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Vertex Operator Representation of the Soliton Tau Functions in the $A_n^{(1)}$ Toda Models by Dressing Transformations

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Abstract

Affine Toda theories based on simple Lie algebras $\mathcal{G}$ are known to possess soliton solutions. Toda solitons has been found by Olive, Turok and Underwood within the group–theoretical approach to the integrable field equations. Single solitons are created by exponentials of special elements of the underlying affine Lie algebra which diagonalize the adjoint action of the principal Heisenberg subalgebra. When $\mathcal{G}$ is simply laced and level one representations are considered, the generators of the affine Lie algebra are expressed in terms of the principal Heisenberg oscillators. This representation is known as vertex operator construction. It plays a crucial role in the string theory as wel as in the conformal field theory. Alternatively, solitons can be generated from the vacuum by dressing transformations. The problem to relate dressing symmetry to the vertex operator representation of the tau functions for the sine–Gordon model was previously considered by Babelon and Bernard. In the present paper, we extend this relation for arbitrary $A_n^{(1)}$ Toda field theory.

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1 Introduction

Exact localizable solutions of a relativistic field theory with finite energy are known in the literature as (depending on the dimension of the space–time) solitons (kinks) or monopoles [1]. Such solutions play an important role, since they are interpreted as new particles, which appear in the spectrum of the theory. As argued in [1, 2], the particles corresponding to solitons (or monopoles) are of nonperturbative nature. Usually, the solitons (monopoles) are also characterized by nontrivial topological charges. This provides a deep connection between field theory and geometry [3]. There exists [4] a remarkable duality which exchanges the sine–Gordon model with the massive Thirring model. Under this duality transformation perturbative particles are mapped into solitons and vice versa. The duality also exchanges Noether currents with topological ones. Recently [5], Seiberg and Witten has shown that there exist supersymmetric field theories in four dimensions which exhibit exact duality.

Within the Inverse Scattering Method (ISM) [6], soliton equations are required to admit Lax or zero curvature representation. This representation guarantees that the spectrum of the Lax operator is constant in time. The leading idea of the ISM is to consider the time evolution as an evolution of the scattering data of the corresponding Lax operator. Due to the zero curvature condition, the time evolution of the associated scattering data can be found explicitly. In the framework of the ISM, solitons correspond to vanishing reflection coefficients. Imposing this condition, the inverse spectral transformation reduces to a linear algebraic system. The integrability (in the sense of Liouville) also ensures that the interaction between the single is elastic. This intriguing property survives the quatization and was used to get exact quantum S matrices in various integrable models [7].

The propagation of waves in one–dimensional lattices with exponential nearest–neighbour interaction was studied by Toda [8]. In this pioneering paper, exact soliton solutions were also studied. The Toda lattice equations admit a field theoretical analogue in $1+1$ dimensions [9] which exhibit both the integrability and the relation to the Lie algebras. Due to the last sequence of papers, it became clear that (generalized) Cartan matrices can be used to construct integrable exponential interactions in two dimensions. It was also shown that the field equations admit zero curvature representation of a Lax connection whose components belong to a certain Lie algebra. A deep relation between integrable hierarchies and the Kac–Mood (or affine) Lie algebras [10] has been clarified by Drinfeld and Sokolov [11]. In the last paper it was also explained the *crucial* role of Heisenberg subalgebras and the related to them gradations in constructing integrable evolution equations [12, 13].

Soliton solutions of the sine-Gordon model are known since the early days of the ISM [6]. N soliton solutions in the $A_n^{(1)}$ Toda models are found by the use of the Hirota method in [14]. It also became clear that physical observables, when evalueted on solitons, take finite quantities for imaginary values of the coupling constant only. These results has been extended by using group theoretical methods [15, 16], which also provide a bridge between the integrable models and the Kac–Moody algebras. To be more precise, the soliton tau functions corresponding to a fundamental highest weight state $|\Lambda>$ of the affine Lie algebra admit the representation

$$\frac{\tau_\Lambda(\Phi)}{\tau_\Lambda(\Phi_0)} = <\Lambda|\prod_{i=1}^{N}(1 + X_i F^{r_i}(\mu_i))|\Lambda> \tag{1.1}$$

where Φ_0 is the vacuum solution, X_i are numerical factors depending exponentially on the light cone coordinates and F^{r_i} are elements of the affine Lie algebra which diagonalize the adjoint action of the principal Heisenberg subalgebra (for details see [10, 15]) and N is the number of the solitons. The above expression is analogous to the representation obtained by the Kyoto group [17] for the tau function of the KP hierarchy. Integrable models possess a dressing symmetry [18]. The dressing group acts by gauge transformations on the Lax connection and preserves its form. An important property of the dressing symmetry is that it is a Lie–Poisson group. Applications of this symmetry to the Toda field theories has been done in [19]. Using the dressing symmetry, an alternative expression for the soliton tau functions arises

$$\frac{\tau_\Lambda(\Phi)}{\tau_\Lambda(\Phi_0)} = <\Lambda|g_-^{-1}(x^+, x^-).g_+(x^+, x^-)|\Lambda> \tag{1.2}$$

where g_-^{-1} and g_+ are triangular elements of the affine group which generate solitons from the vacuum. Babelon and Bernard demonstrated that the expressions (1.1) and (1.2) are equivalent, at least for affine sine–Gordon solitons. In a previous paper [21] we obtained explicit expressions for the elements $g_\pm$ (1.2). In the present work, by using the results of [21], we show that the vertex operator representation of the tau functions (1.1) corresponding to fundamental representations of the affine Lie algebra is a consequence of the dressing group expression (1.2) for arbitrary $A_n^{(1)}$ Toda solitons.

The paper is organized as follows: In Sec. 2 we introduce the $A_n^{(1)}$ Toda models, the soliton solutions and briefly comment the results of [21]. Sec. 3 is also complementary: the vertex operator representation of the $A_n^{(1)}$ algebras in the principal gradation is derived. In Sec. 4 we obtain the vertex operator representation of the monosoliton tau functions starting from the dressing symmetry. Sec.5 generalizes this result to generic soliton solutions.

