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Anyonic Construction of the $sl_{q,s}(2)$ Algebra

by

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Abstract

Considering anyonic oscillators in a two-dimensional lattice, we realize the quantum semi-group $sl_{(q,s)}(2)$ by means of a generalized Schwinger construction. We find that the parameter q of the algebra is connected to the statistical parameter, whereas the s parameter is related to a s-deformed oscillator introduced at each point of the lattice.

Key-words: Deformed algebras, Anyons.

1 Introduction

Quasitriangular Hopf Algebras, also called Quantum Groups [1–3], have attracted a lot of attention from physicists and mathematicians in the last years. They have found applications in several areas of physics, such as: the inverse scattering method, vertex models, anisotropic spin chains Hamiltonians, knot theory, conformal field theory, heuristic phenomenology of deformed molecules and nuclei, non-commutative approach to quantum gravity and anyon physics [4] (references therein) and [5–9].

In the last case, an interesting connection between the quantum envelopping algebras $sl_q(2)$ and anyons [10–13] was found [6]. It was shown to be possible to realize $sl_q(2)$ via a generalized Schwinger construction [14], using non-local, intrinsically two-dimensional objects. These anyonic oscillators are defined on a square lattice Ω and interpolate between bosonic and fermionic oscillators. The analysis of all deformed classical Lie algebra [15–16] was also done, and the deformation parameter q is related to the statistical parameter ν as $q = \exp(i\pi\nu)$ ($q = \exp(2i\pi\nu)$ for $\mathcal{U}_q(C_n)$).

These anyonic oscillators are hard core objects, and should not be confused with q-oscillators, since these objects are local and can live in any dimension. The connection of q-oscillators with quantum algebras was recently investigated [17–21] thus permitting the discussion of the thermal properties of systems with quantum group symmetry [22–26] and the analysis of the possible application to physical phenomena.

The aim of this letter is to construct the $sl_{(q,s)}(2)$ [27, 28] algebra with these anyonic oscillators. In the next section we review the main results concerning anyonic oscillators, in section 3 we follow ref.[16] to construct a non-local set of generators for $sl_{(q,s)}(2)$. Section 4 is devoted to constructing anyonic oscillators and to showing how they are connected to the generators, introduced in section 3, by the Schwinger method. Then we shall also see that in order to realize $sl_{(q,s)}(2)$ with anyonic oscillators we had to introduce a sort of s-oscillator at each point of the lattice Ω. We make some final remarks in the conclusion.

2 Lattice Angle Functions and Anyonic Oscillators

In this section we are going to review the construction of anyonic oscillators defined on a two-dimensional square lattice Ω of spacing one as has been done in ref.[6].

Anyonic oscillators are intrinsically non-local two-dimensional objects [29–33] which interpolate between fermionic and bosonic oscillators, that can be constructed on a square lattice Ω by means of a Jordan-Wigner [34] construction which in our case transmutes fermionic oscillators into anyonic ones.

To each point $\mathbf{x} = (x_1, x_2)$ of the lattice Ω we associate a cut γ_x, made of bonds of the dual lattice $\tilde{\Omega}$ from minus infinity to $x^* = x + o^*$ along the x (horizontal) axis, $o^* = (\frac{1}{2}, \frac{1}{2})$ being the origin of the dual lattice $\tilde{\Omega}$. We denote by $\mathbf{x}_\gamma$ the point $\mathbf{x}$ and its associated cut γ_x.

The angle function between two different points $\mathbf{x}$ and $\mathbf{y}$ belonging to the lattice Ω is denoted by $\Theta_{\gamma_x}(\mathbf{x}, \mathbf{y})$ and is defined as the angle of the point $\mathbf{x}$ measured from the point y^* belonging to $\tilde{\Omega}$ with respect to a line parallel to the positive x-axis.

