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BRS Cohomology of Zero Curvature Systems
II. The Noncomplete Ladder Case

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Abstract

The Yang-Mills type theories and their BRS cohomology are analysed within the zero curvature formalism.

Key-words: BRS, Cohomology; Anomalies.

1 Introduction

In the first part (I) of this work we have studied the zero curvature formulation of systems described by means of a complete ladder field, the components of which span all possible form degrees. The present paper is devoted to analyse the zero curvature equation in the case in which the completeness condition for the generalized ladder field is relaxed. This means that we shall deal with a gauge ladder $\widetilde{\mathcal{A}}$ for which the form degree of the highest component is strictly lower than the space-time dimension D, *i.e.*

$$\widetilde{\mathcal{A}} = c + A + \varphi_2^{-1} + ... + \varphi_q^{1-q} \ , \qquad 1 \leq q < D \ . \tag{1.1}$$

As we shall see in the following, the noncomplete case will display a set of remarkable features which will make it quite different from the previous complete case. The first interesting aspect, as already mentioned in the introduction of part I, is that the consistency of the zero curvature condition

$$\widetilde{\mathcal{F}} = \widetilde{d}\widetilde{\mathcal{A}} - i\widetilde{\mathcal{A}}^2 = 0 \tag{1.2}$$

implies now the existence of a set of new operators $(\mathcal{G}_k^{1-k}, \ 2 \leq k \leq D)$ which are in involution, according to the algebra

$$\begin{aligned} \mathcal{G}_2^{-1} &= \frac{1}{2}[\, \delta \, , \, d \,] \ , \\ \mathcal{G}_k^{1-k} &= \frac{1}{k}\left[\, \delta \, , \, \mathcal{G}_{k-1}^{2-k} \,\right] \ , \qquad k > 2 \ , \end{aligned} \tag{1.3}$$

δ being the operator which together with the BRS operator b decomposes the exterior space-time derivative d as

$$d = -\,[b, \delta] \ . \tag{1.4}$$

The second interesting feature of the noncomplete case is that the cohomology of the BRS operator b is richer than the corresponding one of the complete case. Indeed, the noncompleteness of $\widetilde{\mathcal{A}}$ will allow us to introduce a set of curvatures $(R_{m+1}^{1-m}, \ 1 \leq m \leq q)$ which are a generalization of the familiar two-form gauge field strength $F = dA - iA^2$. It follows then that, in addition to the usual ghost cocycles $(\mathrm{Tr}\, c^{2n+1})$ of the complete case (see Sect. 4 of I), the cohomology of b now includes also invariant polynomials in the highest curvature (R_{q+1}^{1-q}).

As a consequence of these new features, the expressions of the polynomials ω_j^{G+D-j} $(0 \leq j \leq D)$ which solve the descent equations

$$\begin{aligned} & b\,\omega_{D-j}^{G+j} + d\,\omega_{D-j-1}^{G+j+1} = 0 \ , \qquad 0 \leq j \leq (D-1) \ , \\ & b\,\omega_0^{G+D} = 0 \ , \end{aligned} \tag{1.5}$$

will get modified with respect to the complete case. This modification will result in the appearence of a set of local polynomials Ω_j^{G+D-j} $(q+1 \leq j \leq D)$ in the curvatures (R^{1-m}_{m+1}) which have to be added to the cocycles obtained from the expansion of the generalized terms $(\mathrm{Tr}\,\widetilde{\mathcal{A}}^{G+D})$. These polynomials, as already observed in [1, 2] in the case of Yang-Mills, turn out to be characterized by a set of consistency conditions involving the operators $\mathcal{G}^{1-k}_k$.

The second part of the work is organized as follows. In Sect. 2 we present the zero curvature condition for the noncomplete gauge ladder. Sect. 3 is devoted to the study of the cohomology of the BRS operator. In Sect. 4 we solve the descent equations. Sect. 5 and Sect. 6 are finally devoted to the discussion of several examples among which one finds the zero curvature formulation of the pure Yang-Mills gauge theory.

2 The zero curvature condition

In part I (cf. Sect. 2) the BRS transformations of the various components of the gauge ladder $\widetilde{\mathcal{A}}$ have been obtained by constraining the latter to obey a zero curvature condition. Equivalently, as we have seen in Sect. 3 of I, once the BRS transformations of the fields have been given, the zero curvature condition becomes a consequence of the existence of the operator δ which realizes the decomposition (1.4). This second procedure will be taken as the starting point for the discussion of the zero curvature condition in the present noncomplete case. The gauge ladder $\widetilde{\mathcal{A}}$ takes now the following form

$$\widetilde{\mathcal{A}} = c + A + \varphi_2^{-1} + ... + \varphi_q^{1-q} \ , \qquad 1 \le q < D \ , \tag{2.1}$$