2 $A_n^{(1)}$ Toda Solitons and the Related Dressing Transformations

Solitons in the $A_n^{(1)}$ Toda models has been found by Hollowood [14] who used the Hirota equations. Physically relevant solutions appear only for imaginary values of the coupling constant, nevertheless the components of the Toda field, are in general complex. Further, affine Toda solitons were studied from the point of view of the ISM [22]. Due to the last paper, it became clear that the Jost solutions [6] and the elements of the transition matrix loose, in general, their nice analyticity properties. Similar phenomenon occurs when one considers the scattering data related to a linear differential operator of an arbitrary order [23] . In [21] we developed an elegant method to get $A_n^{(1)}$ Toda solitons. It strongly exploits the fact that the Lax connection belongs to the $A_n^{(1)}$ Lie algebra in the *principal* gradation (for a summary of the Lie algebraic background, see the Appendix). The dynamical variables which we use to describe the soliton dynamics, appeared previously in the study of the periodic solution of the KdV equation and of the periodic Toda chain [24]. Similar representation for the sine–Gordon solitons was recently used to compute form factors in the quantum theory [25].

To get the $A_n^{(1)}$ Toda equations, we impose the zero curvature condition

$$\partial_+ A_- - \partial_- A_+ + [A_+, A_-] = 0 \tag{2.1a}$$

on the Lax connection

$$A_\pm \;=\; \pm\partial_\pm \Phi + m e^{\pm ad\Phi}\mathcal{E}_\pm \tag{2.1b}$$

$$0 \tag{2.1c}$$

$$\partial_\pm \;=\; \frac{\partial}{\partial x^\pm} \tag{2.1d}$$

which belongs to the loop Lie algebra $\widetilde{\mathcal{G}} = \widetilde{SL}(n+1)$; $x^\pm = x \pm t$ are the light cone coordinates in the two dimensional Minkowski space. The elements $\mathcal{E}_\pm = \mathcal{E}_{\pm 1}$ are generators of grade ± 1 of the principal Heisenberg subalgebra (cf.(A.7)).

$$\mathcal{E}_k = \lambda^k \sum_{p\in\mathbb{Z}_{n+1}} |p><p+k| \tag{2.1e}$$

and Φ is a diagonal matrix

$$\Phi = \frac{1}{2}\sum_{i\in\mathbb{Z}_{n+1}} \varphi_i E^{ii} \tag{2.1f}$$

The $A_n^{(1)}$ Toda models has a conformally invariant extension [26] known as the A_n Conformal Afine Toda (CAT) model. It also admits a zero curvature representation (2.1a) for a connection of the form (2.1d) in the *affine* Lie algebra $\hat{\mathcal{G}} = \hat{sl}(n+1) = A_n^{(1)}$ [26]. The affine algebra analogue of (2.1f) is

$$\Phi \to \Phi + \eta\hat{d} + \frac{\hat{c}}{2(n+1)}\zeta \tag{2.2}$$

where (see the Appendix) $\hat{d}$ and $\hat{c}$ are the derivation and the central element respectively. The A_n CAT equations of motion are

$$\begin{aligned}
&\partial_+\partial_-\varphi_i = m^2 e^{2\eta}(e^{\varphi_i-\varphi_{i+1}} - e^{\varphi_{i-1}-\varphi_i})\\
&\partial_+\partial_-\eta = 0\\
&\partial_+\partial_-\zeta = m^2 e^{2\eta}\sum_{k\in\mathbb{Z}_{n+1}} e^{\varphi_i-\varphi_{i+1}}
\end{aligned} \tag{2.3}$$

The first of the above equations coincides with the $A_n^{(1)}$ Toda equations provided that $\eta = 0$. As noted in [20], affine Toda solitons arise after imposing the last restriction. In this case [14], (2.3) admit a Hirota bilinear representation

$$\partial_+\tau_k\partial_-\tau_k - \tau_k\partial_+\partial_-\tau_k = m^2(\tau_{k+1}\tau_{k-1} - \tau_k^2) \tag{2.4a}$$

$$e^{-\varphi_k} = \frac{\tau_k}{\tau_{k-1}} \quad , k \in \mathbb{Z}_{n+1} \tag{2.4b}$$

$$e^{\zeta_0 - \zeta} = \prod_{k\in\mathbb{Z}_{n+1}} \tau_k \tag{2.4c}$$

where

$$\zeta_0 = (n+1)m^2x^+x^- \tag{2.4d}$$

The above value of the field ζ together with $\varphi_k = 0, k \in \mathbb{Z}_{n+1}$ corresponds to the *vacuum* solution of the CAT model. The corresponding $A_n^{(1)}$ Toda vacuum is obtained by ignoring the additional field ζ. In [21] we used an alternative procedure to get $A_n^{(1)}$ Toda solitons. The approach used by us strongly relies on the work [27] where several soliton equations, including the sine–Gordon equation, has been studied. The dynamics of the $A_n^{(1)}$ Toda solitons is governed by the following *algebraic* equations

$$\prod_{l=1}^{N} \frac{\epsilon_{kl} + \omega^{r_j}\mu_j}{\epsilon_{kl} + \mu_j} = c_j\omega^{r_j(1-k)}\frac{e(\omega^{r_j}\mu_j)}{e(\mu_j)}$$

$$e(\lambda) \;=\; exp\{m(\lambda x^+ + \frac{x^-}{\lambda})\} \tag{2.5a}$$

$$e^{-\varphi_k} = (-)^N \prod_{j=1}^{N} \frac{\epsilon_{kj}}{\mu_j}, \qquad k \in \mathbb{Z}_{n+1} \tag{2.5b}$$

The integer N stands for the number of solitons; $\mu_j, j = 1, \ldots, N$ are (complex) parameters related to the soliton rapidities and r_j are discrete parameters which take nonvanishing values in the cyclic group $\mathbb{Z}_{n+1}$. The last are also known in the literature [14, 15, 16] as *soliton species.* The relation between solitons and N–body integrable relativistic systems was studied in details [28, 29]. In order to simplify our analysis, only solitons with $|\mu_i| \neq |\mu_j|$ $(i \neq j)$ will be considered. Certain particular solutions which violate this restriction are also known [30]. Let us also note that in contrast to [14, 15, 16], we are working with a real value of the coupling constant. This wants to say that the solutions (2.5a), (2.5b) are solitons in *algebraic* sence.

