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Point-Splitting as a Regularisation Method for $\lambda\varphi^4$-type Vertices: Abelian case

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Abstract

We obtained regularised Abelian Lagrangeans containing $\lambda\varphi^4$-type vertices by means of a suitable point-splitting procedure. The calculation is developed in details for a general Lagrangean, whose fields (gauge and matter ones) satisfy certain conditions. We illustrate our results by considering some special cases, such as the Abelian Higgs, the $(\overline{\psi}\psi)^2$ and the Avdeev-Chizov (real and rank-2 antisymmetric tensor as matter fields) models. We also discuss some features of the obtained Lagrangean such as the regularity and non-locality of its new interacting terms. Moreover, the resolution of the Abelian case may teach us some useful technical aspects when dealing with the non-Abelian one.

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Introduction

In the Quantum Field Theory framework of the modern Physics the products of fields at the same space and/or time point are not well-defined since these fields are taken as operator-valued distributions (whose product, in this way, is not mathematically well-defined, in general). As a consequence of such an ill-defined product we are lead to divergent results (e.g., the ultra-violet ones) when we are calculating some relevant quantities in physical theories. Physically speaking, such divergences arise because we describe elementary particles as if they were point-like entities and, consequently, carrying infinity density of physical quantities: mass, charge, etc.

Even though there have several regularisation methods to deal with such problems, those based on point-splitting may offer some advantages to others when performed in a suitable way. Essentialy, the procedure works by taking the field products, initially at the same point, now in diferent points (by *splitting* them). The result is such that the new Lagrangean contains only regularised interaction terms. Already in 1934, Dirac [1] employed such idea in order to split same point products of quantities contained in density matrices of eletronic (and positronic) physical distributions.

Recently, several results have been obtained by means of this method for both Abelian (QED) and non-Abelian (Standard Model) cases. For example, the values of some important physical parameters such as the quark top and Higgs' scalar masses have been got free of divergences and were shown to be in good accordance with other procedures as well as practical results. This procedure was also shown to respect the gauge invariance of the theories (for details see the papers listed in [2, 3]).

Nevertheless, these works did not pay attention to the explicit construction (and form) of the new *point-split* gauge transformations themselves. Such issue was the subject of a more recent paper, by Gastmans, Newton and Wu [4], where the Abelian infinitesimal forms of these new transformations (so-called *generalized gauge transformations*, denoted by ggt's) was proven to exist for all orders in the coupling constant. The explicit forms of such ggt's as well as of a generalized QED-Lagrangean were presented up to fourth order. This new Lagrangean was shown to be regularized, i.e., its interaction terms (including some new ones which appear from the splitting) presented no product of fields at the same point; on the other hand, those new terms also displayed non-locality property. As expected, as we set the point-splitting parameter to zero, we recover the original results.

Even though those ggt's have been built for QED, we do not see any restriction of their using in other Abelian theories containing usual vector gauge field coupled to matter fields (scalar, spinorial or tensorial) in a suitable way. Therefore, we intend here to obtain a *generalized* Lagrangean (point-split version) within which we have, among others, the $\lambda\varphi^4$-type vertex. This new Lagrangean is explicitly constructed up to the second order in the coupling constant. The application of our results to some $\lambda\varphi^4$ Abelian models is drawn within some details.

1 The Lagrangean and the point-splitting procedure

We shall start this section by considering the following Lagrangean[1] (which has the form of the massive scalar Electrodynamics with self-interaction term, or the Abelian Higgs model if $m = \mu^2 < 0$):

$$\mathcal{L}(x) = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + (D_\mu\varphi)^\dagger(D^\mu\varphi) - \frac{m}{2}\varphi^\dagger\varphi - \frac{\lambda}{4}(\varphi^\dagger\varphi)^2\,, \tag{1}$$

with $D_\mu = \partial_\mu + ieA_\mu$ and $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$. Clearly, the matter fields are considered to be complex[2] and their product are taken at the same space-time point, say, x. This Lagrangean is invariant under the usual local gauge transformations:

$$\delta A_\mu(x) = -\partial_\mu\Lambda(x)\,; \qquad \delta\varphi(x) = +ie\Lambda(x)\varphi(x)\,; \qquad \delta\varphi^\dagger(x) = -ie\Lambda(x)\varphi^\dagger(x)\,. \tag{2}$$

[1]We shall use Minkowski metric $diag(\eta_{\mu\nu}) = (+,-,-,-)$ and greek letters running 0,1,2,3.