D being the dimension of the space-time. We will assume therefore that the nilpotent BRS transformations of the components φ_j^{1-j} $(0 \le j \le q)$ of (2.1) will be the same as those of the corresponding complete case (see Sect. 2 of I), *i.e.*

$$\begin{aligned} bc &= ic^2 \ , \\ bA &= -dc + i[c, A] \ , \\ b\varphi_j^{1-j} &= -d\varphi_{j-1}^{2-j} + \frac{i}{2}\sum_{m=0}^{j}\left[\varphi_m^{1-m} \, , \, \varphi_{j-m}^{1-j+m}\right] \ , \qquad 2 \le j \le q \ , \end{aligned} \tag{2.2}$$

where, as usual, $[a, b] = ab - (-1)^{|a||b|}ba$ denotes the graded commutator and, as done in I, we shall work in the functional space $\mathcal{V}$ of form-valued polynomials built up with the fields φ_j^{1-j} and their differentials $d\varphi_j^{1-j}$, *i.e.*

$$\mathcal{V} = \text{polynomials in } \left(\varphi_j^{1-j}, \, d\,\varphi_j^{1-j} \, ; \; 0 \le j \le q\right) \ . \tag{2.3}$$

Having assigned the BRS transformations, let us turn to the introduction of the decomposition (1.4). To this purpose we define the operator δ as

$$\begin{aligned} &\widetilde{\mathcal{A}} = e^{\delta} c \;, \\ &\delta\, \varphi_j^{1-j} = (j+1)\varphi_{j+1}^{-j} \;, \qquad 0 \le j \le q-1 \;, \\ &\delta \varphi_q^{1-q} = 0 \;, \end{aligned} \tag{2.4}$$

and

$$\begin{aligned} &\delta\, d\, \varphi_m^{1-m} = (m+1)\, d\, \varphi_{m+1}^{-m} \;, \qquad 0 \le m \le q-2 \;, \\ &\delta\, d\, \varphi_{q-1}^{2-q} = q\, d\, \varphi_q^{1-q} - (q+1)\left(d\, \varphi_q^{1-q} - \frac{i}{2}\sum_{j=1}^{q}\left[\varphi_j^{1-j}, \varphi_{q-j+1}^{j-q}\right]\right) \;, \\ &\delta\, d\, \varphi_q^{1-q} = \frac{i}{2}(q+1)\sum_{j=1}^{q}\left[\varphi_{q+2-j}^{-1-q+j}, \varphi_j^{1-j}\right] \;. \end{aligned} \tag{2.5}$$

One easily checks that, on the functional space $\mathcal{V}$, the operators b and δ realize the decomposition (1.4), *i.e.*

$$d = -\,[b, \delta] \;. \tag{2.6}$$

Comparing now equations (2.4), (2.5) with the corresponding ones of the complete ladder case (see Sect. 2 of I) one sees that, while the action of the operator δ on the components (φ_j^{1-j}) is the same, the transformations of the differentials of higher form-degree, *i.e.* $(d\varphi_{q-1}^{2-q})$ and $(d\varphi_q^{1-q})$, are now nonvanishing. This fact implies that, contrary to the complete case, the operator δ does not commute anymore with the exterior derivative d,

$$[\delta, d] \neq 0 \;. \tag{2.7}$$

In addition, depending on the dimension of the space-time D and on the number q of components of the gauge ladder $\widetilde{\mathcal{A}}$, the commutators

$$[\delta, [\delta, [\delta, \ldots\ldots, d]]] \tag{2.8}$$

turn out to be nonvanishing as well.

This algebraic structure, which generalizes that of ref. [1, 2], will have important consequences on the zero curvature condition. The latter, repeating the same argument of Sect. 3 of I, is obtained by applying the operator e^{δ} on the BRS transformation of the zero-form ghost field c, *i.e.*

$$e^{\delta}\, b\, e^{-\delta} e^{\delta} c = e^{\delta}\, i\, c^2 \;. \tag{2.9}$$

Recalling now that $\widetilde{\mathcal{A}} = e^{\delta} c$ and defining the generalized operator $\widetilde{d}$ as

$$\widetilde{d} = e^{\delta}\, b\, e^{-\delta} \;, \tag{2.10}$$

we get the zero curvature condition

$$\widetilde{d}\widetilde{\mathcal{A}} = i\widetilde{\mathcal{A}}^2 \tag{2.11}$$

for the noncomplete ladder case. Equation (2.11) is, however, only apparently similar to the corresponding condition of the complete ladder case. In fact, due to eqs.(2.7) and (2.8), the operator $\tilde{d}$ is now given by

$$\tilde{d} = b + d + \sum_{n\geq 2}^{D} \frac{1}{n!} \underbrace{[\delta, [\delta, [\delta, ..., d]]]}_{\text{(n-1)-times}} , \tag{2.12}$$

so that, defining the operators

$$\begin{aligned} \mathcal{G}_2^{-1} &= \frac{1}{2}[\,\delta\,,\,d\,]\ , \\ \mathcal{G}_3^{-2} &= \frac{1}{3!}\,[\,\delta\,[\delta, d]] = \frac{1}{3}\left[\delta, \mathcal{G}_2^{-1}\right]\ , \\ \mathcal{G}_4^{-3} &= \frac{1}{4!}\,[\delta, [\delta, [\delta, d]]] = \frac{1}{4}\left[\delta, \mathcal{G}_3^{-2}\right]\ , \\ &\quad .... \end{aligned} \tag{2.13}$$

we have

$$\tilde{d} = b + d + \sum_{k\geq 2}^{D} \mathcal{G}_k^{1-k} \tag{2.14}$$

with

$$\begin{aligned} \mathcal{G}_2^{-1} &= \frac{1}{2}[\,\delta\,,\,d\,]\ , \\ \mathcal{G}_k^{1-k} &= \frac{1}{k}\left[\,\delta\,,\,\mathcal{G}_{k-1}^{2-k}\right]\ , \qquad k > 2\ . \end{aligned} \tag{2.15}$$