Transformation groups for soliton equations were introduced in [17]. They has been also studied in relation to the underlying Riemann problem [31] and are known as groups of dressing transformations. In the present paper we will not comment the Poisson–Lie properties [18, 19] of the dressing group. For dressing group elements which create monosoliton solutions from the vacuum in the sine–Gordon model this problem has been discussed in [32]. One believes that [20] in the limit $N \to \infty$, the N–soliton solutions are dense in the dressing group orbit of the vacuum, and therefore, the soliton parameters are expected to provide a convenient coordinate system on the dressing group.

To introduce the group of dressing transformations we first recall that the zero curvature condition (2.1a) can be equivalently written as the compatibility condition of the linear system

$$(\partial_\pm + A_\pm)T = 0 \tag{2.6}$$

We shall also impose the normalization condition $T(0) = 1$. An element of the dressing group is represented by a pair of triangular elements $(g_+(x), g_-(x))$ of the corresponding loop (or affine) Lie group. This want to say that $g_\pm(x) = e^{\mathcal{H}} e^{\mathcal{N}_\pm}$ where $\mathcal{H}$ is the Cartan subalgebra and $\mathcal{N}_\pm$ contain all the elements of positive (negative) grade. The gradation is introduced by the derivation $\hat{d}$ of the loop (affine) Lie algebra (A.18), (A.19). The dressing group acts on the components of the Lax connection by gauge transformations

$$A_\mu \to A_\mu^g = -\partial_\mu g_\pm g_\pm^{-1} + g_\pm A_\mu g_\pm^{-1} \tag{2.7a}$$

or equivalently

$$T(x) \to T(x)^g = g_\pm(x) T(x) g_\pm^{-1}(0) \tag{2.7b}$$

The last factor in the above expression is added to ensure the normalization condition $T^g(0) = 1$. The gauge transformations are also required to preserve the form of the Lax connection (2.1d). Therefore, the dressing transformations form a symmetry group of the corresponding field equations (2.3). Since the group elements g_+ and g_- produce the same result (2.7a), it turns out that they are solution of the factorization problem.

$$g_-^{-1}(x) g_+(x) = T(x) g_-^{-1}(0) g_+(0) T^{-1}(x) \tag{2.8}$$

Comparing (2.1d) with (2.7a), one observs that g_+ and g_- have oposite components on the Cartan torus $e^{\mathcal{H}}$. Therefore, the solution of (2.8) is unique. The multiplication in the dressing group is the same as in the dual group. Therefore the map $(g_+, g_-) \to g_-^{-1} g_+$ is not a group isomorphism of Lie groups.

In the CAT models based on an arbitrary simple Lie algebras, one associates a tau function to each highest weight vector $|\Lambda>$

$$\tau_\Lambda(\Phi) = <\Lambda| e^{-2\Phi} |\Lambda> \tag{2.9}$$

Suppose that a solution Φ (2.2), (2.3) with $\eta = 0$ is related to the vacuum $\phi_0 = \frac{m^2}{2} x^+ x^-$ by dressing transformation (g_+, g_-). The following relation [19, 20]

$$\frac{\tau_\Lambda(\Phi)}{\tau_\Lambda(\phi_0)} = <\Lambda| g_-^{-1}(x) g_+(x) |\Lambda> \tag{2.10}$$

between the tau functions of the two solutions is valid. In [20] it was observed that, after factorizing out the contribution which belongs to the center of the affine group, one can perform the calculation of the element $g_\pm$ in the corresponding loop group:

$$g_\pm(x) = e^{\pm \frac{\zeta_0 - \zeta}{2(n+1)} \hat{c}} \widetilde{g}_\pm(x) \tag{2.11}$$

where $\widetilde{g}_\pm$, considered as elements of the loop group, generate the $A_n^{(1)}$ Toda solution $\Phi = \frac{1}{2} \sum_{i \in \mathbb{Z}_{n+1}} \varphi_i |i><i|$ from the vacuum $\Phi = 0$. We recall that the relation (2.11) has been established in [20] for the affine Lie algebra $A_1^{(1)} = \hat{sl}(2)$. However, it is easy to check that the proof can be easily generalized for an arbitrary affine Lie algebra.

The representation (2.5a), (2.5b) of the N–soliton solutions was used [21] to calculate the elements $\widetilde{g}_\pm$ which by (2.7a) generate these solutions from the vacuum.

It turns out that the dressing group elements $\widetilde{g}_\pm$ can be factorized into a product of monossoliton factors

$$\widetilde{g}_\pm = \widetilde{g}_\pm(N)\widetilde{g}_\pm(N-1)\ldots\widetilde{g}_\pm(1) \tag{2.12}$$

where

$$\widetilde{g}_\pm(i) = e^{K(F_i)+P_i}e^{W_\pm(i)} \tag{2.13a}$$

In the above expression, $F_i, K(F_i)$ and P_i are diagonal traceless matrices

$$\begin{aligned} F_i &= \frac{1}{2}\sum_{k\in\mathbb{Z}_{n+1}} f_{ki}|k><k| \qquad P_i = \frac{1}{2}\sum_{k\in\mathbb{Z}_{n+1}} p_{ki}|k><k| \\ K(F_i) &= \sum_{k\in\mathbb{Z}_{n+1}} K_k(F_i)|k><k| \\ \sum_{k\in\mathbb{Z}_{n+1}} f_{ki} &= \sum_{k\in\mathbb{Z}_{n+1}} p_{ki} = \sum_{k\in\mathbb{Z}_{n+1}} K_k(F_i) = 0 \end{aligned} \tag{2.13b}$$