One can show that

$$\Theta_{\gamma_x}(\mathbf{x},\mathbf{y}) - \Theta_{\gamma_y}(\mathbf{y},\mathbf{x}) = \begin{cases} \pi\ sgn(x_2,y_2) & \text{for} \quad x_2 \neq y_2 \\ \pi\ sgn(x_1,y_1) & \text{for} \quad x_2 = y_2. \end{cases} \tag{1}$$

In fact, to arrive at this result it is necessary to neglect a term that depends on the distance between $\mathbf{x}$ and $\mathbf{y}$, vanishing when they are far apart. This typical lattice feature can be eliminated by embedding Ω into a lattice Λ whose lattice spacing ϵ is much smaller than 1, then all quantities defined on Ω can be seen as restrictions to Ω of quantities defined on Λ. The above result is obtained when we let ϵ go to zero, since in this limit all the points $\mathbf{x}$ and $\mathbf{y}$ of Ω are far apart, from the point of view of the lattice Λ.

Eq.(1) can be used to endow Ω with an ordering, which will be very useful when dealing with anyonic oscillators. One chooses the positive sign in eq.(1) and then one defines

$$\mathbf{x} > \mathbf{y} = \begin{cases} x_2 > y_2 \\ x_2 = y_2, x_1 > y_1, \end{cases} \tag{2}$$

and eq(1) becomes

$$\Theta_{\gamma_x}(\mathbf{x},\mathbf{y}) - \Theta_{\gamma_y}(\mathbf{y},\mathbf{x}) = \pi \qquad \text{for} \quad \mathbf{x} > \mathbf{y}. \tag{3}$$

Even if unambiguous, this theta angle function introduced is not unique, as it depends on the particular choice of the cuts γ. Another fundamental choice can be obtained if we consider the cut δ made with bonds on $\tilde{\Omega}$ from plus infinity to ${}^*x = y - o^*$ parallel to the x-axis [6]. With this cut δ one can define another lattice angle $\tilde{\Theta}_{\delta_x}(\mathbf{x},\mathbf{y})$ which is the angle of $\mathbf{x}$ as seen from ${}^*y = y - o^*$. With the ordering given by eq.(2) it can be shown that

$$\overline{\Theta}_{\delta_x}(\mathbf{x},\mathbf{y}) - \overline{\Theta}_{\delta_y}(\mathbf{y},\mathbf{x}) = -\pi \qquad \text{for} \quad \mathbf{x} > \mathbf{y}. \tag{4}$$

One can also get from their definitions, a relation between these two angle functions

$$\overline{\Theta}_{\delta_x}(\mathbf{x},\mathbf{y}) - \Theta_{\gamma_x}(\mathbf{x},\mathbf{y}) = \begin{cases} -\pi & \text{for} \quad \mathbf{x} > \mathbf{y} \\ \pi & \text{for} \quad \mathbf{x} < \mathbf{y}, \end{cases} \tag{5}$$

and for any $\mathbf{x}$ and $\mathbf{y}$ (even if $\mathbf{x} = \mathbf{y}$) one has

$$\overline{\Theta}_{\delta_y}(\mathbf{y},\mathbf{x}) - \Theta_{\gamma_x}(\mathbf{x},\mathbf{y}) = 0. \tag{6}$$

One can use the theta function introduced above to define anyonic oscillators, which are related by a parity transformation. One defines them as follows:

$$a_i(\mathbf{x}_\alpha) = K_i(\mathbf{x}_\alpha) c_i(\mathbf{x}), \tag{7}$$

with $\alpha_x = \gamma_x$ or δ_x, $i = 1, ..., N$; the disorder operator given by

$$K_i(x_\alpha) = \exp(i\rho \sum_{\substack{\mathbf{y}\in\Omega \\ \mathbf{y}\neq\mathbf{x}}} \Theta_{\alpha_x}(\mathbf{x},\mathbf{y}) c_i^\dagger(\mathbf{y}) c_i(\mathbf{y})), \tag{8}$$

and $c_i(x)$ are fermionic oscillators obeying

$$\begin{aligned} \{c_i(\mathbf{x}), c_i(\mathbf{y})\} &= 0 \\ \left\{c_i(\mathbf{x})^\dagger, c_i(\mathbf{y})\right\} &= \delta_{ij}\ \delta(\mathbf{x},\mathbf{y}), \end{aligned} \tag{9}$$

and their hermitean conjugate counterparts (which are going to be omitted always in this letter). In the above formula $\delta(\mathbf{x},\mathbf{y})$ is the delta function on Ω, i.e.

