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GENERAL EXACT SOLUTION FOR HOMOGENEOUS TIME-DEPENDENT SELF-GRAVITATING PERFECT FLUIDS

by

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ABSTRACT

A procedure to obtain the general exact solution of Einstein equations for a self-gravitating spherically- symmetric static perfect fluid obeying an arbitrary equation of state, is applied to time-dependent Kantowsky-Sachs line elements (with spherical, planar and hyperbolic symmetry). As in the static case, the solution is generated by an arbitrary function of the independent variable and its first derivative. To illustrate the results, the whole family of (plane-symmetric) solutions with a "gamma-law" equation of state is explicitly obtained in terms of simple known functions. It is also shown that, while in the static plane-symmetric line element, every metric is in one to one correspondence with a "partner-metric" (both originated from the same generatrix function) , in this case every generatrix function univocally determines one metric.

Key-words: General relativity; Perfect fluids; Exact solutions.

I. INTRODUCTION

In this article we extend a procedure originally conceived to find the general static solution of self gravitating perfect fluids [1,2] to the case of time-dependent distributions of matter.

The approach consists in looking at the differential equations for the metric coefficients without appealing "a priori" to any equation of state for the self-gravitating perfect fluid. This allows the introduction of an arbitrary function G in terms of which it is possible to determine all the relevant unknown functions. Thus, the explicit form of the equation of state becomes fixed only after the integration has been carried out.

It is not claimed that the method necessarily provides a useful tool to find new solutions. Rather, it should be understood as a device to get extra insight about the structure of the solutions of Einstein equations or as a possible alternative way to classify them.

Nonetheless, due to its physical importance and to get some acquaintance with the method, an example is exhibited.

Apart from the general form of the solution, two items regarding the features of the used mechanism deserve special mention.

First, it should be remembered that if two solutions have the same equation of state they may correspond to the same solution written in different coordinates. However, as it will be seen, solutions labeled by a different function G trully correspond to different solutions as they do induce different equations of state.

Second, in the static case with planar symmetry the solutions appeared in couples: every solution induced a partner that in turn generated back its own seed. However, in this case an important deviation from such situation occurs. The particular combination of factors in the quadrature is such that the integrals, although similar, are simpler than in the static case and the problem is reduced to a single integral depending only on t and G (and not on its first derivative).

In the second section we write the Einstein field equations for spherical, plane and hyperbolic symmetry when the metric coefficients and the matter parameters exhibit time dependence only. Then, in section III, we mimic the method already used in the static case to get the general solution of the equations. In section IV the plane-symmetric case is studied in more detail and the most general plane symmetric metric obeying a gamma-law equation of state is obtained using the proposed scheme. Last section are conclusions.

II.FIELD EQUATIONS

Consider the Kantowsky-Sachs type line element

$$ds^2 = dt^2 - A(t)dr^2 - B(t)\left[d\theta^2 + \Sigma(\theta,K)d\varphi^2\right] \tag{II.1}$$

where the metric coefficients A , B depend only on t and

$$\Sigma(\theta,K) = \begin{cases} \sin^2\theta & , K=1 \\ 1 & , K=0 \\ \sinh^2\theta & , K=-1 \end{cases} \tag{II.2}$$

The parameter K says wether the matter distribution has spherical (K=1), plane (K=0) or hyperbolic (K=-1) symmetry.

A slightly different form of the metric is obtained if B(t) (instead of t itself) is used as independent variable:

$$ds^2 = g^2(t)\,dt^2 - h^2(t)\,dr^2 - t^2\left[d\theta^2 + \Sigma(\theta,K)\,d\varphi^2\right] \tag{II.3}$$

(the new time variable has been renamed as t).

If the gravitational field is generated by a perfect fluid, the associated energy-momentum tensor reads

$$T_{\mu\nu} = (p+\rho)\,u_\mu u_\nu - p\,g_{\mu\nu} \tag{II.4}$$

where p is the matter pressure, ρ is the energy density and

$$u^\mu = \frac{1}{g}\,\delta^\mu_0 \tag{II.5}$$

is the matter four-velocity.