One thus sees that in the noncomplete case the zero curvature condition is accompanied by a set of operators $\mathcal{G}_k^{1-k}$ which are in involution, according to eq.(2.15). We underline, in particular, that the origin of the operators $\mathcal{G}_k^{1-k}$ actually relies on the noncomplete character of the gauge ladder (2.1). It is very easy, using the equations (2.4) and (2.5), to derive the explicit form of the various operators $\mathcal{G}_k^{1-k}$ appearing in the eq. (2.11). In particular, as we shall show later on in the examples, the number of operators $\mathcal{G}_k^{1-k}$ which do not identically vanish depends both on the dimension D of the space-time and on the number q of components of the gauge ladder $\tilde{\mathcal{A}}$. We also notice that these operators are absent when $q = D$, *i.e.* they are not present in the case in which the ladder is complete.

Moreover their existence implies that the cohomology of the operator $\tilde{d}$ is no more directly related to that of the operator $(d + b)$. Therefore the cohomology classes of $\tilde{d}$ do not immediately provide solutions of the descent equations (1.5). It turns out indeed that in order to obtain a solution of the tower (1.5) we must add to the cohomology classes of $\tilde{d}$, *i.e.* $\mathrm{Tr}\,\tilde{\mathcal{A}}^{2n+1}$, certain polynomials Ω_j^{G+D-j} $(q+1 \leq j \leq D)$ which obey a set of consistency conditions involving the operator $\mathcal{G}_k^{1-k}$. In other words, the presence of the $\mathcal{G}_k^{1-k}$'s requires a modification of the solution of the descent equations with respect to the complete ladder case (see Sect. 5 of I).

Let us conclude this section with the following remark. Instead of having assumed the BRS transformations (2.2) we could have started directly with the zero curvature condition (2.11). It is easily verified then that the introduction of the operators $\mathcal{G}_k^{1-k}$ is needed in order to avoid the appearence of constraints among the components of the noncomplete ladder field $\widetilde{\mathcal{A}}$.

3 Cohomology of the BRS operator

The first step in order to solve the descent equations (1.5) is that of computing the cohomology of the BRS operator b. This task, due to the noncomplete character of $\widetilde{\mathcal{A}}$, will turn out to be simplified by the introduction of the following curvatures R_{m+1}^{1-m} of total degree two:

$$R_{m+1}^{1-m} = d\,\varphi_m^{1-m} - \frac{i}{2}\sum_{k=1}^{m} \left[\varphi_k^{1-k}, \varphi_{m+1-k}^{k-m}\right] ; \qquad 1 \leq m \leq q \ . \tag{3.1}$$

In particular, for $m = 1$ the expression (3.1) reduces to

$$R_2^0 = dA - iA^2 = F \ , \tag{3.2}$$

i.e. one recovers the familiar two-form gauge field strength. We also remark that, for $m > 1$, the curvatures R_{m+1}^{1-m} possess the property of having negative ghost number.

The great advantage of working with the curvatures R_{m+1}^{1-m} relies on the fact that they transform covariantly under the action of the BRS operator, *i.e.*

$$b\,R_{m+1}^{1-m} = i\left[c, R_{m+1}^{1-m}\right] \ . \tag{3.3}$$

This feature, following the well known Yang-Mills case [3, 4, 5, 6], suggests that it is convenient to use the curvatures R_{m+1}^{1-m} as independent variables instead of the differentials $d\varphi_m^{1-m}$, *i.e.* we replace everywhere the variables $d\varphi_m^{1-m}$ by R_{m+1}^{1-m} making use of eq.(3.1). Consequently, for the functional space $\mathcal{V}$ we have

$$\mathcal{V} = \text{polynomials in } \Big(c \ , A \ , \varphi_m^{1-m}, \ \ 2 \leq m \leq q \ ; dc \ , R_{j+1}^{1-j}, \ \ 1 \leq j \leq q \Big) \ , \tag{3.4}$$

and, for the nilpotent BRS transformations,

$$\begin{aligned}
& b\,c = ic^2 \ , \\
& b\,A = -dc + i[c, A] \ , \\
& b\,\varphi_m^{1-m} = i\,[c, \varphi_m^{1-m}] - R_m^{2-m} \ , \qquad 2 \leq m \leq q \ , \\
& b\,d\,c = i\,[c, dc] \ , \\
& b\,R_{j+1}^{1-j} = i\left[c, R_{j+1}^{1-j}\right] \ , \qquad 1 \leq j \leq q \ .
\end{aligned} \tag{3.5}$$