$$\delta(\mathbf{x},\mathbf{y}) = \begin{cases} 0 & \text{if } \mathbf{x} \neq \mathbf{y} \\ 1 & \text{if } \mathbf{x} = \mathbf{y}. \end{cases} \tag{10}$$

The anyonic oscillators of type γ obey the following generalized commutation relations for $\mathbf{x} > \mathbf{y}$

$$\begin{aligned} a_i(\mathbf{x}_\gamma)a_i(\mathbf{y}_\gamma) + q^{-1}a_i(\mathbf{y}_\gamma)a_i(\mathbf{x}_\gamma) = 0 \\ a_i(\mathbf{x}_\gamma){a_i}^\dagger(\mathbf{y}_\gamma) + q{a_i}^\dagger(\mathbf{y}_\gamma)a_i(\mathbf{x}_\gamma) = 0, \end{aligned} \tag{11}$$

where $q = \exp(i\pi\rho)$. For $\mathbf{x} = \mathbf{y}$ one has

$$\begin{aligned} (a_i(\mathbf{x}_\gamma))^2 = 0, \\ a_i(\mathbf{x}_\gamma){a_i}^\dagger(\mathbf{x}_\gamma) + {a_i}^\dagger(\mathbf{y}_\gamma)a_i(\mathbf{x}_\gamma) = 1. \end{aligned} \tag{12}$$

Thus, as one can see from the above discussion, anyonic oscillators are hard core objects which obey q-commutation relations at different points of the lattice but standard anticommutation relations at the same point.

The commutation relation among the anyonic oscillators of type δ can be obtained from eqs.(11-12) by replacing q by q^{-1} and of course the cuts γ by δ. This is due to the fact that type δ-oscillators can be obtained from type γ-oscillators by a parity transformation which, as is well known, changes the braiding parameter q to q^{-1} [13].

Commutaion relatins among different types of oscillators can also be computed, and they read

$$\begin{aligned} \{a_i(\mathbf{x}_\gamma), a_i(\mathbf{y}_\delta)\} = 0 \\ \left\{{a_i}^\dagger(\mathbf{y}_\delta), a_i(\mathbf{x}_\gamma)\right\} = 0, \end{aligned} \tag{13}$$

and

$$\left\{a_i(\mathbf{x}_\delta), {a_i}^\dagger(\mathbf{y}_\gamma)\right\} = q^{\left(\sum_{\mathbf{y}<\mathbf{x}} - \sum_{\mathbf{y}>\mathbf{x}}\right) {c_i}^\dagger(\mathbf{y})c_i(\mathbf{y})}. \tag{14}$$

Finally, we should mention that different anyonic oscillators (those made up of different types of fermions) anticommute. With the above defined anyonic oscillators one can realize all the classical deformed algebras (with an introduction of a background term in the disorder operator $K_i(\mathbf{x}_\alpha)$ for the B and D series) [15, 16].

3 The $sl_{(q,s)}(2)$ Quantum Semi-Group and its Non-Local Realization

The commutation relations among the generators of the two-parametric quantum algebra $sl_{(q,s)}(2)$ [27]

$$\begin{aligned} [j_0, j_\pm] = \pm j_\pm, \\ [j_+, j_-]_s \equiv s^{-1}j_+j_- - sj_-j_+ = s^{-2j_0}[2j_0], \end{aligned} \tag{15}$$

where $[x] = \frac{q^x - q^{-x}}{q - q^{-1}}$, can be derived from the R matrix [35]

$$R = \begin{pmatrix} q & 0 & 0 & 0 \\ 0 & s & 0 & 0 \\ 0 & q - q^{-1} & s^{-1} & 0 \\ 0 & 0 & 0 & q \end{pmatrix} \tag{16}$$

which is a solution of the constant Yang-Baxter equation

$$R_{12}R_{13}R_{23} = R_{23}R_{13}R_{12}. \tag{17}$$

The comultiplication structure of the algebra [27]

$$\begin{aligned} \Delta(qs)^{-j_0} &= (qs)^{-j_0} \otimes (qs)^{-j_0}, \\ \Delta(j_\pm) &= (qs)^{-j_0} \otimes j_\pm + j_\pm \otimes (qs^{-1})^{j_0}, \end{aligned} \tag{18}$$

together with the compatibility equations convert $sl_{(q,s)}(2)$ into a bialgebra. It is not possible to find an antipode function for this algebra, and thus $sl_{(q,s)}(2)$ is more properly called a quantum semi-group. In the limit $s \to 1$, $sl_{(q,s)}(2)$ goes to $sl_{(q)}(2)$.