The Einstein field equations

$$G^{\mu}{}_{\nu} = T^{\mu}{}_{\nu} \qquad \text{(II.6)}$$

are

$$\frac{K}{t^2} + \frac{2}{tg^2}\left(\frac{\dot h}{h} + \frac{1}{2t}\right) = \rho$$

$$-\frac{K}{t^2} + \frac{2}{tg^2}\left(\frac{\dot g}{g} - \frac{1}{2t}\right) = p \qquad \text{(II.7)}$$

$$-\frac{1}{g^2}\left[\frac{\ddot h}{h} - \frac{\dot h \dot g}{hg} + \frac{1}{t}\left(\frac{\dot h}{h} - \frac{\dot g}{g}\right)\right] = p$$

whereas the hydrodynamic equation

$$T^{\mu}{}_{\nu;\mu} = 0 \qquad \text{(II.8)}$$

is

$$\dot\rho = -(p+\rho)\left(\frac{\dot h}{h} + \frac{2}{t}\right) \qquad \text{(II.9)}$$

and as usual, it can be also deduced from the Bianchi identities for the curvature tensor constructed from the line element (II.1).

The system of equations (II.7) serves -in principle- to determine both the metric coefficients (g^2 , h^2) and the thermodynamic variables (p , ρ).

However, unless an equation of state

$$p = p(\rho) \qquad \text{(II.10)}$$

linking p and ρ be introduced, the problem remains undetermined (only three equations for four unknowns).

Solving (II.7) together with (II.9) can become extremely difficult. Usually, when handling this kind of systems, a functional dependence of one of the unknown functions (say ρ) on the independent variable (t, in the present case) is given by hand.

In such a way, the three remaining differential field equations can be solved for g^2 , h^2 and p and an equation of state is determined "a posteriori".[3]. For instance, take ρ as given. Then,

$$12\, m\dot{m} - 2t\left(\rho t^2 - 4K\right)\dot{m} + 2t^2\left(2\dot{\rho}t + 3\rho\right)m - t^3\left(\rho^2 t^2 + 4K\rho + 2Kt\dot{\rho}\right) = 0 \qquad \text{(II.11)}$$

Nonetheless, there are few choices of ρ leading to an exactly solvable differential equation for m.

III.THE METHOD

Surprisingly, there exists another prescription by means of which it is possible to get the general exact solution of the problem in terms of quadratures.

In fact, Eq.(II.7.a) can be rewritten as

$$\frac{1}{t^2}\frac{d}{dt}\left[t\left(K + \frac{1}{g^2}\right)\right] = -p \qquad \text{(III.1)}$$

and then integrated, giving

$$\frac{1}{g^2} = -\left[K + \frac{2m(t)}{t}\right] \quad , \qquad \text{(III.2)}$$

where

$$\frac{dm(t)}{dt} = \frac{1}{2} p t^2 \qquad \text{(III.3)}$$

If Eq. (III.2) and $\frac{\dot{h}}{h}$ from Eq. (II.9) are introduced into Eq.(II.7.a), one gets

$$(2m + Kt)\left(\frac{3}{t^3} + \frac{2}{t^2}\ \frac{\dot{\rho}}{p+\rho}\right) = \left(\rho - \frac{K}{t^2}\right) \qquad \text{(III.4)}$$

Now define

$$G(t) \equiv \frac{2m(t) + Kt}{\rho(t) - \frac{K}{t^2}} \qquad \text{(III.5)}$$

Then, Eq.(III.4) becomes

$$\dot{\rho} - \frac{(t^3 - 3G)(t^2 + \dot{G})}{G(t^3 + 3G)}\rho + \frac{K(t^3 - 3G)(t^3 + t\dot{G} - 2G)}{t^3 G(t^3 + 3G)} = 0 \qquad \text{(III.6)}$$

The last equation is first order and linear in ρ and can be integrated at once if G is a given function of t. In such a case

$$\rho(t) = e^{I(t)}\left[\rho_0 + K\int dt\ \frac{(3G - t^3)(t^3 + t\dot{G} - 2G)}{t^3 G(t^3 + G)}\, e^{-I(t)}\right], \qquad \text{(III.7)}$$

where ρ_0 is an integration constant and

$$I(t) = \int dt\ \frac{(t^3 - 3G)(t^2 + \dot{G})}{G(t^3 + 3G)} \qquad \text{(III.8)}$$