[2]For further applications to fermionic fields, the Hermitian conjugation must be changed to Dirac conjugation. On the other hand, if the matter fields are rank-2 tensors, then additional attention must be paid to their indices. See Section 2 for details.

Now, in order to obtain a point-split version of the former Lagrangean, i.e., a form free of same point product of fields in interaction terms, we begin by writing the generalized version of the gauge transformations, ggt's (denoted by δ_g) up to e^2 [4]:

$$\delta_g A_\mu(x) = -\partial_\mu \Lambda(x) = \delta A_\mu(x), \tag{3}$$

$$\delta_g \varphi(x) = +ie\Lambda(1)\varphi(2) + \frac{1}{2}(ie)^2[\Lambda(1) + \Lambda(3)](1,3)\varphi(4) + \mathcal{O}(e^3)\,, \tag{4}$$

$$\delta_g \varphi^\dagger(x) = -ie\Lambda(-1)\varphi^\dagger(-2) + \frac{1}{2}(ie)^2[\Lambda(-1) + \Lambda(-3)](-1,-3)\varphi^\dagger(-4) + \mathcal{O}(e^3)\,, \tag{5}$$

where we have defined:

$$\Lambda(\pm n) = \Lambda(x \pm na); \qquad \varphi(\pm n) = \varphi(x \pm na); \qquad \varphi^\dagger(\pm n) = \varphi^\dagger(x \pm na); \tag{6}$$

$$(\pm m, \pm n) = \lim_{b \to 0^+} \int_{x \pm ma + b}^{x \pm na - b} A_\mu(\eta) d\eta^\mu\,. \tag{7}$$

From the last definition we see the first price to be paid in order to avoid the same point product of fields: the non-locality of the new model, which will be seen in more details later.

These ggt's can be shown to satisfy the *generalized Abelian condition* up to e^2, i.e., the commutator of two distinct ggt's (each of them with its respective parameter a_1 and a_2) vanishes up to such order:

$$[\delta_{g1}, \delta_{g2}]\varphi(x) = \mathcal{O}(e^3); \qquad [\delta_{g1}, \delta_{g2}]\varphi^\dagger(x) = \mathcal{O}(e^3).$$

It is noteworth that as the parameter a is set to zero, all the above results recover the usual ones (hereafter, by consistency, the same should happen to all poin-split results). Notice also that, the point-splitting acts only in transformations which present same point product, what is the case for $\delta\varphi$ and $\delta\varphi^\dagger$ but is not for δA_μ.[3]

Now, we discuss the invariance of the ordinary Lagrangean (1) under the above ggt's (more precisely, up to order e^2). The kinetic gauge term is clearly invariant since $\delta_g A_\mu =$

[3] In the Abelian case $\delta_g A_\mu = \delta A_\mu$ holds, but in the non-Abelian scenario, where the ordinary gauge transformation for A^a_μ involves same point product, the point-splitting will act on it, and its non-Abelian ggt will be different from the usual. Indeed, such ggt's were already worked out for $SU(2)$ case, see Ref. [5].

δA_μ. The mass term for matter fields can be shown to be invariant in its action form, $\int m\varphi^\dagger\varphi d^4x$, with suitable change of variables within the integration [4]. To the contrary, the other terms are not invariant and must have their points split. We choose to do the point-splitting (P.S) in the following way (as in (6), $A_\mu(\pm n)$ stands for $A_\mu(x\pm n)$):

$$\begin{aligned}(D_\mu\varphi(x))^\dagger(D^\mu\varphi(x)) \xrightarrow{P.S} \;& (D_\mu\varphi)^\dagger(D^\mu\varphi)_{P.S} = && (8)\\ &= \left[\partial_\mu\varphi^\dagger(x) - ieA_\mu(-1)\varphi^\dagger(-2)\right]\left[\partial^\mu\varphi(x) + ieA_\mu(1)\varphi(2)\right] ,\\ \left(\varphi^\dagger(x)\varphi(x)\right)^2 \xrightarrow{P.S} \;& \left(\varphi^\dagger\varphi\right)^2_{P.S} = \varphi^\dagger(-1)\varphi(1)\varphi^\dagger(-2)\varphi(2)\,. && (9)\end{aligned}$$

And the 'split Lagrangean' takes the form:

$$\mathcal{L}^{(0)}_{P.S} = -\frac{1}{4}F_{\mu\nu}(x)F^{\mu\nu}(x) - \frac{m}{2}\varphi^\dagger(x)\varphi(x) + (D_\mu\varphi)^\dagger(D^\mu\varphi)_{P.S} - \frac{\lambda}{4}(\varphi^\dagger\varphi)^2_{P.S}\,. \tag{10}$$