Let us turn now to the computation of the cohomology of b. Introducing the filtering operator $\mathcal{N}$

$$
\begin{aligned}
&\mathcal{N} c = c \;, \qquad \mathcal{N} A = A \;, \\
&\mathcal{N} \varphi^{1-m}_m = \varphi^{1-m}_m \;, \qquad 2 \le m \le q \;, \\
&\mathcal{N}\, d\, c = dc \;, \qquad \mathcal{N} R^{1-j}_{j+1} = R^{1-j}_{j+1} \;, \qquad 1 \le j \le q \;,
\end{aligned}
\tag{3.6}
$$

the BRS operator decomposes as

$$
b = b_0 + b_1 \;, \tag{3.7}
$$

with

$$
\begin{aligned}
&b_0\, c = 0 \;, \\
&b_0\, A = -dc \;, \qquad b_0\, d\, c = 0 \;, \\
&b_0\, \varphi^{1-m}_m = -R^{2-m}_m \;, \qquad b_0 R^{2-m}_m = 0 \;, \qquad 2 \le m \le q \;, \\
&b_0\, R^{1-q}_{q+1} = 0 \;, \\
&b_0^2 = 0 \;.
\end{aligned}
\tag{3.8}
$$

Equations (3.8) show that all the variables except the zero-form ghost c and the highest curvature R^{1-q}_{q+1} are grouped in BRS doublets. This implies that the cohomology of b_0 and, in turn, that of the full BRS operator b depend only on c and R^{1-q}_{q+1}. More precisely, using the general results of refs. [3, 4, 5, 6], it follows that the cohomology of b on the functional space $\mathcal{V}$ is spanned by invariant polynomials in the variables (c, R^{1-q}_{q+1}) built up with factorized monomials of the form

$$
\left(\mathrm{Tr}\, \frac{c^{2n+1}}{(2n+1)!} \right) \cdot \left(\mathrm{Tr}\, (R^{1-q}_{q+1})^m \right) \;, \qquad n, m = 1, 2, ... \;. \tag{3.9}
$$

One sees that in the noncomplete ladder case the cohomology of the BRS operator b, in addition of the usual ghost cocycles $(\mathrm{Tr}\, c^{2n+1})$, includes also polynomials in the highest curvatures R^{1-q}_{q+1}. Notice finally that, being the ghost number of the highest curvature R^{1-q}_{q+1} negative for $q > 1$, the cohomology classes of b are nonvanishing in the negative charged sectors.

We conclude this section by remarking that the highest curvature R^{1-q}_{q+1} is actually related to the ghost field c through the action of the operator $\mathcal{G}^{-q}_{q+1}$,

$$
\mathcal{G}^{-q}_{q+1}\, c = (\mathrm{const})\, R^{1-q}_{q+1} \;, \tag{3.10}
$$

the proportionality factor being easily computed by means of the eqs.(2.4), (2.5).

4 Solution of the descent equations

Having characterized the cohomology of the BRS operator b, let us focus on the cohomology of b modulo d, *i.e.* let us try to solve the descent equations

$$
\begin{aligned}
& b\,\omega^{G+j}_{D-j} + d\,\omega^{G+j+1}_{D-j-1} = 0 \ , \qquad 0 \le j \le (D-1) \ , \\
& b\,\omega^{G+D}_{0} = 0 \ .
\end{aligned} \tag{4.1}
$$

As mentioned before and as already observed in the case of pure Yang-Mills (*i.e.* $q=1$), the presence of the operators $\mathcal{G}^{1-k}_{k}$ in the zero curvature condition (2.11) requires a slight modification of the climbing procedure presented in the previous complete ladder case (see I).

Repeating indeed the same argument of [1], it is easy to convince oneself that, once a solution ω^{G+D}_0 of the last equation of (4.1) has been obtained, an explicit expression for the higher polynomials ω^{G+D-j}_j is provided by the generalized cocycle $\tilde{\omega}^{G+D}$ of total degree $(G+D)$

$$
\begin{aligned}
\tilde{\omega}^{G+D} &= \sum_{j=0}^{D} \omega^{G+D-j}_{j} \ , \\
\tilde{\omega}^{G+D} &= e^{\delta} \left(\omega^{G+D}_0 + \sum_{j=q+1}^{D} \Omega^{G+D-j}_{j} \right) \ ,
\end{aligned} \tag{4.2}
$$

where ω^{G+D}_0 is

$$
\omega^{G+D}_0 = \mathrm{Tr}\, \frac{c^{G+D}}{(G+D)!} \ , \tag{4.3}
$$

and the quantities Ω^{G+D-j}_j are determined recursively by means of the consistency conditions

$$
\left\{
\begin{array}{l}
b\,\Omega^{G+D-j}_{j} = (j-1)(-1)^j\, \mathcal{G}^{1-j}_{j} \omega^{G+D}_0 + \displaystyle\sum_{k=2}^{(j-1)} (k-1)(-1)^k \mathcal{G}^{1-k}_{k} \Omega^{G+D-j+k}_{j-k} \ , \\
\Omega^{G+D-j+k}_{j-k} = 0 \qquad \text{if} \qquad (j-k) < q+1 \ .
\end{array}
\right. \tag{4.4}
$$

As we shall see, the latters turn out to be easily disentangled by using the results (3.9) on the BRS cohomology. Moreover, setting $q=1$, equations (4.4) are seen to reproduce those already met in the pure Yang-Mills case [1]. In particular, from equations (4.2) and (4.3), we see that the solution of the tower (4.1) in the noncomplete case turn out to be deformed with respect to the corresponding solution of the complete ladder case (see Sect. 5 of I) by the inclusion of the cocycles Ω^{G+D-j}_j.

