class. We find for $n, m \neq 0:$ $\{\Phi_n, p_0\} = 0$, $\{\Phi_n, \Phi_m\} = -2ik_n\delta_{m+n,0}$, $\{\Phi_n, \beta\} = \{p_n, \beta\} = -(1/\sqrt{R})\int dx\; [V''(\phi) - V''([\omega + q_0]/\sqrt{R})]\, e^{-ik_n x} \equiv -\alpha_n/\sqrt{R}$, $\{p_0, \beta\} = -(1/\sqrt{R})\int dx\; V''(\phi) \equiv -\alpha/\sqrt{R}$, $\{p_0, p_0\} = \{\beta, \beta\} = 0$. Implement first the pair of constraints $p_0 \approx 0, \beta \approx 0$ by modifying the Poisson brackets to the star bracket $\{\}^*$ defined by $\{f,g\}^* = \{f,g\} - [\{f,p_0\}\;\{\beta,g\} - (p_0 \leftrightarrow \beta)]\,(\alpha/\sqrt{R})^{-1}$. We may then set $p_0 = 0$ and $\beta = 0$ as strong equalities. We find by inspection that the brackets $\{\}^*$ of the remaining variables coincide with the standard Poisson brackets except for the ones involving q_0 and p_n $(n \neq 0)$: $\{q_0, p_n\}^* = \{q_0, \Phi_n\}^* = -(\alpha^{-1}\alpha_n)$. For example, if $V(\phi) = (\lambda/4)\,(\phi^2 - m^2/\lambda)^2$, $\lambda \geq 0, m \neq 0$ we find $\{q_0, p_n\}^*\; [\{\, 3\lambda\,(\omega + q_0/\sqrt{R})^2 - m^2\,\}R + 6\lambda(\omega + q_0/\sqrt{R})\int\; dx\varphi + 3\lambda\int\; dx\,\varphi^2\,] = -\,3\lambda\,[\,2(\omega + q_0/\sqrt{R})\,\sqrt{R}q_{-n} + \int\; dx\,\varphi^2\, e^{-ik_n x}\,]$.

Implement next the constraints $\Phi_n \approx 0$ with $n \neq 0$. We have $C_{nm} = \{\Phi_n, \Phi_m\}^* = -2ik_n\delta_{n+m,0}$ and its inverse is given by ${C^{-1}}_{nm} = (1/2ik_n)\delta_{n+m,0}$. The Dirac bracket which takes care of all the constraints is then given by

$$\{f,g\}_D = \{f,g\}^* - \sum\nolimits'_n \frac{1}{2ik_n}\{f,\, \Phi_n\}^*\, \{\Phi_{-n},\, g\}^*$$

where we may now in addition write $p_n = ik_n q_{-n}$. It is easily shown that $\{q_0, q_0\}_D = 0, \{q_0, p_n\}_D = \{q_0,\, ik_n q_{-n}\}_D = \frac{1}{2}\,\{q_0, p_n\}^*, \{q_n, p_m\}_D = \frac{1}{2}\delta_{nm}$.

The limit to the continuum, $R \to \infty$ is taken as usual: $\Delta = 2\,(\pi/R) \to dk$, $k_n =$

$n\Delta \to k$, $\sqrt{R}\, q_{-n} \to lim_{R\to\infty} \int_{-R/2}^{R/2} dx\, \varphi(x)\, e^{ik_n x} \equiv \int_{-\infty}^{\infty} dx\, \varphi(x)\, e^{ikx} = \sqrt{2\pi}\tilde{\varphi}(k)$ for all n, $\sqrt{2\pi}\varphi(x) = \int_{-\infty}^{\infty} dk\, \tilde{\varphi}(k)\, e^{-ikx}$, and $(q_0/\sqrt{R}) \to 0$. From $\{\sqrt{R}q_m, \sqrt{R}q_{-n}\}_D = R\,\delta_{nm}/(2ik_n)$ following from $\{q_n, p_m\}_D$ for $n, m \neq 0$ we derive, on using $R\delta_{nm} \to \int_{-\infty}^{\infty} dx e^{i(k-k')x} = 2\pi\delta(k-k')$, that $\{\tilde{\varphi}(k), \tilde{\varphi}(-k')\}_D = \delta(k-k')/(2ik)$ where $k, k' \neq 0$. If we use the integral representation of the sgn function the well known LF Dirac bracket $\{\varphi(x,\tau), \varphi(y,\tau)\}_D = -\frac{1}{4}\epsilon(x-y)$ is obtained. The expressions of $\{q_0, p_n\}_D$ (or $\{q_0, \varphi'\}_D$) show that the DLCQ is harder to work with here. The continuum limit of the *constraint eq.* $\beta = 0$ is