Here, it is worth to notice that while the kinetic matter term, $\partial_\mu\varphi^\dagger(x)\partial^\mu\varphi(x)$, involves same point product it does not need undergone splitting because the action of the ggt's on it will already produce regularised terms. Now, taking δ_g of such split terms up to order e we get:

$$\delta_g\left((D_\mu\varphi)^\dagger(D^\mu\varphi)_{P.S}\right) = (ie)\left[\Lambda(1)\partial_\mu\varphi^\dagger(x)\partial^\mu\varphi(2) - \Lambda(-1)\partial_\mu\varphi^\dagger(-2)\partial^\mu\varphi(x)\right] + \mathcal{O}(e^2)\,.$$

At a first glance, this term is non-null, but if we take its action form (inside $\int d^4x$) we can make a change of variables to show that a integral cancels another, exactly. Now, for the self-interaction term we get:

$$\begin{aligned}\delta_g\left((\varphi^\dagger\varphi)^2_{P.S}\right) = \;& ie\left[\Lambda(2)\varphi^\dagger(-1)\varphi(3) - \Lambda(-2)\varphi^\dagger(-3)\varphi(1)\right]\varphi^\dagger(-2)\varphi(2) && (11)\\ &+ie\varphi^\dagger(-1)\varphi(1)\left[\Lambda(3)\varphi^\dagger(-2)\varphi(4) - \Lambda(-3)\varphi^\dagger(-4)\varphi(2)\right] + \mathcal{O}(e^2)\,.\end{aligned}$$

To the contrary the former, this term seems to be intrinsically non-vanishing; in fact, we did not see any way to set it to zero (its action form, more precisely) either by suitable change of variables or partial integration. Therefore, we must seach for a new term, $\Omega^{(1)}_{P.S}$, such that $(\varphi^\dagger\varphi)^2_{P.S} + \Omega^{(1)}_{P.S}$ be invariant under δ_g at leat up to order e. This term exists and can be explicitly written as:

$$\Omega^{(1)}_{P.S} = -ie\left(\{-2,2\}\varphi^\dagger(-2)\varphi(2) + \{-3,3\}\varphi^\dagger(-1)\varphi(1)\right)\,, \tag{12}$$

with the definition:

$$\{-n,+n\} = \lim_{b\to 0^+}\int_{x-na+b}^{x+na-b} dy^\mu\partial_\mu\left[\varphi^\dagger(\frac{y}{n}+\frac{n-1}{n}x-na)\,\varphi(\frac{y}{n}+\frac{n-1}{n}x+na)\,(-\infty,y)\right],\quad(13)$$

where $(-\infty,y)$ stands for $\int_{-\infty}^{y} A^\nu(\eta)d\eta_\nu$.

Therefore the 'split Lagrangean', whose action is invariant under δ_g up to first order, $\mathcal{L}^{(1)}_{P.S}$, is the sum of $\mathcal{L}^{(0)}_{P.S}$ and $-\frac{\lambda}{4}\Omega^{(1)}_{P.S}$, equations (10) and (12).

Now, calculating $\delta_g\mathcal{L}^{(1)}_{P.S}$ at order e^2 we can get (after suitable change of variables in the action forms of the terms):

$$\delta_g\left((D_\mu\varphi)^\dagger(D^\mu\varphi)_{P.S}\right)\Big\}_{e^2} = (ie)^2\left[\Lambda(1)A_\mu(3)-\Lambda(3)A_\mu(1)\right]\varphi^\dagger(x)\overleftrightarrow{\partial^\mu}\varphi(4)\,,\qquad(14)$$

(with $U\overleftrightarrow{\partial}V = U\partial V-(\partial V)U$). Again, we cannot set this term to zero. Instead, according to $\Omega^{(1)}_{P.S}$ we must search for a new term, $\Sigma^{(2)}_{P.S}$, such that $(D_\mu\varphi)^\dagger(D^\mu\varphi)_{P.S}+\Sigma^{(2)}_{P.S}$ be invariant under δ_g at least up to order e^2. Such term can be found and its simplest form is:

$$\Sigma^{(2)}_{P.S} = -(ie)^2\,\Sigma_\mu\left[\varphi^\dagger(x)\overleftrightarrow{\partial^\mu}\varphi(4)\right],$$

with Σ_μ being a function of Λ and A_μ. In fact, Σ_μ must be an object such that $\delta_g\Sigma_\mu = \Lambda(1)A_\mu(3)-\Lambda(3)A_\mu(1)$. It is easy to check that the following expression satisfies such requiriment:

$$\Sigma^{(2)}_{P.S} = -(ie)^2\left\{[1]A_\mu(3)-[3]A_\mu(1)-\frac{1}{2}\left(\frac{[1]^2}{\Lambda(1)}\partial_\mu\Lambda(3)-\frac{[3]^2}{\Lambda(3)}\partial_\mu\Lambda(1)\right)\right\}\varphi^\dagger(x)\overleftrightarrow{\partial^\mu}\varphi(4)\,,\quad(15)$$

with $[\pm n] = \frac{1}{2}\left[(-\infty,\pm n)+(\infty,\pm n)\right]$. It is easy to check that as $a\to 0$, then $\Sigma^{(2)}_{P.S}$ vanishes. Moreover, it is worth to notice that quantities like $[\pm n]^2$ do not involve same point product because the space-times within which the integrals are performed (see (7) for a better understanding) are taken to be different ones. The same statement will be valid for other quantities which will appear hereafter.

Now, for the self-interaction sector we may get:

$$\delta_g\left((\varphi^\dagger\varphi)^2_{P.S}+\Omega^{(1)}_{P.S}\right)\Big\}_{e^2} = \frac{1}{2}(ie)^2\left\{2\{-2,2\}\left[\Lambda(-3)\varphi^\dagger(-4)\varphi(2)-\Lambda(3)\varphi^\dagger(-2)\varphi(4)\right]+\right.$$
$$+2\{-3,3\}\left[\Lambda(-2)\varphi^\dagger(-3)\varphi(1)-\Lambda(2)\varphi^\dagger(-1)\varphi(3)\right]+$$

$$
\begin{aligned}
&+[\Lambda(-2)+\Lambda(-4)]\,(-2,-4)\left[\varphi^{\dagger}(-5)\varphi(1)\varphi^{\dagger}(-2)\varphi(2)+\varphi^{\dagger}(x)\varphi(2)\varphi^{\dagger}(-5)\varphi(3)\right]+\\
&+[\Lambda(2)+\Lambda(4)]\,(2,4)\left[\varphi^{\dagger}(-1)\varphi(5)\varphi^{\dagger}(-2)\varphi(2)+\varphi^{\dagger}(-2)\varphi(x)\varphi^{\dagger}(-3)\varphi(5)\right]+\\
&-2\,\Lambda(4)\,(-\infty,2)\left[\varphi^{\dagger}(-1)\varphi(5)\varphi^{\dagger}(-2)\varphi(2)+\varphi^{\dagger}(-3)\varphi(5)\varphi^{\dagger}(-2)\varphi(x)\right]+\\
&-2\,\Lambda(-4)\,(-\infty,-2)\left[\varphi^{\dagger}(-5)\varphi(1)\varphi^{\dagger}(-2)\varphi(2)+\varphi^{\dagger}(-5)\varphi(3)\varphi^{\dagger}(x)\varphi(2)\right]+\\
&+2\,\left[\Lambda(2)\,(-\infty,-2)+\Lambda(-2)\,(-\infty,2)\right]\varphi^{\dagger}(-3)\varphi(3)\varphi^{\dagger}(-2)\varphi(2)+\\
&+2\,\left[\Lambda(3)\,(-\infty,-3)+\Lambda(-3)\,(-\infty,3)\right]\varphi^{\dagger}(-4)\varphi(4)\varphi^{\dagger}(-1)\varphi(1)\,\Big\}\;. \qquad (16)
\end{aligned}
$$

The non-vanishing of this term is evident. The searching for a new term, $\Omega^{(2)}_{P.S}$, such that $(\varphi^{\dagger}\varphi)^2_{P.S}+\Omega^{(1)}_{P.S}+\Omega^{(2)}_{P.S}$ be invariant under δ_g at least up to order e^2, is more difficult than for the former ones ($\Omega^{(1)}_{P.S}$ and $\Sigma^{(2)}_{P.S}$). The difficulty arises from its more complicated structure, but once more, a explicit expression may be found. For that, we notice that the six last terms have similar structure, say, $\Lambda(\pm n)\,(\pm m,\pm p)$ times $\varphi^{\dagger}\varphi\varphi^{\dagger}\varphi$ factors. In fact, for such terms, the simplest $\Omega^{(2)}_{P.S}$-type counter-terms have the general form:

$$
\frac{1}{2}(ie)^2\left(\frac{1}{2}\frac{\Lambda(\pm n)}{\Lambda(\pm p)-\Lambda(\pm m)}\,(\pm m,\pm p)^2\right).
$$

By remembering the definitions of the above quantities, it is easy to see that such form vanishes as $a\rightarrow 0$.