Moreover, from Eq.(III.3) and the definition (III.5) of G,

$$p(t) = \frac{1}{t^2}\left[\rho\dot{G} + G\dot{\rho} - \frac{K}{t^3}\left(t^3 + t\dot{G} - 2G\right)\right] \qquad \text{(III.9)}$$

Also

$$g^2 = \left[\frac{1}{t}\left(\frac{K}{t^2} - \rho\right)G\right]^{-1} \qquad \text{(III.10)}$$

and

$$h^2 = \frac{h_o^2}{t}\, e^{-J(t)} \qquad \text{(III.11)}$$

where

$$J(t) = \int dt\, \frac{t^2}{G} \qquad \text{(III.12)}$$

and h_o is an integration constant.

Considering G as a given function of t is in a sense equivalent to postulate an equation of state. In fact, once a choice for G in terms of t is made, Eq.(III.5) links p and m. Such a relationship cannot be directly understood as an equation of state because it lacks the invariance property under arbitrary coordinate transformation it should have. (In any case, it would be more satisfactory to have a relationship involving p and ρ rather than p and m).

IV. AN EXAMPLE : PLANE SYMMETRY.

Consider now K = 0.
Then, from Eq.(III.7)

$$\rho = \rho_o\, e^{I(t)} \qquad \text{(IV.1)}$$

where I is given by Eq. (III.8).
A little algebra shows that

$$I(t) = \ln \frac{G}{(t^3 + 3G)^2} + J(t) \qquad \text{(IV.2)}$$

Then, from Eq.(IV.1)

$$\rho = \rho_0 \frac{G}{(t^3 + 3G)^2} e^{J(t)} \qquad \text{(IV.3)}$$

Also, from Eqs. (III.6) and (III.9) (for $K=0$)

$$p = \frac{t^3 + 2t\dot{G} - 3G}{t^3 + 3G} \rho \qquad \text{(IV.4)}$$

and from Eq. (III.10)

$$g^2 = - \frac{t}{\rho G} \qquad \text{(IV.5)}$$

To illustrate the above formulae let us find - following reference [4] - the plane symmetric line element generated when the space is filled with a perfect fluid obeying a "gamma-law" equation of state:

$$p = (\gamma - 1) \rho \quad , \qquad \text{(IV.6)}$$

for any γ such that $1 < \gamma \leq 2$.
Substituting Eq.(IV.6) into Eq. (IV.4),

$$2t\dot{G} - 3\gamma G + (2-\gamma)t^3 = 0 \quad , \qquad \text{(IV.7)}$$

which is solved by

$$G = \begin{cases} -\frac{1}{3}t^3 + G_0 \, t^{\frac{3\gamma}{2}} & \gamma \neq 2 \\ G_0 t^3 & \gamma = 2 \end{cases} \quad , \qquad \text{(IV.8)}$$

where G_0 is an integration constant.
Inserting (IV.8) into Eq.(III.12)

$$\exp J(t) = \begin{cases} t^{\frac{3\gamma}{2-\gamma}} \; G^{\frac{2}{\gamma-2}} & \gamma \neq 2 \\ t^{G_0} & \gamma = 2 \end{cases} \qquad \text{(IV.9)}$$

Then, from equations (III.11), (IV.3), (IV.6) and (IV.5) one gets respectively

$$h^2(t) = \begin{cases} h_0^2 \; t^{\frac{2(\gamma+1)}{\gamma-2}} \; G^{\frac{2}{2-\gamma}} & \gamma \neq 2 \\ h_0^2 \, t^{-(1+G_0)} & \gamma = 2 \end{cases} \qquad \text{(IV.10)}$$

$$(\gamma-1)\,\rho(t)=p(t)=\begin{cases}\dfrac{\rho_0(\gamma-1)}{9G_0^2}\,t^{\frac{3\gamma(\gamma-1)}{2-\gamma}}\,G^{\frac{\gamma}{\gamma-2}} & \gamma\neq 2\\[2ex] \dfrac{\rho_0 G_0(\gamma-1)}{(1+3G_0)^2}\;t^{G_0-3} & \gamma=2\end{cases}\tag{IV.11}$$

$$g^2(t)=\begin{cases}-\dfrac{9G_0^2}{\rho_0}\,t^{\frac{2+2\gamma-3\gamma^2}{2-\gamma}} & \gamma\neq 2\\[2ex] -\dfrac{(1+3G_0)^2}{\rho_0 G_0}\;t^{-G_0} & \gamma=2\end{cases}\tag{IV.12}$$