The entries of matrix $K_k(F_i)$ obey the recursion relations

$$K_k(F_i) - K_{k+1}(F_i) = \frac{f_{ki}+f_{ki+1}}{2} \tag{2.13c}$$

which agree with the periodicity property $K_k(F_i) = K_{k+n+1}(F_i)$ since F_i is traceless. The loop algebra elements $W_\pm(i)$ are given by:

$$\begin{aligned} W_\pm(i) &= -K(F_i) + \sum_{k=1}^{n} f_{ki}\mathcal{S}^k_\pm(\mu_i) \\ \mathcal{S}^k_\pm(\mu_i) &= \mathcal{B}^k_\pm(\mu_i) - \mathcal{B}^{n+1}_\pm(\mu_i) \\ \mathcal{B}^k_\pm(\mu_i) &= \mp\left(\frac{1}{2} + \frac{(\frac{\lambda}{\mu_i})^{\pm(n+1)}}{1-(\frac{\lambda}{\mu_i})^{\pm(n+1)}}\right)E^{kk} \mp \sum_{l\gtrless k} \frac{(\frac{\lambda}{\mu_i})^{l-k\pm(n+1)}}{1-(\frac{\lambda}{\mu_i})^{\pm(n+1)}}E^{kl} \\ &\mp \sum_{l\gtrless k} \frac{(\frac{\lambda}{\mu_i})^{l-k}}{1-(\frac{\lambda}{\mu_i})^{\pm(n+1)}}E^{kl} \end{aligned} \tag{2.13d}$$

In the above expressions λ stands for the loop (or spectral) parameter (see the Appendix). We also note that the elements $\mathcal{B}^k_\pm(\mu_i)$, in contrast to $\mathcal{S}^k_\pm(\mu_i)$, are not in the loop algebra $\widetilde{sl}(n+1)$. Due to the diagonal (λ–depending) contributions with nonvanishing trace, $\mathcal{B}^k_\pm(\mu_i)$ are in the loop algebra $\widetilde{gl}(n+1)$. Taking into account the loop algebra analogue of (A.13) together (A.23) one gets

$$\mathcal{S}^k_\pm(\mu_i) \;\;=\;\; \mp\sum_{r\in\mathbb{Z}_{n+1}} \omega^r(\omega^{-rk}-1)\left(\frac{F^r_0}{2} + \sum_{p\gtrless 0}\frac{F^r_p}{\mu_i^p}\right) \tag{2.14}$$

The elements of the Cartan subalgebra $K(F_i)$ (2.13b), (2.13c) can also be written in alternative basis (A.13). Therefore, from (2.13d) and the above expansions, we obtain

$$W_\pm(i) = \sum_{k\in\mathbb{Z}_{n+1}} f_{ki}W^k_\pm(\mu_i) \tag{2.15a}$$

where

$$W_+^k(\mu_i) = -\sum_{r=1}^{n}\left(\frac{1-\omega^{-rk}}{1-\omega^{-r}}F_0^r + \omega^{r(1-k)}\sum_{p\geqslant 0}\frac{F_p^r}{\mu_i^p}\right) \tag{2.15b}$$

$$W_-^k(\mu_i) = \sum_{r=1}^{n}\left(-\frac{1-\omega^{r(1-k)}}{1-\omega^{-r}}F_0^r + \omega^{r(1-k)}\sum_{p\leqslant -1}\frac{F_p^r}{\mu_i^p}\right) \tag{2.15c}$$

Note that neither $W_+^k(\mu_i)$ nor $W_-^k(\mu_i)$ contain contributions belonging to the principal Heisenberg subalgebra. From the above expressions one also derives the identities

$$W_+^k(\mu_i) - W_-^k(\mu_i) = -\sum_{r=1}^{n}\omega^{r(1-k)}\sum_{p\in\mathbb{Z}}\frac{F_p^r}{\mu_i^p} \tag{2.16}$$

The above expression together with (2.15a)–(2.15c) provides a hint of how to relate the dressing group approach with the group theoretical methods, developed in [15, 16]. This relation has been conjectured for general integrable hierarchies which admit a vacuum solution [13].

Up to now we discussed the properties of the factors $\widetilde{g}_\pm(k)$ (2.12) which produce single solitons without specifying the relation to the N–soliton solution (2.5a), (2.5b). This relation has been discussed in details in [21]. Here we only list the results. First of all, it has been shown that the diagonal matrices F_l, $l = 1,\ldots,N$ (2.13b) satisfy the identities

$$\begin{aligned}
&\sum_{k\in\mathbb{Z}_{n+1}}\omega^{r(1-k)}\left(\frac{\rho_{kl}(\omega^r\mu_{l+1})}{\rho_{kl}(\mu_{l+1})} - \omega^{-r}\frac{\rho_{k+1l}(\omega^r\mu_{l+1})}{\rho_{k+1l}(\mu_{l+1})}\right)\beta_k(F_{l+1}) = \delta_{r,r_{l+1}}(1-\omega^r)\times \\
&\times\prod_{a\neq l+1}\frac{\omega^r\mu_{l+1}-\mu_a}{\mu_{l+1}-\mu_a}\prod_{a=1}^{N}\frac{\mu_{l+1}+\epsilon_{1a}}{\omega^r\mu_{l+1}+\epsilon_{1a}}\times \\
&\times\sum_{k\in\mathbb{Z}_{n+1}}\left(1+\mu_{l+1}\frac{d}{d\lambda}\ln\frac{\rho_{kl}}{\rho_{k+1l}}(\mu_{l+1})\right)\beta_k(F_{l+1})
\end{aligned} \tag{2.17a}$$