5 Example I: Pure Yang-Mills theory as a zero curvature system

As a first important example of a noncomplete ladder system let us present here the zero curvature formulation of the pure Yang-Mills gauge theory in any space-time dimension, corresponding to a generalized ladder with $q = 1$, *i.e.*

$$\widetilde{\mathcal{A}} = c + A \ . \tag{5.1}$$

It is worthy to recall that, since the Yang-Mills theories are power-counting nonrenormalizable for space-time dimensions greater than four, the fields A and c, unlike the three dimensional Chern-Simons case discussed in I, are now regarded as unquantized external fields coupled to currents of quantum matter fields. Therefore, the existence of gauge anomalies at the quantum level, will correspond to a violation of the conservation law of the matter currents and to the appearence of Schwinger terms in the corresponding current algebra.

It is easily checked that in this case the consistency of the zero curvature condition (2.11) requires that only the first operator $\mathcal{G}_2^{-1}$ of eq.(2.14) is nonvanishing. Therefore for the operator $\widetilde{d}$ we get

$$\widetilde{d} = b + d + \mathcal{G}_2^{-1} \ , \tag{5.2}$$

and from

$$\widetilde{d}\widetilde{\mathcal{A}} \ = \ i \, \widetilde{\mathcal{A}}^2 \ , \tag{5.3}$$

we obtain

$$\begin{aligned} b\, c &= ic^2 \ , \\ b\, A &= -d\, c + i[c, A] \ , \end{aligned} \tag{5.4}$$

and

$$\begin{aligned} \mathcal{G}_2^{-1}\, c &= -d\, A + i\, A^2 = -F \ , \\ \mathcal{G}_2^{-1}\, d\, c &= i[A, F] \ , \\ \mathcal{G}_2^{-1}\, A &= \mathcal{G}_2^{-1}\, F = 0 \ . \end{aligned} \tag{5.5}$$

From equations (2.4), (2.5), for the operator δ we have

$$\begin{aligned} &\delta\, c = A \ , \qquad \delta\, dc = -d\, A + 2iA^2 \ , \\ &\delta A = 0 \ , \qquad \delta\, dA = 0 \ , \end{aligned} \tag{5.6}$$

and

$$\begin{aligned} &d = -[d, \delta] \ , \qquad \mathcal{G}_2^{-1} = \frac{1}{2}[\delta, d] \ , \\ &[\delta, \mathcal{G}_2^{-1}] = \ [b, \mathcal{G}_2^{-1}] = \ [d, \mathcal{G}_2^{-1}] = 0 \ . \end{aligned} \tag{5.7}$$

For what concerns the solutions of the descent equations (4.1) here we shall limit ourselves only to state the final result, reminding the reader to the detailed discussion and proofs already given in [1]. We underline in particular that, as proven in [2], the cocycles Ω_j^{G+D-j} appearing in eq.(4.2) can be summed up into a unique closed generalized expression which collects both the gauge anomalies and the Chern-Simons terms. The latters are given respectively by

$$\omega_{2n}^1 = \sum_{p=0}^{n} \frac{i^{(n-p)}}{(2n-p+1)!p!} \left(\mathcal{P}\left(c, F^p, (A^2)^{n-p}\right) + i(n-p)\mathcal{P}\left([c,A], F^p, A, (A^2)^{n-p}\right) \right) , \tag{5.8}$$

and

$$\omega_{2n+1}^0 = \sum_{p=0}^{n} \frac{i^{(n-p)}}{(2n-p+1)!p!} \mathcal{P}\left(F^p, A, (A^2)^{n-p}\right) , \tag{5.9}$$

where the integer $n = 1, 2, ...$ labels the various dimensions of the space-time and $\mathcal{P}(\mathcal{J}_1, \mathcal{J}_2, ..., \mathcal{J}_n)$ denotes the symmetric invariant polynomials defined as

$$\mathcal{P}(\mathcal{J}_1, \mathcal{J}_2, ..., \mathcal{J}_n) = \mathcal{J}_1^{a_1} \mathcal{J}_2^{a_2} ......\mathcal{J}_n^{a_n} \, S\mathrm{Tr}\left(T^{a_1} T^{a_2} .....T^{a_n}\right) , \tag{5.10}$$

STr being the symmetrized trace [7] and, following Zumino's notations [8], we have used

$$\mathcal{P}(\mathcal{J}_1, \mathcal{J}_2, ..., \mathcal{J}^p) = \mathcal{P}(\mathcal{J}_1, \mathcal{J}_2, \mathcal{J}_3, \underbrace{\mathcal{J}, ..., \mathcal{J}}_{\text{p-times}}) . \tag{5.11}$$

It is worthy to emphasize that, actually, the formulas (5.8), (5.9) represent one of the most compact expression for the gauge anomaly and for the Chern-Simons term in any space-time dimension.