On the other hand, for the first two terms (proportional to $\{-n,+n\}$), the task of finding $\Omega^{(2)}_{P.S}$-type counter-terms become very easy if we take into account that:

$$
\delta_g\{-n,+n\}\Big\}_{e^0}=\Lambda(-n)\varphi^{\dagger}(-n-1)\varphi(n-1)-\Lambda(n)\varphi^{\dagger}(-n+1)\varphi(n+1)
$$

In fact, as can be easily checked, those first two terms have the following $\Omega^{(2)}_{P.S}$-type counter-term:[4]

$$
\frac{1}{2}(ie)^2\,(-2\,\{-2,+2\}\,\{-3,+3\})\,.
$$

Therefore, the full $\Omega^{(2)}_{P.S}$ term takes the form:

$$
\Omega^{(2)}_{P.S}=\;-\frac{1}{2}(ie)^2\left[\quad 2\,\{-2,+2\}\,\{-3,+3\}\quad+\right.
$$

[4]In fact, if φ (and $\varphi^{\dagger}$) are considered as fermionic fields, it is noteworth that while φ (or $\varphi^{\dagger}$) has anticommutative property, the bilinear $\varphi^{\dagger}\varphi$ has commutative behavior. Therefore, even for fermionic fields, we can change the order of $\{-2,+2\}$ by $\{-3,+3\}$ and vice-versa without any extra minus sign.

$$
\begin{aligned}
&+\left(\frac{\Lambda(2)+\Lambda(4)}{\Lambda(2)-\Lambda(4)}\right)\frac{(2,4)^2}{2}\left[\varphi^\dagger(-1)\varphi(5)\varphi^\dagger(-2)\varphi(2)+\varphi^\dagger(-2)\varphi(x)\varphi^\dagger(-3)\varphi(5)\right]+\\
&+\left(\frac{\Lambda(-2)+\Lambda(-4)}{\Lambda(-2)-\Lambda(-4)}\right)\frac{(-2,-4)^2}{2}\left[\varphi^\dagger(-5)\varphi(1)\varphi^\dagger(-2)\varphi(2)+\varphi^\dagger(x)\varphi(2)\varphi^\dagger(-5)\varphi(3)\right]+\\
&+\frac{\Lambda(4)}{\Lambda(2)}(-\infty,2)^2\left[\varphi^\dagger(-1)\varphi(5)\varphi^\dagger(-2)\varphi(2)+\varphi^\dagger(-3)\varphi(5)\varphi^\dagger(-2)\varphi(x)\right]+\\
&+\frac{\Lambda(-4)}{\Lambda(-2)}(-\infty,-2)^2\left[\varphi^\dagger(-5)\varphi(1)\varphi^\dagger(-2)\varphi(2)+\varphi^\dagger(-5)\varphi(3)\varphi^\dagger(x)\varphi(2)\right]+\\
&-\left(\frac{\Lambda(2)}{\Lambda(-2)}(-\infty,-2)^2+\frac{\Lambda(-2)}{\Lambda(2)}(-\infty,2)^2\right)\varphi^\dagger(-3)\varphi(3)\varphi^\dagger(-2)\varphi(2)+\\
&-\left(\frac{\Lambda(3)}{\Lambda(-3)}(-\infty,-3)^2+\frac{\Lambda(-3)}{\Lambda(3)}(-\infty,3)^2\right)\varphi^\dagger(-4)\varphi(4)\varphi^\dagger(-1)\varphi(1)\Bigg]\,. \qquad (17)
\end{aligned}
$$

Finally, the $\mathcal{L}^{(2)}_{P.S}$ Lagrangean, whose action is invariant under δ_g up to order e^2 may be written as (here we shall not write its explicit form due its length):

$$
\mathcal{L}^{(2)}_{P.S}=\mathcal{L}^{(0)}_{P.S}+\Sigma^{(2)}_{P.S}-\frac{\lambda}{4}(\Omega^{(1)}_{P.S}+\Omega^{(2)}_{P.S})\,, \qquad (18)
$$

with the expressions for the above terms being given by (10), (12), (15), and (17).