In the above system r is discrete parameter which belongs to $\mathbb{Z}_{n+1}$; $r_i = 1,\ldots,n$ stands for the soliton species (2.5a), (2.5b) related to the soliton with the rapiditylike parameter μ_i for $i = 1,\ldots N$. It was noted in [21] that (2.17a) is a system of algebraic equations which determines the variables

$$\beta_k(F_l) = e^{K_k(F_l)-\frac{f_{kl}}{2}} \tag{2.17b}$$

The quantities $K_k(F_l)$ and f_{kl} which appear in the above definition are introduced by (2.13a) and (2.13c). The functions on λ $\rho_{al}(\lambda)$ (2.17a), $a\in\mathbb{Z}_{n+1}$, $l = 1,\ldots,N$ can be considered as components of $n+1$–dimensional column vectors $\rho_l(\lambda)$. The following relations are valid

$$\begin{aligned}
\rho_l(\lambda) &= \mathcal{D}^{(l\,l-1)}(\lambda)\rho_{l-1}(\lambda) \\
\rho_{l-1}(\lambda) &= \mathcal{D}^{(l-1\,l)}(\lambda)\rho_l(\lambda)
\end{aligned} \tag{2.17c}$$

where $\mathcal{D}^{(l\,l-1)}(\lambda)$ and $\mathcal{D}^{(l-1\,l)}(\lambda)$ are $(n+1)\times(n+1)$ matrices ($\mathcal{D}^{(l\,l-1)}(\lambda)\mathcal{D}^{(l-1\,l)}(\lambda) = 1$). Their matrix elements are given by

$$\mathcal{D}^{(l\,l-1)}_{ab}(\lambda) = \frac{\gamma_l\beta_b(F_l)}{(n+1)\rho_{bl-1}(\mu_l)}\sum_{q\in\mathbb{Z}_{n+1}}\omega^{q(a-b)}\frac{\lambda-\omega^q e^{f_{bl}}\mu_l}{\lambda-\omega^q\mu_l}$$
$$\mathcal{D}^{(l-1\,l)}_{ab}(\lambda) = \frac{\rho_{al-1}(\mu_l)}{(n+1)\gamma_l\beta_b(F_l)}\sum_{q\in\mathbb{Z}_{n+1}}\omega^{q(a-b)}\frac{\lambda-\omega^q e^{-f_{bl}}\mu_l}{\lambda-\omega^q\mu_l}$$
$$\gamma_l = \left(\prod_{p\in\mathbb{Z}_{n+1}}\rho_{pl-1}(\mu_l))\right)^{\frac{1}{n+1}} \tag{2.17d}$$

The first relation (2.17c) together with

$$\rho_{j0}(\lambda) = \frac{1}{n+1} \tag{2.17e}$$

determines uniquely the vectors $\rho_l(\lambda)$. The entries of the diagonal matrices P_l are given by [21]

$$e^{K_a(F_{l-1})+\frac{p_{al-1}}{2}} = \frac{\rho_{al-1}(\mu_l)}{\gamma_l} \tag{2.17f}$$

Therefore, the equations (2.17a)–(2.17f) provide a recursive method to obtain the factorized expression (2.12) for a dressing group element in the loop group which generate the N–soliton solution (2.5a), (2.5b) from the vacuum. In view of the close relation (2.11) between the dressing group elements in the affine group and in the loop group, the factorization property (2.12) together the recurrence relations (2.17a)– (2.17f), will play a crucial role in obtaining the vertex operator representation for the soliton tau function. Note also that in view of (2.13c), the quantities $\beta_k(F_i)$ (2.17b) satisfy the relation

$$e^{-f_{kl}} = \frac{\beta_k(F_l)}{\beta_{k-1}(F_i)} \tag{2.18}$$

which resembles (2.4b). However β_k, as it can be seen from (2.17a) and (2.17c), (2.17d), do not satisfy the Hirota bilinear equations (2.4a) in general. This wants to say that the fields F_l (2.13b) *do not* satisfy the $A^{(1)}_n$ Toda equations for arbitrary value of l.

The components $\rho_{al}(\lambda)$ of the vectors $\rho_l(\lambda)$ (2.17a), (2.17c), (2.17d) satisfy algebraic and differential equations which will be important in what follows. First of all, combining (2.17c) with (2.17d), we get

$$\sum_{a,r\in\mathbb{Z}_{n+1}}\frac{\omega^{r(a'-a)}}{\lambda-\omega^r\mu_l}\left(\frac{\lambda-\omega^r e^{-f_{al}}\mu_l}{\lambda-\omega^r\mu_l}\frac{\rho_{al}(\lambda)}{\gamma_l\beta_a(F_l)} - \frac{\rho_{al-1}(\lambda)}{\rho_{al-1}(\mu_l)}\right) = 0 \tag{2.19}$$

for any $a'\in\mathbb{Z}_{n+1}$. From the above identity and the second equation (2.17d) we deduce the relations

$$\sum_{a\in\mathbb{Z}_{n+1}}\frac{d\,\mathcal{D}^{(l-1\,l)}_{a'a}}{d\lambda}(\lambda)\rho_{al}(\lambda) = \frac{\rho_{a'l-1}(\mu_l)}{n+1}\sum_{a,r\in\mathbb{Z}_{n+1}}\frac{\omega^{r(a'-a)}}{\lambda-\omega^r\mu_l}\left(\frac{\rho_{al}(\lambda)}{\gamma_l\beta_a(F_l)} - \frac{\rho_{al-1}(\lambda)}{\rho_{al-1}(\mu_l)}\right) \tag{2.20}$$

In view of (2.17c), the following representation

$$\rho_j(\lambda) = \mathcal{D}^{(j\,j-1)} \ldots \mathcal{D}^{(1\,0)} \rho_0(\lambda) \tag{2.21a}$$

takes place. Differentiating the above expression with respect to λ and taking into account that $\rho_0(\lambda)$ (2.17e) does not depend on the spectral parameter, one obtains

$$\frac{d\,\rho_j}{d\lambda}(\lambda) = \sum_{l=1}^{j} \mathcal{D}^{(j\,l)}(\lambda) \frac{d\,\mathcal{D}^{(l\,l-1)}}{d\lambda}(\lambda)\rho_{l-1}(\lambda) \tag{2.21b}$$

where the matrices $\mathcal{D}^{(j\,l)}(\lambda)$ for $j \geq l$ are given by

$$\begin{aligned} \mathcal{D}^{(j\,l)}(\lambda) &= \mathcal{D}^{(j\,j-1)}(\lambda) \ldots \mathcal{D}^{(l+1\,l)}(\lambda) \\ \rho_j(\lambda) &= \mathcal{D}^{(j\,j-1)}(\lambda)\rho_l(\lambda) \\ \mathcal{D}^{(j\,j)}(\lambda) &= 1 \end{aligned} \tag{2.21c}$$