An important fact about $sl_{(q,s)}(2)$ is that Pauli matrices are its two-dimensional representation, thus the fundamental representation is the same as for the $sl(2)$ algebra, and all its representations can be obtained from the fundamental one by the use of the comultiplication rules given by eq.(18).

Let us now go back to the lattice Ω introduced in the last section and assign to each point $\mathbf{x} \in \mathbf{\Omega}$ a fundamental representaion of $sl_{(q,s)}(2)$, its generators satisfying the local algebra

$$\begin{aligned} [j_0(\mathbf{x}), j_\pm(\mathbf{x})] &= \pm j_\pm(\mathbf{x}) \\ [j_+(\mathbf{x}), j_-(\mathbf{x})]_s &= s^{-2j_0(\mathbf{x})}[2j_0(\mathbf{x})]. \end{aligned} \tag{19}$$

As the fundamental representation of $sl_{(q,s)}(2)$ is the same as $sl(2)$, the q-deformed structure of this equation is only formal, thus we just write them in this way for future use.

With the local generators $j_0(\mathbf{x}), j_\pm(\mathbf{x})$ one can define

$$\begin{aligned} J_0(\mathbf{x}) &= \textstyle\prod^\otimes_{\mathbf{y}<\mathbf{x}} \mathbf{1}_\mathbf{y} \otimes j_0(\mathbf{x}) \otimes \prod^\otimes_{\mathbf{z}>\mathbf{x}} \mathbf{1}_\mathbf{z} \\ J_\pm(\mathbf{x}) &= \textstyle\prod^\otimes_{\mathbf{y}<\mathbf{x}} (qs)^{-j_0(\mathbf{y})} \otimes j_\pm(\mathbf{x}) \otimes \prod^\otimes_{\mathbf{z}>\mathbf{x}} (qs^{-1})^{j_0(\mathbf{z})}, \end{aligned} \tag{20}$$

(hereafter we shall drop the symbol of the direct product) and the generators

$$\begin{aligned} J_\pm &= \sum_{\mathbf{x}\in\Omega} J_\pm(\mathbf{x}) \\ J_0 &= \sum_{\mathbf{x}\in\Omega} J_0(\mathbf{x}), \end{aligned} \tag{21}$$

obey the algebra of $sl_{(q,s)}(2)$, eq.(15), as they are obtained by the iterated coproduct of the envelopping algebra.

The generators $J_0(\mathbf{x}), J_\pm(\mathbf{x})$ defined above obey the commutation relations

$$
\begin{aligned}
[J_0(\mathbf{x}), J_\pm(\mathbf{y})] &= \pm\delta(\mathbf{x},\mathbf{y})\; J_\pm(\mathbf{x}) \\
[J_+(\mathbf{x}), J_-(\mathbf{y})] &= 0 \qquad \mathbf{x} \neq \mathbf{y} \\
[J_+(\mathbf{x}), J_-(\mathbf{y})]_s &= \prod_{\mathbf{y}<\mathbf{x}} (qs)^{-2j_0(\mathbf{y})} [j_+(\mathbf{x}), j_-(\mathbf{y})]_s \prod_{\mathbf{z}>\mathbf{x}} (qs^{-1})^{2j_0(\mathbf{z})},
\end{aligned} \tag{22}
$$

and the densities $J_\pm(\mathbf{x})$ obey the braiding relations

$$
\begin{aligned}
J_+(\mathbf{x})J_+(\mathbf{y}) &= q^2 J_+(\mathbf{y})J_+(\mathbf{x}) \\
J_-(\mathbf{x})J_-(\mathbf{y}) &= q^{-2} J_-(\mathbf{y})J_-(\mathbf{x}),
\end{aligned} \tag{23}
$$

which could be used to prove directly that $J_0, J_\pm$ obey the $sl_{(q,s)}(2)$ algebra eq.(15).