2 Applications to some models

Here, in order to illustrate our results, we shall deal with some $\lambda\varphi^4$-type models. When necessary, we shall pay attention to specific points which were not still presented.

i) **The Abelian Higgs model**

Due to the scalar character of its matter fields, this is the simplest model we may deal with. In fact, its Lagrangean may be directly obtained from (1) with $m=\mu^2<0$ (in order to realise the spontaneous symmetry breaking). Therefore, its $\mathcal{L}^{(2)}_{P.S}$ is identical that we obtained in previous Section, equation (18). No differences nor special care need be taken, except for the negativity of the mass parameter.

ii) **The $(\overline{\psi}\psi)^2$ model**

The model is described by the following Lagrangean:[5]

$$\mathcal{L}_{\psi}(x) = -\frac{1}{4}F^{\mu\nu}F_{\mu\nu} + \overline{\psi}\,(iD_{\mu}\gamma^{\mu} - m_f)\,\psi - g(\overline{\psi}\psi)^2\,, \tag{19}$$

with D_μ and $F_{\mu\nu}$ previously defined.

Now, the matter fields are fermionic ones, what brings us a novel feature (its anti-commutative nature) and lead us to take special care with their order. Also, the kinetic term is slightly different from that for scalar field and must be taken apart. However, such term was already studied in Ref.[4] and, if we perform the following splitting:

$$\begin{aligned}\overline{\psi}(x)\,iD_{\mu}\gamma^{\mu}\psi(x) \xrightarrow{P.S} \quad & (\overline{\psi}\,iD_{\mu}\gamma^{\mu}\psi)_{P.S} = \\ & \overline{\psi}(x)\,i\partial_{\mu}\gamma^{\mu}\psi(x) - e\overline{\psi}(x-a)A_{\mu}(x)\gamma^{\mu}\psi(x+a)\,,\end{aligned}$$

it is easy to show that $\int d^4x\,(\overline{\psi}\,iD_{\mu}\gamma^{\mu}\psi)_{P.S}$ is invariant up to order e. At second order, such variation does not vanishes, but it is exactly canceled by the following term (see equation (24) in [4]):

$$\Sigma^{(2)}_{\psi,P.S} = -\frac{ie^2}{2}\overline{\psi}(-2)\gamma^{\mu}\psi(2)\left\{[A_{\mu}(-1)+A_{\mu}(1)](-1,+1) + ([-1]+[1])\int_{x-a}^{x+a} d\eta^{\nu}F_{\mu\nu}(\eta)\right\}. \tag{20}$$

Now, the $(\overline{\psi}\psi)^2$-term is split in the same way of $(\varphi^{\dagger}\varphi)^2$:

$$(\overline{\psi}(x)\psi(x))^2 \xrightarrow{P.S} (\overline{\psi}\psi)^2_{P.S} = \overline{\psi}(-1)\psi(1)\overline{\psi}(-2)\psi(2).$$

So the $\mathcal{L}^{(0)}_{P.S}$ for ψ reads:

$$\mathcal{L}^{(0)}_{\psi,P.S} = -\frac{1}{4}F^{\mu\nu}(x)F_{\mu\nu}(x) - m_f\overline{\psi}(x)\psi(x) + i(\overline{\psi}D_{\mu}\gamma^{\mu}\psi)_{P.S} - g(\overline{\psi}\psi)^2_{P.S}\,. \tag{21}$$

To get $\mathcal{L}^{(2)}_{\psi,P.S}$ we may use the $\Omega^{(1)}_{P.S}$ and $\Omega^{(2)}_{P.S}$ obtained in the previous Section with suitable change of φ by ψ and $\varphi^{\dagger}$ by $\overline{\psi}$. Indeed, as we kept the original order of those matter fields in previous results, we may write:

$$\Omega^{(1)}_{\psi,P.S} = \Omega^{(1)}_{P.S}\Big\{_{\varphi\to\psi,\,\varphi^{\dagger}\to\overline{\psi}} \quad \text{and} \quad \Omega^{(2)}_{\psi,P.S} = \Omega^{(2)}_{P.S}\Big\{_{\varphi\to\psi,\,\varphi^{\dagger}\to\overline{\psi}}\,. \tag{22}$$

[5]It is worth to notice the non-renormalizable property of this self-interaction vertice: $[g] = [mass]^{-1}$.

Finally, we get:

$$\mathcal{L}^{(2)}_{\psi,P.S} = \mathcal{L}^{(0)}_{\psi,P.S} + \Sigma^{(2)}_{\psi,P.S} + g(\Omega^{(1)}_{\psi,P.S} + \Omega^{(2)}_{\psi,P.S})\,. \tag{23}$$

iii)**The Avdeev-Chizhov model**

Recently, Avdeev and Chizhov[6] proposed a renormalizable Abelian model which includes antisymmetric rank-2 real tensors that describe matter rather than gauge degrees of freedom. They are coupled to a usual vector gauge field, as well as interacting with fermions. The model has revealed several interesting properties, e.g., these new matter fields have played an important rôle in connection with extended electroweak models in order to explain some recent observable decays like $\pi^- \to e^- + \overline{\nu} + \gamma$ and $K^+ \to \pi^0 + e^+ + \nu$ [7]; on the other hand, a classical analysis of its dynamics has shown that some longitudinal excitations may carry physical degrees of freedom [8]. In addition, some works have been devoted to the study of its supersymmetric generalization [9], as well as its connection with non-linear sigma models [10].