Since $\mathcal{D}^{(l-1\,l)}$ is the inverse of $\mathcal{D}^{(l\,l-1)}$, we can rewrite (2.21b) as follows

$$\frac{d\,\rho_j}{d\lambda}(\lambda) = -\sum_{l=1}^{j} \mathcal{D}^{(j\,l-1)}(\lambda) \frac{d\,\mathcal{D}^{(l-1\,l)}}{d\lambda}(\lambda)\rho_l(\lambda) \tag{2.21d}$$

Inserting (2.20) into the above identity we obtain

$$\frac{d\,\rho_{kj}}{d\lambda}(\lambda) = -\frac{1}{n+1} \sum_{l=1}^{j} \sum_{a,a',r \in \mathbb{Z}_{n+1}} \frac{\omega^{r(a'-a)}}{\lambda - \omega^r \mu_l} \mathcal{D}^{(j\,l-1)}_{ka'}(\lambda) \left(\frac{\rho_{al}(\lambda)}{\gamma_l \beta_a(F_l)} - \frac{\rho_{al-1}(\lambda)}{\rho_{al-1}(\mu_l)} \right) \tag{2.22}$$

This result will be used in Sec. 5 to demonstrate the equivalence between the expressions (1.1) and (1.2) for the N–soliton tau functions.

3 Free field realization of the $A_n^{(1)}$ Lie algebras in the Principal Gradation.

Realizations of infinite dimensional Lie algebras in terms of harmonic oscillators play a crucial role in the representation theory. They are also important in the applications to the string theory and the two dimensional conformal models (for a review, see [33]). In the simplest case of $A_1^{(1)} = \hat{sl}(2)$, the free field (or vertex operator) construction was obtained in [34]. This result has been generalized for affine Lie algebras $\hat{g}$ in the principal gradation [35]. This occurs when the underlying classical Lie algebras g is simply laced and the level of the corresponding representation is one. The generalization of the vertex operator realisation of $\hat{g}$ with g being non–simply laced as well as for twisted affine Lie algebras, is also known [15, 16]. In the present Section we limit our attention on the $A_n^{(1)}$ Lie algebras in the principal gradation only. A rather involved exposition on the subject is given in [10].

The construction of solutions of CAT models (2.1a), (2.1d), (2.2), (2.3) can be done by using irreducible highest weight representations of the corresponding affine algebra. The representation spaces are generated by the action of arbitrary products of negative grade elements (A.19) on the highest–weght state $|\Lambda>$. The last is annihilated by the elements of positive grade

$$F_k^r|\Lambda>=0\ , \qquad k\geqslant 1\ ,\quad r\in\mathbb{Z}_{n+1} \tag{3.1a}$$

where we have used the notation (A.15). We also recall that $F_{p(n+1)}^0\equiv 0\ ,\quad p\in\mathbb{Z}$ since these elements are not in the affine algebra. The highest–weight state is also an eigenvector of the subalgebra $\hat{g}_0$ of the elements of $\mathbb{Z}$ grade zero (A.18), (A.19)

$$F_0^r|\Lambda>=\Lambda(F_0^r)|\Lambda> \quad ,\hat{d}|\Lambda>=\Lambda(\hat{d}) \quad ,\hat{c}|\Lambda>=\Lambda(\hat{c})|\Lambda> \tag{3.1b}$$

In view of (A.23), the generators $\mathcal{E}_k=(n+1)F_k^0\quad(k\neq 0\,\mathrm{mod(n+1)})$ form a Heisenberg subalgebra. It is also known as the *principal* Heisenberg subalgebra. Since $\mathcal{E}_k$ are bosonic oscillators, there is a Fock representation, built up on the Fock vacuum

$$\mathcal{E}_k|0>=0,\qquad k\leqslant 1$$
$$<0|\mathcal{E}_k=0,\qquad k\geqslant -1 \tag{3.2}$$

It is known that when the value of the central charge $\hat{c}$ is 1, all the irreducible highest representations of the Lie algebras $A_n^{(1)}$ are expressed in terms of the elements of the (principal) Heisenberg subalgebra only [10, 34, 35]. To proceed we shall need the following notations

$$\mathcal{E}(\mu)=\sum_{\substack{k\in\mathbb{Z}\\ k\neq 0\,(\mathrm{mod(n+1)})}}\frac{\mathcal{E}_k}{\mu^k} \tag{3.3a}$$

$$\theta_r(\mu)=i\sum_{\substack{k\in\mathbb{Z}\\ k\neq 0\,(\mathrm{mod(n+1)})}}\frac{\omega^{-rk}-1}{l}\,\frac{\mathcal{E}_k}{\mu^k}$$
$$\theta_r(\mu)=\Phi_{r+n+1}(\mu)\ ,\quad \theta_0(\mu)=0 \tag{3.3b}$$

where i is the imaginary unit $i^2=-1$. The above expressions are related by the identity

$$\mathcal{E}(\mu)=\frac{i}{n+1}\mu\frac{d}{d\mu}\sum_{r\in\mathbb{Z}_{n+1}}\theta_r(\mu) \tag{3.3c}$$