Let us now use the angles $\Theta_{\gamma_x}(\mathbf{x},\mathbf{y})$ and $\bar{\Theta}_{\delta_x}(\mathbf{x},\mathbf{y})$ introduced in the last section to construct new non-local densities $J_0(\mathbf{x})$, $J_\pm(\mathbf{x})$

$$
\begin{aligned}
J_+(\mathbf{x}) &= \prod_{\mathbf{y}<\mathbf{x}} q^{-\frac{2}{\pi}\theta_{\gamma_\mathbf{x}}(\mathbf{x},\mathbf{y}) j_0(\mathbf{y})}\; s^{-2j_0(\mathbf{y})}\; j_+(\mathbf{x}) \prod_{\mathbf{z}>\mathbf{x}} q^{-\frac{2}{\pi}\theta_{\gamma_\mathbf{x}}(\mathbf{x},\mathbf{z}) j_0(\mathbf{z})}\; s^{-2j_0(\mathbf{z})} \\
J_-(\mathbf{x}) &= \prod_{\mathbf{y}<\mathbf{x}} q^{\frac{2}{\pi}\theta_{\delta_\mathbf{x}}(\mathbf{x},\bar{\mathbf{y}}) j_0(\mathbf{y})}\; s^{-2j_0(\mathbf{y})}\; j_-(\mathbf{x}) \prod_{\mathbf{z}>\mathbf{x}} q^{\frac{2}{\pi}\bar{\theta}_{\delta_\mathbf{x}}(\mathbf{x},\mathbf{z}) j_0(\mathbf{z})}\; s^{-2j_0(\mathbf{z})} \\
J_0(\mathbf{x}) &= \prod_{\mathbf{y}<\mathbf{x}} 1_\mathbf{y}\; j_0(\mathbf{x}) \prod_{\mathbf{z}>\mathbf{x}} 1_\mathbf{z}.
\end{aligned} \tag{24}
$$

Using the relations obeyed by the theta-angle functions and the local algebra eq.(19), we can prove that these densities obey the commutation relations eq.(22) as well as the braiding relations eq.(23) and then realize the algebra $sl_{(q,s)}(2)$, eq.(15).

4 Anyonic Realization of $sl_{(q,s)}(2)$

In this section we are going to show that the anyonic oscillators defined in section 1, with a suitable choice of the disorder operator $K_i(\mathbf{x}_\alpha)$, realize, via a Schwinger like construction the algebra of densities eq.(22) and the braiding relations eq.(22), and consequently also the $sl_{(q,s)}(2)$ algebra eq.(15).

We begin by recalling that all classical Lie algebras can be constructed *à la* Schwinger on a manifold Ω in terms of fermionic oscillators. In particular, for the $sl(2)$ algebra one can define at each point $\mathbf{x}$ of Ω

$$
\begin{aligned}
j_+(\mathbf{x}) &= {c_1}^\dagger(\mathbf{x})c_1(\mathbf{x}) \\
j_0(\mathbf{x}) &= \tfrac{1}{2}\left({c_1}^\dagger(\mathbf{x})c_1(\mathbf{x}) - {c_2}^\dagger(\mathbf{x})c_2(\mathbf{x})\right) \\
j_-(\mathbf{x}) &= {c_2}^\dagger(\mathbf{x})c_2(\mathbf{x}),
\end{aligned} \tag{25}
$$

where $c_i(\mathbf{x})$ are fermionic oscillators. These operators obey a local $sl(2)$ algebra

$$
\begin{aligned}
[j_0(\mathbf{x}), j_\pm(\mathbf{y})] &= \pm\,\delta(\mathbf{x},\mathbf{y})\, j_\pm(\mathbf{x}) \\
[j_+(\mathbf{x}), j_-(\mathbf{y})] &= 2\, j_0(\mathbf{x})\,\delta(\mathbf{x},\mathbf{y}).
\end{aligned} \tag{26}
$$