6 Example II: the case $D = 6$, $G = 1$, $q = 3$

In order to clarify the role of the operators $\mathcal{G}_k^{1-k}$ and of the generalized curvatures R_{m+1}^{1-m}, let us discuss in this second example the solution of the descent equations (1.5) in the six dimensional case $D = 6$ with ghost number $G = 1$ and a gauge ladder with $q = 3$, *i.e.*

$$\widetilde{\mathcal{A}} = c + A + \varphi_2^{-1} + \varphi_3^{-2} . \tag{6.1}$$

From eqs.(3.5), for the BRS transformations we have

$$\begin{aligned} b\, c &= ic^2 , \\ b\, A &= -dc + i[c, A] , \\ b\, \varphi_2^{-1} &= i\left[c, \varphi_2^{-1}\right] - R_2^0 , \\ b\, \varphi_3^{-2} &= i\left[c, \varphi_3^{-2}\right] - R_3^{-1} , \end{aligned} \tag{6.2}$$

where R_2^0, R_3^{-1} are the generalized curvatures of eq.(3.1) whose expressions are given

$$\begin{aligned} R_2^0 &= F = dA - iA^2 , \\ R_3^{-1} &= d\varphi_2^{-1} - i\left[A, \varphi_2^{-1}\right] . \end{aligned} \tag{6.3}$$

In particular, for the highest curvature R_4^{-2} we have

$$R_4^{-2} = d\varphi_3^{-2} - i\left[A, \varphi_3^{-2}\right] - \frac{i}{2}\left[\varphi_2^{-1}, \varphi_2^{-1}\right] \tag{6.4}$$

and

$$b\, R_{m+1}^{1-m} = i\left[c, R_{m+1}^{1-m}\right] \ , \qquad 1 \le m \le 3 \ . \tag{6.5}$$

The curvatures $(R_2^0, R_3^{-1}, R_4^{-2})$ obey the following generalized Bianchi identities

$$\begin{aligned} d\,R_2^0 &= i\,[A, R_2^0] \\ d\,R_3^{-1} &= i\left[A, R_3^{-1}\right] + i\left[\varphi_2^{-1}, R_2^0\right] \\ d\,R_4^{-2} &= i\left[A, R_4^{-2}\right] + i\left[\varphi_2^{-1}, R_3^{-1}\right] + i\left[\varphi_3^{-2}, R_2^0\right] \ . \end{aligned} \tag{6.6}$$

They transform under the operator δ of eqs. (2.4), (2.5) as

$$\begin{aligned} \delta\,R_2^0 &= 2R_3^{-1} \\ \delta\,R_3^{-1} &= -R_4^{-2} - \frac{i}{2}\left[\varphi_2^{-1}, \varphi_2^{-1}\right] \\ \delta\,R_4^{-2} &= -i\left[\varphi_3^{-2}, \varphi_2^{-1}\right] \ . \end{aligned} \tag{6.7}$$

For what concerns the operators $\mathcal{G}_k^{1-k}$ of eq.(2.15) it is easily seen that in the present example the zero curvature equation (2.11) implies the existence of a set of five nonvanishing operators $(\mathcal{G}_2^{-1}, \mathcal{G}_3^{-2}, \mathcal{G}_4^{-3}, \mathcal{G}_5^{-4}, \mathcal{G}_6^{-5})$. Their action on the fields and on the curvatures is given respectively by

$$\left\{ \begin{array}{l} \mathcal{G}_2^{-1}\, c = 0 \ , \qquad \mathcal{G}_2^{-1}\, A = 0 \ , \qquad \mathcal{G}_2^{-1}\, \varphi_2^{-1} = -2R_4^{-2} \ , \\ \mathcal{G}_2^{-1}\, \varphi_3^{-2} = 2i\left[\varphi_3^{-2}, \varphi_2^{-1}\right] \ , \\ \mathcal{G}_2^{-1}\, dc = 0 \ , \qquad \mathcal{G}_2^{-1}\, R_2^0 = 0 \ , \\ \mathcal{G}_2^{-1}\, R_3^{-1} = 2i\Big(\left[\varphi_2^{-1}, R_3^{-1}\right] + \left[\varphi_3^{-2}, R_2^0\right] \Big) \ , \\ \mathcal{G}_2^{-1}\, R_4^{-2} = 2i\left[\varphi_3^{-2}, R_3^{-1}\right] \ , \end{array} \right. \tag{6.8}$$

$$\left\{ \begin{array}{l} \mathcal{G}_3^{-2}\, c = 0 \ , \qquad \mathcal{G}_3^{-2}\, A = \dfrac{4}{3} R_4^{-2} \\ \mathcal{G}_3^{-2}\, \varphi_2^{-1} = -\dfrac{4i}{3}\left[\varphi_3^{-2}, \varphi_2^{-1}\right] \ , \\ \mathcal{G}_3^{-2}\, \varphi_3^{-2} = 2i\left[\varphi_3^{-2}, \varphi_3^{-2}\right] \ , \\ \mathcal{G}_3^{-2}\, dc = 0 \ , \\ \mathcal{G}_3^{-2}\, R_2^0 = -\dfrac{4i}{3}\Big(\left[\varphi_2^{-1}, R_3^{-1}\right] + \left[\varphi_3^{-2}, R_2^0\right] \Big) \ , \\ \mathcal{G}_3^{-2}\, R_3^{-1} = 4i\left[\varphi_3^{-2}, R_3^{-1}\right] \ , \\ \mathcal{G}_3^{-2}\, R_4^{-2} = 0 \ , \end{array} \right. \tag{6.9}$$