Starting off from these interesting features, it was shown that the coupling between tensorial and fermionic fields generates anomalies in the quantum version of the model and could also spoil its renormalizability [11]. The removal of the fermions has the additional usefulness of allowing us to write the new Lagrangean in a shorter form by means of complex field tensors, $\varphi_{\mu\nu}$ and $\varphi^{\dagger}_{\mu\nu}$ [12]:[6]

$$\mathcal{L}_{AC}(x) = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + (D_\mu\varphi^{\mu\nu})(D^\alpha\varphi_{\alpha\nu})^\dagger - \frac{\lambda}{4}\varphi^\dagger_{\mu\nu}\varphi^{\nu\kappa}\varphi^\dagger_{\kappa\lambda}\varphi^{\lambda\mu}\,, \tag{24}$$

with D_μ and $F_{\mu\nu}$ already defined. Once $\varphi_{\mu\nu}$ is taken to satisfy a complex self-dual relation:

$$\varphi_{\mu\nu}(x) = +i\tilde{\varphi}_{\mu\nu}(x)\,, \qquad \tilde{\varphi}_{\mu\nu} = \frac{1}{2}\epsilon_{\mu\nu\alpha\beta}\varphi^{\alpha\beta},$$

then it can be split into two (real tensors) parts:

$$\varphi_{\mu\nu}(x) = T_{\mu\nu}(x) + i\tilde{T}_{\mu\nu}(x) \quad \text{and} \quad \varphi^\dagger_{\mu\nu}(x) = T_{\mu\nu}(x) - i\tilde{T}_{\mu\nu}(x),$$

[6]Notice that because $\varphi_{\mu\nu}$ (or $T_{\mu\nu}$ and $\tilde{T}_{\mu\nu}$) is massless it describes spin_0 particles.

where $T_{\mu\nu}$ and $\tilde{T}_{\mu\nu}$ are real and antisymmetric fields (the Avdeev-Chizhov's matter fiels).

Now, making similar splittings in $\mathcal{L}_{AC}(x)$ as were made in former cases, we get, after some calculation, $\mathcal{L}^{(2)}_{P.S}(\varphi^{\mu\nu})$:[7]

$$\mathcal{L}^{(2)}_{P.S}(\varphi^{\mu\nu}) = \mathcal{L}^{(0)}_{P.S}(\varphi^{\mu\nu}) + \Sigma^{(2)}_{P.S}(\varphi^{\mu\nu}) - \frac{\lambda}{4}\left(\Omega^{(1)}_{P.S}(\varphi^{\mu\nu}) + \Omega^{(2)}_{P.S}(\varphi^{\mu\nu})\right) \tag{25}$$

where the above terms have the following expressions:

$$\mathcal{L}^{(0)}_{P.S}(\varphi^{\mu\nu}) = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + (D_\mu\varphi^{\mu\nu})_{P.S}(D^\alpha\varphi_{\alpha\nu})^\dagger_{P.S} - \frac{\lambda}{4}(\varphi^\dagger_{\mu\nu}\varphi^{\nu\kappa}\varphi^\dagger_{\kappa\lambda}\varphi^{\lambda\mu})_{P.S} \tag{26}$$

with the splittings previously made;

$$\begin{aligned}\Sigma^{(2)}_{P.S}(\varphi^{\mu\nu}) = -(ie)^2 \ &\left[[1]A^\mu(3) - [3]]A^\mu(1) - \frac{1}{2}\left(\frac{[1]^2}{\Lambda(1)}\partial^\mu\Lambda(3) - \frac{[3]^2}{\Lambda(3)}\partial^\mu\Lambda(1)\right)\right]\cdot \\ &\cdot\left[\varphi^\dagger_{\mu\nu}(x)\partial_\alpha\varphi^{\alpha\nu}(4) - \varphi_{\mu\nu}(4)\partial_\alpha\varphi^{\dagger\alpha\nu}(x)\right] \ .\end{aligned} \tag{27}$$