Once more, global generators $J_\pm$, J_0 can be defined from the densities $J_\pm(\mathbf{x})$, $J_0(\mathbf{x})$,

$$\begin{aligned} J_\pm &= \sum_{\mathbf{x}\in\Omega} J_\pm(\mathbf{x}) \\ J_0 &= \sum_{\mathbf{x}\in\Omega} J_0(\mathbf{x}), \end{aligned} \tag{27}$$

where

$$\begin{aligned} J_0(\mathbf{x}) &= \prod_{\mathbf{y}<\mathbf{x}} \mathbf{1}_\mathbf{y}\; j_0(\mathbf{x}) \prod_{\mathbf{z}>\mathbf{x}} \mathbf{1}_\mathbf{z} \\ J_\pm(\mathbf{x}) &= \prod_{\mathbf{y}<\mathbf{x}} \mathbf{1}_\mathbf{y}\; j_0(\mathbf{x}) \prod_{\mathbf{z}>\mathbf{x}} \mathbf{1}_\mathbf{z}, \end{aligned} \tag{28}$$

and it is very easy to see that $J_\pm$, J_0 closes under $sl(2)$. The spin-0 and spin-1/2 representations of the local algebra can be combined to give all the unitary representations of $sl(2)$.

As was stated in the last section, the $sl_q(2)$ algebra can also be generated by this local $sl(2)$ algebra, just by changing the comultiplication rules for the representations. From the point of view of the Schwinger construction, this is equivalent to changing the oscillators of eq.(26) into the anyonic oscillators introduced in section 2. In fact, with the choices of densities

$$\begin{aligned} J_+(\mathbf{x}) &= {a_1}^\dagger(\mathbf{x}_\gamma) a_2(\mathbf{x}_\gamma) \\ J_0(\mathbf{x}) &= \tfrac{1}{2}\left({a_1}^\dagger(\mathbf{x}_\gamma) a_1(\mathbf{x}_\gamma) - {a_2}^\dagger(\mathbf{x}_\gamma) a_2(\mathbf{x}_\gamma)\right) \\ j_-(\mathbf{x}) &= {a_2}^\dagger(\mathbf{x}_\delta) a_1(\mathbf{x}_\delta), \end{aligned} \tag{29}$$

and with the help of eq.(14) it is possible to see that $J_\pm(\mathbf{x})$, $J_0(\mathbf{x})$, obey eq.(22-23) for $s = 1$ [6]. Thus the global generators defined as the direct sum of the densities will obey the $sl_q(2)$ algebra. We notice here that the choice of the cut γ in J_0 is immaterial, since the product ${a_i}^\dagger(\mathbf{x}_\alpha) a_i(\mathbf{x}_\alpha)$ can be written in terms of fermionic oscillators without any dependence on the disorder operator $K_i(\mathbf{x}_\alpha)$.

The Schwinger construction of $sl_{(q,s)}(2)$ algebra has, however, a subtlety due to the presence of the s-commutator in the local algebra eq.(19). Let us now define the local generators

$$\begin{aligned} j_+(\mathbf{x}) &= {c_1}^\dagger(\mathbf{x}) s^{-\frac{1}{2}({c_1}^\dagger(\mathbf{x}) c_1(\mathbf{x}) - {c_2}^\dagger(\mathbf{x}) c_2(\mathbf{x}))} c_1(\mathbf{x}) \\ j_0(\mathbf{x}) &= \tfrac{1}{2}\left({c_1}^\dagger(\mathbf{x}) c_1(\mathbf{x}) - {c_2}^\dagger(\mathbf{x}) c_2(\mathbf{x})\right) \\ j_-(\mathbf{x}) &= {c_2}^\dagger(\mathbf{x}) s^{\frac{1}{2}({c_1}^\dagger(\mathbf{x}) c_1(\mathbf{x}) - {c_2}^\dagger(\mathbf{x}) c_2(\mathbf{x}))} c_1(\mathbf{x}). \end{aligned} \tag{30}$$

It is easy to see that these local generators close under the algebra eq.(19).

The anyonic oscillators can be defined as

$$A_i(\mathbf{x}_\alpha) = K_i(\mathbf{x}_\alpha) b_i(\mathbf{x}_\alpha), \tag{31}$$

with

$$K_i(\mathbf{x}_\gamma) = \exp \sum_{\mathbf{y}\neq\mathbf{x}} \Big(i\rho\Theta_{\gamma_x}(\mathbf{x},\mathbf{y}) c_i^\dagger(\mathbf{y}) c_i(\mathbf{y}) + \frac{i}{2}\nu\pi c_i^\dagger(\mathbf{y}) c_i(\mathbf{y}) \Big)$$
$$K_i(\mathbf{x}_\delta) = \exp \sum_{\mathbf{y}\neq\mathbf{x}} \Big(i\rho\bar{\Theta}_{\delta_x}(\mathbf{x},\mathbf{y}) c_i^\dagger(\mathbf{y}) c_i(\mathbf{y}) - \frac{i}{2}\nu\pi c_i^\dagger(\mathbf{y}) c_i(\mathbf{y}) \Big), \tag{32}$$