We also define the bosonic normal product :: in the standard way: $\mathcal{E}_k$ with $k\geqslant 1$ are moved to the right, while $\mathcal{E}_k$ with $\leqslant -1$ are moved to the left in each normal ordered monomial containing generators of the Heisenberg subalgebra only. Since the Heisenberg subalgebra does not contain elements of grade zero, it turns out that each normal ordered monomial of positive (negative) grade annihilates the Fock vacuum $|0>$ $(<0|)$ (3.2). Taking into account (3.2) and the Heisenberg commutation relations (A.23) we get the contraction identities

$$\theta_r(\mu_1)\theta_s(\mu_2)=<\theta_r(\mu_1)\theta_s(\mu_2)>+:\theta_r(\mu_1)\theta_s(\mu_2):$$
$$\mathcal{E}(\mu_1)\theta_s(\mu_2)=<\mathcal{E}(\mu_1)\theta_s(\mu_2)>+:\mathcal{E}(\mu_1)\theta_s(\mu_2): \tag{3.4a}$$

where :: is the bosonic normal ordering. The vacuum expectation values are given by

$$
\begin{aligned}
< \theta_r(\mu_1)\theta_s(\mu_2) >&= \ln\frac{(\mu_1-\omega^{-\mathrm{r}}\mu_2)(\mu_1-\omega^{\mathrm{s}}\mu_2)}{(\mu_1-\mu_2)(\mu_1-\omega^{\mathrm{s-r}}\mu_2)} \\
< \mathcal{E}(\mu_1)\theta_r(\mu_2) >&= i\frac{(1-\omega^r\mu_2)\mu_1\mu_2}{(\mu_1-\mu_2)(\mu_1-\omega^r\mu_2)}
\end{aligned}
\tag{3.4b}
$$

To obtain a realization of the affine Lie algebras $A_n^{(1)}$ we introduce the *vertex* operators

$$
\begin{aligned}
V^r(\mu) &= \ :e^{i\theta_r(\mu)} := e^{\sum_{k\leqslant -1}\frac{1-\omega^{-kr}}{k}\ \frac{\mathcal{E}_k}{\mu^k}}e^{\sum_{k\geqslant 1}\frac{1-\omega^{-kr}}{k}\ \frac{\mathcal{E}_k}{\mu^k}} \\
r &= 1,...,n
\end{aligned}
\tag{3.5}
$$

From the (bosonic) wick theorem one deduces the following operator products:

$$
\begin{aligned}
\mathcal{E}(\mu_1)V^r(\mu_2) &= \frac{(\omega^r-1)\mu_1\mu_2}{(\mu_1-\mu_2)(\mu_1-\omega^r\mu_2)}V^r(\mu_2)+:\mathcal{E}(\mu_1)V^r(\mu_2): \\
V^r(\mu_1)V^s(\mu_2) &= \frac{(\mu_1-\mu_2)(\mu_1-\omega^{s-r}\mu_2)}{(\mu_1-\omega^{-r}\mu_2)(\mu_1-\omega^s\mu_2)}\ :e^{i(\theta_r(\mu_1)+\theta_s(\mu_2))}: \\
|\mu_1| &> |\mu_2|
\end{aligned}
\tag{3.6}
$$

In view of (A.23) and (3.2) we also get

$$
\begin{aligned}
\mathcal{E}(\mu_1)\mathcal{E}(\mu_2) &= \frac{\mu_1\mu_2}{(\mu_1-\mu_2)^2}-(n+1)\frac{\mu_1^{n+1}\mu_2^{n+1}}{(\mu_1^{n+1}-\mu_2^{n+1})^2}+:\mathcal{E}(\mu_1)\mathcal{E}(\mu_2): \\
|\mu_1| &> |\mu_2|
\end{aligned}
\tag{3.7}
$$

By using the notion of radial ordering [33] which is an analogue of the standard time ordering in the field theory, one can extend the operator products (3.6) and (3.7) to the region $|\mu_1|<|\mu_2|$. Note also that the products $A(\mu_1)B(\mu_2) = A(\mu_2)B(\mu_1)$ $(|\mu_1|\neq|\mu_2|)$ with $A(\mu)B(\mu)$ being $\mathcal{E}(\mu)$ or $V^r(\mu)$, become meromorphic operator valued functions on the complex variables μ_1 and μ_2. Therefore, one can use the Laurent expansions

$$
V^r(\mu)=\sum_{l\in\mathbb{Z}}\frac{V_l^r}{\mu^l} \quad , \qquad V_l^r=\oint_{S^1}\frac{\mathrm{d}\mu}{2\pi\mathrm{i}\mu}\mu^l V(\mu) \tag{3.8}
$$

Note also that normal ordered terms which appear in the right–hand side of (3.6) and (3.7) has finite matrix elements when evaluated between normalizable states in the bosonic Fock space.

To compute the algebra of commutators of the Laurent modes $\mathcal{E}_k,\ k\in\mathbb{Z}\backslash(n+1)\mathbb{Z}$ and F_k^r we shall use the *contour* deformation technique [33]. Let us briefly recall it: suppose that $A(\mu)$ and $B(\mu)$ are meromorphic operator valued functions on μ which *locally* commute $A(\mu_1)B(\mu_2)=A(\mu_2)B(\mu_1)$ for $(|\mu_1|\neq|\mu_2|)$. Then the commutator between the Laurent coefficients

$$
A_k=\oint_{S^1}\frac{\mu^k\mathrm{d}\mu}{2\pi\mathrm{i}\mu}A(\mu) \quad , \qquad B_l=\oint_{S^1}\frac{\mu^l\mathrm{d}\mu}{2\pi\mathrm{i}\mu}B(\mu) \tag{3.9a}
$$

is given by

$$[A_k, B_l] = \left(\oint_{|\mu_1|>|\mu_2|} - \oint_{|\mu_1|<|\mu_2|} \right) \mu_1^k \frac{\mathrm{d}\mu_1}{2\pi \mathrm{i}\mu_1} \ \mu_2^l \frac{\mathrm{d}\mu_2}{2\pi \mathrm{i}\mu_2} A(\mu_1) B(\mu_2) \qquad (3.9\mathrm{b})$$