Its slightly difference with respect to $\Sigma^{(2)}_{P.S}$, eq. (15), is due to the tensorial indices;

$$\Omega^{(1)}_{P.S}(\varphi^{\mu\nu}) = (-ie)\left(\{-2,+2\}^\mu_\nu \ \varphi^\dagger_{\mu\alpha}(-2)\varphi^{\alpha\nu}(2) + \{-3,+3\}^\mu_\nu \ \varphi^\dagger_{\mu\alpha}(-1)\varphi^{\alpha\nu}(1)\right) \ . \tag{28}$$

And, finally, $\Omega^{(2)}_{P.S}(\varphi^{\mu\nu})$ is easily obtained from $\Omega^{(2)}_{P.S}$ by making the interchanges:

$$\begin{aligned}&\{-2,+2\}\{-3,+3\} \longrightarrow \{-2,+2\}^\mu_\nu \ \{-3,+3\}^\nu_\mu \\ &\varphi^\dagger\varphi\varphi^\dagger\varphi \longrightarrow \varphi^\dagger_{\mu\nu}\varphi^{\nu\kappa}\varphi^\dagger_{\kappa\alpha}\varphi^{\alpha\mu} \ ,\end{aligned}$$

where we have defined $\{-n,+n\}^\nu_\mu$ in the same way as $\{-n,+n\}$ with $\varphi^\dagger\varphi$ changed by $\varphi^\dagger_{\mu\alpha}\varphi^{\alpha\nu}$ in its definition, eq. (13).

Concluding Remarks

Point-splitting procedure combined with generalized gauge transformations have yielded regularized Lagrangeans which includes $\lambda\varphi^4$-type interaction. The result is such that the generalized Lagrangeans have their interacting terms defined in different space-time

[7]We have already found a similar expression for this model in [13]. There, a slightly modified splitting was employed, as well as a lengther form to $\Sigma^{(2)}_{P.S}(\varphi^{\mu\nu})$.

points. Nevertheless, this property came together with the non-locality one.
In general, non-local theories cannot be quantised in the canonical ways and the interpretation of their results are not quite obvious. Moreover, we know that non-locality can lead to troubles concerning to the causality of the theory.

Nevertheless, Osland and Wu [2] obtained some standard results in QED starting by a split Lagrangean (with regularity and non-locality properties)[8]. Their method works by calculating the quantities with a dependence in the splitting parameter (that ensures the regularity) which is set to zero, at the end of calculations, in order to get the standard results.

What we may learn from these calculations is that when point-splitting is taken together generalized gauge transformations in order to obtain regularized Abelian Lagrangeans, the task becomes more difficult with the increasing of the number of matter fields within the same vertex; in general, those complications which arise from the presence of extras Abelian gauge fields are minor ones. So, the calculation involving $\lambda\varphi^4$-type vertices is harder to be performed than for 'lower vertices', as φ^3-like, $\varphi A_\mu \varphi$, $\varphi A_\mu A^\mu \varphi$, and so forth. In addition, higher order terms in the coupling constant are, in general, more complicated to be handled than for lower ones.

Another point that should be stressed is that this procedure is independent of the dimension of the space-time, and so, of the canonical dimension of the fields (matter or gauge ones)[9]. Hence, the expressions for our Σ and Ω terms remain valid in other dimensions. On the other hand, if we are dealing, for example, with a renormalizable theory (scalar, for simplicity) in $(2+1)$ dimensions, an extra $f\varphi^6$-term is allowed. In this case, our results could be applied to the model, including the $\lambda\varphi^4$-term, but the extra term

[8] It is worth to notice that their Lagrangean (eq. (2.7) in Ref.[2]) is different from the '*correct*' generalized QED-Lagrangean, up to fourth order (eq. (24) in Ref.[4]). Such difference may be explained by noticing that in [2], the ganeralized gauge covariance is not taken within its precise meaning.

[9] In fact, the ggt's depend on the splitting parameter and on the Abelian (or non-Abelian) character of the gauge fields.

should be worked out apart.

We also hope that our work in dealing with the Abelian case shall help us when we shall treat the non-Abelian one. On the other hand, it is clear that, in the non-Abelian scenario novel features will arise, mainly because the $\delta_g A_\mu^a$ will take more complicated (and lengther) forms, and they will imply in new ggt's for the matter fields which, unfortunatelly, will also take lengther expressions than those for the Abelian case.

Finally, we claim that some questions concerning this issue should be clearer. For example, how could Feynman rules for such kind of Lagrangean be formulated?
Or still, as we may see, there are some new ' interaction terms' within the generalized Lagrangean. Could these new terms have some physical interpretation and/or relevance? These subjects should be the goal of some forthcoming works, once they need (and deserve) be better studied.

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