and

$$b_i(\mathbf{x}_\gamma) = \exp\Big(\frac{i\nu\pi}{2} {c_i}^\dagger(\mathbf{x}) c_i(\mathbf{x})\Big) c_i(\mathbf{x})$$
$$b_i(\mathbf{x}_\delta) = \exp\Big(\frac{-i\nu\pi}{2} {c_i}^\dagger(\mathbf{x}) c_i(\mathbf{x})\Big) c_i(\mathbf{x}). \tag{33}$$

These operators obey, at each point $\mathbf{x} \in \Omega$, the algebra

$$\begin{aligned} {b_i}^2(\mathbf{x}_\alpha) &= 0 \\ \{b_i(\mathbf{x}_\alpha), {b_i}^\dagger(\mathbf{x}_\alpha)\} &= 1 \\ \{b_i(\mathbf{x}_\alpha), {b_j}^\dagger(\mathbf{x}_\alpha)\} &= 0 \quad i \neq j, \end{aligned} \tag{34}$$

where α can be γ or δ, and also

$$b_i(\mathbf{x}_\gamma) {b_i}^\dagger(\mathbf{x}_\delta) + s\ {b_j}^\dagger(\mathbf{x}_\delta) b_i(\mathbf{x}_\gamma) = s^{N_i(\mathbf{x})}, \tag{35}$$

where $N_i = {c_i}^\dagger c_i$. So the b operators are hard-core objects that obey a s-deformed Heisenberg algebra at each point of the lattice Ω.

With this choice, the densities $J_\pm(\mathbf{x})$, $J_0(\mathbf{x})$ defined by

$$\begin{aligned} J_+(\mathbf{x}) &= {A_1}^\dagger(\mathbf{x}_\gamma) A_2(\mathbf{x}_\gamma) \\ J_0(\mathbf{x}) &= \tfrac{1}{2}\Big({A_1}^\dagger(\mathbf{x}_\gamma) A_1(\mathbf{x}_\gamma) - {A_2}^\dagger(\mathbf{x}_\gamma) A_2(\mathbf{x}_\gamma)\Big) \\ J_-(\mathbf{x}) &= {A_2}^\dagger(\mathbf{x}_\delta) A_1(\mathbf{x}_\delta), \end{aligned} \tag{36}$$

obey the commutation relations eq.(22), the braiding relations (23) and consequently, $J_0, J_\pm$ defined by

$$\begin{aligned} J_\pm &= \sum_{\mathbf{x}\in\Omega} J_\pm(\mathbf{x}) \\ J_0 &= \sum_{\mathbf{x}\in\Omega} J_0(\mathbf{x}), \end{aligned} \tag{37}$$

satisfy the algebra of $sl_{(q,s)}(2)$.

From their definition, it is easy to see that the disorder operators $K_i(\mathbf{x}_\alpha)$ commute among themselves

$$K_i(\mathbf{x}_\alpha)\ K_j(\mathbf{y}_\beta) = K_j(\mathbf{y}_\beta)\ K_i(\mathbf{x}_\alpha), \qquad \text{for all } \mathbf{x},\mathbf{y} \tag{38}$$

for any value for i, j, where the cuts α and β can be either γ or δ, Eq.(38), together with eq.(34,35) give the following relations for the operators $A_i(\mathbf{x}_\alpha)$

$$A_i(\mathbf{x}_\gamma)A_i(\mathbf{y}_\gamma) = -q^{-1}A_i(\mathbf{y}_\gamma)A_i(\mathbf{x}_\gamma)$$
$$A_i(\mathbf{x}_\gamma){A_i}^\dagger(\mathbf{y}_\gamma) = -qA_i(\mathbf{y}_\gamma)A_i(\mathbf{x}_\gamma), \tag{39}$$

for all $\mathbf{x} > \mathbf{y}$. At the same point we have

$$\left\{A_i(\mathbf{x}_\gamma), {A_i}^\dagger(\mathbf{x}_\gamma)\right\} = 1. \tag{40}$$