$$
\left\{
\begin{array}{l}
\mathcal{G}_4^{-3}\, c = -\dfrac{1}{3} R_4^{-2} \ , \qquad \mathcal{G}_4^{-3}\, A = \dfrac{i}{3} \left[\varphi_3^{-2}, \varphi_2^{-1} \right] \ , \\
\mathcal{G}_4^{-3}\, \varphi_2^{-1} = -\dfrac{5i}{2} \left[\varphi_3^{-2} \, , \varphi_3^{-2} \right] \ , \qquad \mathcal{G}_4^{-3}\, \varphi_3^{-2} = 0 \ , \\
\mathcal{G}_4^{-3}\, dc = \dfrac{i}{3} \Big(\left[A, R_4^{-2} \right] + \left[\varphi_2^{-1}, R_3^{-1} \right] + \left[\varphi_3^{-2}, R_2^{0} \right] \Big) \ , \\
\mathcal{G}_4^{-3}\, R_2^{0} = \dfrac{i}{3} \Big(\left[\varphi_2^{-1}, R_4^{-2} \right] - 11 \left[\varphi_3^{-2}, R_3^{-1} \right] \Big) \ , \\
\mathcal{G}_4^{-3}\, R_3^{-1} = 0 \ , \qquad \mathcal{G}_4^{-3}\, R_4^{-2} = 0 \ ,
\end{array}
\right. \tag{6.10}
$$

$$
\left\{
\begin{array}{l}
\mathcal{G}_5^{-4}\, c = 0 \ , \qquad \mathcal{G}_5^{-4}\, A = \dfrac{6i}{5} \left[\varphi_3^{-2}, \varphi_3^{-2} \right] \ , \\
\mathcal{G}_5^{-4}\, \varphi_2^{-1} = 0 \ , \qquad \mathcal{G}_5^{-4}\, \varphi_3^{-2} = 0 \ , \\
\mathcal{G}_5^{-4}\, dc = \dfrac{16i}{5} \left[\varphi_3^{-2}, R_3^{-1} \right] \ , \\
\mathcal{G}_5^{-4}\, R_2^{0} = \mathcal{G}_5^{-4}\, R_3^{-1} = \mathcal{G}_5^{-4}\, R_4^{-2} = 0 \ ,
\end{array}
\right. \tag{6.11}
$$

$$
\left\{
\begin{array}{l}
\mathcal{G}_6^{-5}\, c = -\dfrac{i}{5} \left[\varphi_3^{-2}, \varphi_3^{-2} \right] \ , \\
\mathcal{G}_6^{-5}\, A = \mathcal{G}_6^{-5}\, \varphi_2^{-1} = \mathcal{G}_6^{-5}\, \varphi_3^{-2} = 0 \ , \\
\mathcal{G}_6^{-5}\, dc = \mathcal{G}_6^{-5}\, R_2^{0} = \mathcal{G}_6^{-5}\, R_3^{-1} = \mathcal{G}_6^{-5}\, R_4^{-2} = 0 \ .
\end{array}
\right. \tag{6.12}
$$

Turning now to the descent equations

$$
\begin{array}{l}
b\, \omega_{6-j}^{1+j} + d\, \omega_{5-j}^{2+j} = 0 \ , \qquad 0 \le j \le 5 \ , \\
b\, \omega_0^7 = 0 \ ,
\end{array} \tag{6.13}
$$

we have that, taking into account the result (3.3) on the cohomology of the BRS operator b and the equation (4.2), a solution of the ladder (6.13) is provided by the generalized cocycle of total degree seven

$$
\tilde{\omega}^7 = e^{\delta} \left(\omega_0^7 + \Omega_4^3 + \Omega_5^2 + \Omega_6^1 \right) \tag{6.14}
$$

with

$$
\omega_0^7 = \mathrm{Tr}\, \frac{c^7}{7!} \ , \tag{6.15}
$$

and $(\Omega_4^3,\, \Omega_5^2,\, \Omega_6^1)$ solutions of the equations (4.4), *i.e.*

$$
b\, \Omega_4^3 = 3\, \mathcal{G}_4^{-3} \omega_0^7 \ , \tag{6.16}
$$

$$
b\, \Omega_5^2 = -4\, \mathcal{G}_5^{-4} \omega_0^7 \ , \tag{6.17}
$$

$$
b\, \Omega_6^1 = \mathcal{G}_2^{-1} \Omega_4^3 \, + \, 5\, \mathcal{G}_6^{-5} \omega_0^7 \ . \tag{6.18}
$$