V. CONCLUSIONS

As it has been stated elsewhere [1,2] , expressions (III.7) to (III.11) constitute the whole set of solutions of the equations (II.7). No spurious solutions have been introduced anywhere as it can be proved by direct substitution of the solutions into the field equations. On the other hand, any solution can be accomodated in the present scheme; for suppose you consider a metric obtained using other techniques. Then, by inverting relation (III.9) the generatrix function can be determined in terms of the metric coefficient h and its first derivative:

$$G=-\frac{ht^3}{h_0\,(h+2\dot{h}t)}\tag{V.1}$$

Moreover, the solutions obtained are not merely coordinate transformations, because every solution does produce a different equation of state.

It is interesting that within the present context, some restrictions have to be imposed on the relevant parameters of the solutions if we want them to possess physical meaning. In fact, if t is the time (and that happens to be when the metrics are asymptotically flat) then it ranges from $-\infty$ to $+\infty$. Then, by looking at expression (IV.8) for the generatrix function, it is seen that there are certain forbidden values of γ in order to mantain G (and the physical quantities derived from it) real.

In order to stablish a comparison, let us summarize the main results obtained in the static case with plane symmetry.

The line element to be determined there is:

$$ds^2 = g^2(x)dt^2 - h^2(x)\, dx^2 - x^2\left(dy^2 + dz^2\right) \tag{V.2}$$

If G is defined as

$$G \equiv \frac{m}{p} \tag{V.3}$$

then

$$\begin{aligned} p &= p_0 \exp\left[\int \frac{(x^2+G')(x^3+G)}{G\,(x^3-G)}dx\right] \\ &= p_0 \frac{G}{(x^3-G)^2}\; e^{J(x)}\; e^{8H(x)} \end{aligned} \tag{V.4}$$

$$\begin{aligned} \rho &= \frac{1}{x^2}\left(pG' + p'G\right) \\ &= \frac{x^3 + 2xG' + G}{x^3 - G}\; p \end{aligned} \tag{V.5}$$

$$g^2 = g_0^2\; \frac{e^{-J(x)}}{x} \tag{V.6}$$

$$h^2 = -\frac{x}{pG} \tag{V.7}$$

where

$$J(x) = \int dx\; \frac{x^2}{G} \tag{V.8}$$

and

$$H(x) = \int dx \; \frac{x^2}{x^3 - G} \tag{V.9}$$

Now, suppose that the integrals J and H can be completely carried out in terms of known functions. A companion metric can be determined at once. In fact, consider

$$G^* = x^3 - G \tag{V.10}$$

Then, $J^* = H$ and $H^* = J$. (Notice that $(G^*)^* = G$)
Consequently, a star solution can be written

$$p^* = p_0 \; \frac{x^3 - G}{G^2} \; e^{H(x)} \; e^{8J(x)} \tag{V.11}$$

$$\rho^* = \frac{7x^3 - 2xG' - G}{G} \; p^* \tag{V.12}$$

$$(g^*)^2 = g_0^2 \; \frac{e^{-H(x)}}{x} \tag{V.13}$$

$$h^{*2} = - \frac{x}{p^* (x^3 - G)} \tag{V.14}$$

However, in the present case no other solution can be found with the above prescription : every solution is univocally determined by G and it does not induce any other solution.

The physical meaning of the generatrix function G remains unknown to our knowledge. Some attempts in order to extract additional information about it are presently at work [5], specially regarding the stability of the solution (III.7) to (III.10) under perturbations by a scalar field. In particular, the exact analytical solution for the scalar field in a class of background metrics presented in reference [4] has been found.

In any case, we claim that the method applied in the present article possess a very attractive feature when extracting analytical information from the field equations : it allows to handle a whole family of solutions on the same footing.

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