The relations for the cut δ can be obtained by changing q into q^{-1} in the relations above. We can also find relations among oscillators defined with different cuts, and they read

$$A_i(\mathbf{x}_\gamma)A_i(\mathbf{y}_\delta) = -s^{-1}A_i(\mathbf{y}_\delta)A_i(\mathbf{x}_\gamma), \tag{41}$$

for all $\mathbf{x}$,$\mathbf{y}$ and

$$A_i(\mathbf{x}_\delta){A_i}^\dagger(\mathbf{y}_\gamma) = -s^{-1}{A_i}^\dagger(\mathbf{y}_\gamma)A_i(\mathbf{x}_\delta), \tag{42}$$

for $\mathbf{x} \neq \mathbf{y}$. If $\mathbf{x} = \mathbf{y}$ we have

$$\begin{aligned}
A_i(\mathbf{x}_\gamma){A_i}^\dagger(\mathbf{x}_\delta) &= K_i(\mathbf{x}_\gamma)b_i(\mathbf{x}_\gamma), {K_i}^\dagger(\mathbf{x}_\delta){b_i}^\dagger(\mathbf{x}_\delta) \\
&= {K_i}^\dagger(\mathbf{x}_\delta)b_i(\mathbf{x}_\gamma){b_i}^\dagger(\mathbf{x}_\delta)K_i(\mathbf{x}_\gamma) \\
&= {K_i}^\dagger(\mathbf{x}_\delta)\left(s^{N_i(\mathbf{x})} - s{b_i}^\dagger(\mathbf{x}_\delta)b_i(\mathbf{x}_\gamma)\right)K_i(\mathbf{x}_\gamma) \\
&= -s{A_i}^\dagger(\mathbf{x}_\delta)A_i(\mathbf{x}_\gamma) + s^{N_i(\mathbf{x})}{K_i}^\dagger(\mathbf{x}_\delta)K_i(\mathbf{x}_\gamma),
\end{aligned} \tag{43}$$

implying the relation

$$A_i(\mathbf{x}_\gamma){A_i}^\dagger(\mathbf{x}_\delta) + s{A_i}^\dagger(\mathbf{x}_\delta)A_i(\mathbf{x}_\gamma) = q^{\left(\sum\limits_{\mathbf{y}<\mathbf{x}} - \sum\limits_{\mathbf{y}>\mathbf{x}}\right){c_i}^\dagger(\mathbf{y})c_i(\mathbf{y})}\, s^{\sum\limits_{\mathbf{y}} N(\mathbf{y})}. \tag{44}$$

The construction we have performed in this section for $sl_{(q,s)}(2)$ algebra uses anyonic oscillators made with s-deformed oscillators. This is a special characteristic of the two-parameter algebra, since all deformed one-parameter algebras were constructed with anyonic oscillators built up of fermions [6, 15, 16].

5 Conclusions

In this letter we have realized the quantum semi-group $sl_{(q,s)}(2)$ on a two-dimensional lattice Ω. We have first showed that the generators can be written as a non-local expression made of the lattice angle function on Ω and we have discussed its connection with a Schwinger like construction using anyonic oscillators.

Differently from the cases previously considered [6, 15, 16], we had to consider a different kind of anyonic oscillators. These anyonic oscillators are made with s-deformed fermionic oscillators defined on Ω, instead of pure fermionic oscillators. The parameter q, as in the previous cases, is connected to the statistical parameter ν by $q = \exp(i\pi\nu)$.

We find that it would be interesting to generalize the analysis we have performed for the case of multiparametric deformed algebras in order to see what would be the role of the various parameters, as in the case of two-parameter algebra one being associated to the statistical parameter and the other being related to the parameter of the deformed Heisenberg algebra.

The connection of q-oscillators with quantum algebras permitted the investigation of the possible applications of quantum groups to physical problems through the analysis of the thermal properties of deformed systems [22–28]. We consider that the analysis of the connection between quantum algebras and anyons could, as well, provide another area of research on the possible applications of quantum groups to physical problems through the anyonic interpretation of planar physics.

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