This system can be easily solved by using the cohomology of b. Indeed, beggining with the first equation (6.16) we have from (6.10)

$$\mathcal{G}_4^{-3}\,\omega_0^7 = -\frac{1}{6!}\mathrm{Tr}\,\frac{R_4^{-2}c^6}{3}\ , \tag{6.19}$$

so that Ω_4^3 may be identified with

$$\Omega_4^3 = -\frac{i}{6!}\mathrm{Tr}\ R_4^{-2}c^5\ . \tag{6.20}$$

Concerning now the second equation (6.17), we get from (6.11) that

$$\mathcal{G}_5^{-4}\,\omega_0^7 = 0\ . \tag{6.21}$$

Moreover, since the cohomology of b in the sector of form degree five and ghost number two is empty, we may choose Ω_5^2 to be vanishing as well

$$\Omega_5^2 = 0\ . \tag{6.22}$$

Finally for the last equation (6.18), we get

$$b\,\Omega_6^1 = \frac{2}{6!}\mathrm{Tr}\left(\ \left[\varphi_3^{-2}, R_3^{-1}\right]\ c^5\ -\ i\varphi_3^{-2}\varphi_3^{-2}c^6\right)\ . \tag{6.23}$$

However, from

$$b\left(\mathrm{Tr}\,\varphi_3^{-2}\varphi_3^{-2}c^5\right) = \mathrm{Tr}\left(\ \left[\varphi_3^{-2}, R_3^{-1}\right]\ c^5\ -\ i\varphi_3^{-2}\varphi_3^{-2}c^6\right)\ , \tag{6.24}$$

we obtain

$$\Omega_6^1 = \frac{1}{6!}\mathrm{Tr}\left(\ \left[\varphi_3^{-2}, \varphi_3^{-2}\right]c^5\right)\ . \tag{6.25}$$

Summarizing, an explicit expression for the Ω's is given by

$$\begin{aligned}
\Omega_4^3 &= -\frac{i}{6!}\mathrm{Tr}\ R_4^{-2}c^5\ ,\\
\Omega_5^2 &= 0\ ,\\
\Omega_6^1 &= \frac{1}{6!}\mathrm{Tr}\left(\ \left[\varphi_3^{-2}, \varphi_3^{-2}\right]c^5\right)\ .
\end{aligned} \tag{6.26}$$

Of course, the above expressions are always determined modulo trivial b-cocycles.

Concluding, for the generalized cocycle $\tilde{\omega}^7$ we have

$$\tilde{\omega}^7 = \mathrm{Tr}\left(\frac{\widetilde{\mathcal{A}}^7}{7!}\ +\ \Omega_4^3\ +\ \delta\Omega_4^3\ +\ \frac{\delta^2}{2}\Omega_4^3\ +\ \Omega_6^1\right)\ . \tag{6.27}$$

The expansion of $\tilde{\omega}^7$ in terms of components of different degree and ghost number will give an explicit expression for the cocycles entering the descent equations (6.13).

Conclusion

We have shown that the Yang-Mills type theories can be characterized by means of a noncomplete gauge ladder field constrained to obey a zero curvature condition, which implies the existence of a set of new operators $\mathcal{G}_k^{1-k}$. These operators give rise together with the BRS operator b to a kind of descent equations which are easily solved using the results on the cohomology of b. These solutions provide a deformation of the cohomology of b modulo d with respect to the corresponding complete ladder case presented in the part I.

References

[1] S.P. Sorella, *Comm. Math. Phys.* 157 (1993) 231;

[2] S.P. Sorella and L.Tataru, *Phys. Lett.* B324 (1994) 351;

[3] M. Dubois-Violette, M. Talon and C.M. Viallet, *Comm. Math. Phys.* 102 (1985) 105;

[4] F. Brandt, N. Dragon and M. Kreuzer, *Phys. Lett.* B231 (1989) 263; *Nucl. Phys.* B332 (1990) 224, 250;

[5] M. Dubois-Violette, M. Henneaux, M. Talon and C.M. Viallet, *Phys. Lett.* B289 (1992) 361;

[6] G. Barnich, F. Brandt and M. Henneaux, *Local BRST cohomology in the antifield formalism: I. General theorems*, ULB-TH-94/06, NIKHEF-H 94-13, hepth/9405109;
G. Barnich, F. Brandt and M. Henneaux, *Local BRST cohomology in the antifield formalism: II. Application to Yang-Mills theory*, ULB-TH-94/07, NIKHEF-H 94-15;

[7] J. Thierry-Mieg, *J. Math. Phys.* 21 (1980) 2834; *Nuovo Cim.* 56A (1980) 396;
L. Baulieu, *Nucl. Phys.* B241 (1984) 557;
L. Baulieu and J. Thierry-Mieg, *Phys. Lett.* B145 (1984) 53;
J. Manes, R. Stora and B. Zumino, *Comm. Math. Phys.* 102 (1985) 157;
R. Stora, *Algebraic Structure and Topological Origin of Anomalies*, in " *Progress in Gauge Field Theory* ", ed. 't Hofft et al. Plenum Press, New York, 1984;
R.D. Ball, *Phys. Rev.* 182 (1989) 1;

[8] B. Zumino, Yong-Shi Wu and A. Zee *Nucl. Phys.* B239 (1984) 477;
B. Zumino, *Nucl. Phys.* B53 (1985) 477;