$$lim_{R\to\infty} \frac{1}{R} \int_{-R/2}^{R/2} dx\, V'(\phi) \equiv$$

$$\omega(\lambda\omega^2 - m^2) + lim_{R\to\infty} \frac{1}{R} \int_{-R/2}^{R/2} dx \Big[(3\lambda\omega^2 - m^2)\varphi + \lambda(3\omega\varphi^2 + \varphi^3) \Big] = 0$$

while that for the LF Hamiltonian is ($P^- \equiv H^{l.f.}$)

$$P^- = \int dx\, \Big[\omega(\lambda\omega^2 - m^2)\varphi + \frac{1}{2}(3\lambda\omega^2 - m^2)\varphi^2 + \lambda\omega\varphi^3 + \frac{\lambda}{4}\varphi^4 \Big]$$

These results follow immediately if we worked directly in the continuum formulation; we do have to handle generalized functions now. In the LF Hamiltonian theory we have an additional new ingredient in the form of the *constraint equation*. Elimination of ω using it would lead to a *nonlocal LF Hamiltonian* corresponding to the local one in the equal-time formulation. At the tree or classical level the integrals appearing in in the constraint eq. are convergent and when $R \to \infty$ it leads to $V'(\omega) = 0$. In equal-time theory this is essentially *added* to it as an external constraint based on physical considerations. In the renormalized theory [13] the constraint equation describes the high order quantum corrections to the tree level value of the condensate.

The quantization is performed via the correspondence $i\{f, g\}_D \to [f, g]$. Hence $\varphi(x,\tau) = (1/\sqrt{2\pi}) \int dk\; \theta(k)\; [a(k,\tau)\; e^{-ikx} + a^\dagger(k,\tau) e^{ikx}]/(\sqrt{2k})$, were $a(k,\tau)$ and $a^\dagger(k,\tau)$ satisfy the canonical equal-τ commutation relations, $[a(k,\tau), a(k',\tau)^\dagger] = \delta(k-k')$ etc.. The vacuum state is defined by $a(k,\tau)|vac\rangle = 0$, $k > 0$ and the tree level description of the *SSB* is given as follows. The values of $\omega = \langle|\phi|\rangle_{vac}$ obtained from $V'(\omega) = 0$ the different vacua in the theory. Distinct Fock spaces corresponding to different values of ω are built as usual by applying the creation operators on the corresponding vacuum state. The $\omega = 0$ corresponds to a *symmetric phase* since the Hamiltonian is then symmetric under $\varphi \to -\varphi$. For $\omega \neq 0$ this symmetry is violated and the system is in a *broken or asymmetric phase.*

The extension to 3+1 dimensions and to global continuous symmetry is straightforward [13,2]. Consider real scalar fields $\phi_a (a = 1, 2, ..N)$ which form an isovector of global internal symmetry group $O(N)$. We now write $\phi_a(x, x^\perp, \tau) = \omega_a + \varphi_a(x, x^\perp, \tau)$ and the Lagrangian density is $\mathcal{L} = [\dot{\varphi}_a \varphi'_a - (1/2)(\partial_i \varphi_a)(\partial_i \varphi_a) - V(\phi)]$, where $i = 1, 2$ indicate the transverse space directions. The Taylor series expansion of the constraint equations $\beta_a = 0$ gives a set of coupled eqs. $R\, V'_a(\omega) + V''_{ab}(\omega) \int dx \varphi_b + V'''_{abc}(\omega) \int dx \varphi_b \varphi_c/2 + ... = 0$. Its discussion at the tree level leads to the conventional theory results. The LF symmetry generators are found to be $G_\alpha(\tau) = -i \int d^2x^\perp dx \varphi'_c (t_\alpha)_{cd} \varphi_d = \int d^2k^\perp\, dk\, \theta(k)\, a_c(k, k^\perp)^\dagger (t_\alpha)_{cd} a_d(k, k^\perp)$ where $\alpha, \beta = 1, 2, .., N(N-1)/2$, are the group indices, t_α are hermitian and antisymmetric generators of $O(N)$, and $a_c(k, k^\perp)^\dagger$ $(a_c(k, k^\perp))$ is creation (destruction) operator contained in the momentum space expansion of φ_c. These are to be contrasted with the generators in the equal-time theory, $Q_\alpha(x^0) = \int d^3x\, J^0 = -i \int d^3x (\partial_0 \varphi_a)(t_\alpha)_{ab} \varphi_b - i(t_\alpha \omega)_a \int d^3x (d\varphi_a/dx_0)$. All the symmetry generators thus annihilate the LF vacuum and the SSB is now seen in the broken symmetry of the quantized theory Hamiltonian. The criterian for the counting of the number of Goldstone bosons on the LF is found to be the same as in the conventional theory. In contrast, the first term on the right hand side of $Q_\alpha(x^0)$ does annihilate the conventional theory vacuum but the second term gives now non-vanishing contributions for some of the (broken) generators. The symmetry of the conventional theory vacuum is thereby broken while the quantum Hamiltonian remains invariant. The physical content of SSB in the *instant form* and the *front form*, however, is the same though achieved by differnt descriptions. Alternative proofs on the LF, in two dimensions, can be given of the Coleman's theorem related to the absence of Goldstone bosons and of the pathological nature of massless scalar theory; we are unable to implement the second class constraints over the phase space.

We remark that the simplicity of the LF vacuum is in a sense compensated by the involved nonlocal Hamiltonian. The latter, however, may be treatable using advance computational techniques. In a recent work [13] it was also shown that renormalized theory may be constructed without the need of first solving the constraint eq. for ω. Instead we may perform renormalization and obtain a renormalized constraint equation.

Appendix D: Commutators for equal-x^-

The LF formulation is symmetrical with respect to x^+ and x^- and it is a matter of convention that we take the plus component as the LF *time* while the other as a spatial coordinate. The theory quantized at $x^+ = const.$ hyperplanes, however, does seem to already incorporate in it the information from the equal-x^- quantized theory.

For illustration we consider the two dimensional massive free scalar theory. The LF quantization, assuming x^+ as the LF *time* coordinates, leads to $\omega = 0$ and the equal-x^+ commutator $[\varphi(x^+, x^-), \varphi(x^+, y^-] = -i\epsilon(x^- - y^-)/4$. The commutator can be realized in the momentum space through the expansion (Appendix C)

$$\varphi(x^+, x^-) = \frac{1}{\sqrt{2\pi}} \int_{k^+>0}^{\infty} \frac{dk^+}{\sqrt{2k^+}} \Big[a(k^+) e^{-i(k^+x^- + k^-x^+)} + a^\dagger(k^+) e^{i(k^+x^- + k^-x^+)}\Big]$$

where $[a(k^+), a(l^+)^\dagger] = \delta(k^+ - l^+)$ etc. and $2k^+k^- = m^2$. It is then easy to show

$$[\varphi(x^+, x^-), \varphi(y^+, x^-)] = \frac{1}{2\pi} \int_{k^+>0}^{\infty} \frac{dk^+}{2k^+} \Big[e^{ik^-(y^+ - x^+)} - e^{-ik^-(y^+ - x^+)}\Big].$$

We may change the integration variable to k^- by making use of $k^- dk^+ + k^+ dk^- = 0$. Hence on employing the integral representation $\epsilon(x) = (i/\pi)\mathcal{P} \int_{-\infty}^{\infty} (d\lambda/\lambda)\ exp(-i\lambda x)$ we arrive at the equal-x^- commutator

$$[\varphi(x^+, x^-), \varphi(y^+, x^-)] = -\frac{i}{4}\epsilon(x^+ - y^+)$$

The above field expansion on the LF, in contrast to the equal-time case, does not involve the mass parameter m and the same result follows in the massless case also if we assume that $k^+ = l^+$ implies $k^- = l^-$. Defining the right and the left movers by $\varphi(0, x^-) \equiv \varphi^R(x^-)$, and $\varphi(x^+, 0) \equiv \varphi^L(x^+)$ we obtain $[\varphi^R(x^-), \varphi^R(y^-)] = (-i/4)\epsilon(x^- - y^-)$ while $[\varphi^L(x^+), \varphi^L(y^+)] = (-i/4)\epsilon(x^+ - y^+)$.

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