Using the above identity, the Cauchy theorem and taking into account (3.6) one gets

$$[\mathcal{E}_k, V_l^r] = \left(\oint_{|\mu_1|>|\mu_2|} - \oint_{|\mu_1|<|\mu_2|} \right) \mu_1^k \frac{\mathrm{d}\mu_1}{2\pi \mathrm{i}\mu_1} \ \mu_2^l \frac{\mathrm{d}\mu_2}{2\pi \mathrm{i}\mu_2} \mathcal{E}_k(\mu_1) V^r(\mu_2) =$$

$$= \oint \mu_2^{l+1} \frac{\mathrm{d}\mu_2}{2\pi \mathrm{i}\mu_2} \left(\oint_{C_{\mu_2}} - \oint_{C_{\omega^r \mu_2}} \right) \mu_1^{k+1} \frac{\mathrm{d}\mu_1}{2\pi \mathrm{i}\mu_1} (\omega^r - 1) \frac{V^r(\mu_2)}{(\mu_1 - \mu_2)(\mu_1 - \omega^r \mu_2)} =$$

$$= (\omega^{kr} - 1) V_{k+l}^r \qquad (3.10\mathrm{a})$$

where C_μ stands for a small anti–clockwise oriented contour which surrounds the point μ. Similarly from the second equation (3.6) we obtain the commutators

$$[V_k^r, V_l^s] = \frac{(\omega^r - 1)(\omega^s - 1)}{(\omega^{r+s} - 1)} (\omega^{sk} - \omega^{rl}) V_{k+l}^{r+s}$$

$$r + s \neq 0 \ (\mathrm{mod\ n+1}) \qquad (3.10\mathrm{b})$$

$$\left[V_k^r, V_l^{-r}\right] = -\omega^r (1 - \omega^{-r})^2 \left((\omega^{-rk} - \omega^{rl}) \mathcal{E}_{k+l} + k\omega^{-rk} \delta_{k+l,0} \right) \qquad (3.10\mathrm{c})$$

In deriving the above equations we have also used the identity

$$\theta_r(\omega^p \mu) = \theta_{p+r}(\mu) - \theta_p(\mu)$$

which is a straightforward consequence from (3.3b). Using the contour deformation technique one also concludes that (3.7) is equivalent to the Heisenberg commutators (A.23). Therefore we conclude that $\mathcal{E}_k, \quad k \in \mathbb{Z} \backslash (n+1)\mathbb{Z}$ together with $V_k^r, r = 1, ..., n, k \in \mathbb{Z}$ form a Lie algebra. The last is *isomorfic* to $A_n^{(1)}$. To see this it is sifficient to set

$$F_k^r = \frac{\omega^{rl} V_k^r}{(n+1)(\omega^r - 1)} \qquad (3.11)$$

$$l \in \mathbb{Z}_{n+1}, r = 1, ..., n$$

and to verify that (3.10a)–(3.10c) together with (A.15) coincide with (A.21). In the above equation, $l \in \mathbb{Z}_{n+1}$ is an additional parameter. Note that the map

$$\mathcal{E}_k \longrightarrow \mathcal{E}_k$$

$$V_k^r \longrightarrow \omega^r V_k^r, r = 1, ..., n \qquad (3.12)$$

is an automorphism of order $(n+1)$ of the Lie algebra (3.10a)–(3.10c), (A.23). It is clear that this automorfism is *outer*, and therefore the representation of $A_n^{(1)}$ corresponding to different values of l (3.11) are *inequivalent.* Due to (3.2), (3.5) and (3.11) we obtain highest–weight representations characterized by

$$F_0^r |\Lambda_l> = \frac{\omega^{rl}}{(n+1)(\omega^r - 1)} |\Lambda_l>$$

$$\widehat{c} = |\Lambda_l> = |\Lambda_l>, \quad \widehat{d}|\Lambda_l> = 0 \qquad (3.13)$$

where the highest–weight state coincides with the Fock vacuum (3.2). Since $\widehat{c} = 1$, we conclude that representations constructed by us are the $n+1$ *fundamental* representations [10] of the Lie algebra $A_n^{(1)}$.

We further proceed by calculating the tau functions (2.9) in terms of the components of the field (2.1f), (2.2) with $\eta = 0$. Taking also into account the last identity (A.13), one obtains

$$\Phi = \frac{1}{2} \sum_{\substack{1 \le r \le n \\ k \in \mathbb{Z}_{n+1}}} \omega^{r(1-k)} \varphi_k F_0^r + \frac{\zeta}{2(n+1)} \widehat{c} \tag{3.14}$$

Combining the above expression with (3.13) we derive

$$\langle \Lambda_l | \Phi | \Lambda_l \rangle = \frac{\zeta}{2(n+1)} + \frac{1}{2(n+1)} \sum_{\substack{1 \le r \le n \\ k \in \mathbb{Z}_{n+1}}} \frac{\omega^{r(l+1-k)}}{\omega^r - 1} \varphi_k \tag{3.15}$$

Therefore the group theoretical tau functions satisfy the identities

$$\frac{\tau_{\Lambda_k}(\Phi)}{\tau_{\Lambda_{k-1}}(\Phi)} = e^{-\varphi_k} \tag{3.16a}$$

$$\prod_{k \in \mathbb{Z}_{n+1}} \tau_{\Lambda_k}(\Phi) = e^{-\zeta} \tag{3.16b}$$

Comparing the above identities with the definition (2.4a)–(2.4c) of the Hirota tau funtions we get the relations

$$\tau_{\Lambda_k}(\Phi) = e^{-\frac{\zeta_0}{n+1}} \tau_k(\Phi) \tag{3.17}$$

where ζ_0 was given by (2.14).

4 Derivation of the vertex operator representation for one–soliton solutions.

In the present Section we focus our attention on the equivalence between the representations (1.1) and (1.2) in the case of one–soliton solutions of the A_n CAT equations. The existence of one–solitons, i. e. solitary waves which propagate without changing their shape, is not a distinguishing property of the integrable evolution equations. As a counter–exemple, one can quote the existence of kink solution of the φ^4 model in $1+1$ dimensions [1]. On the other hand, the presence of of multisoliton configurations which describe the elastic collision of an arbitrary number of solitary waves is an idication of the the integrability of the the corresponding system.

In accordance with (2.5a), (2.5b) the evolution of a single soliton is dictated by the equations

$$\begin{aligned} e^{-\varphi_k} &= \frac{1 + X_k}{1 + X_{k-1}} \\ X_k &= -\frac{\omega^{rk}}{c} \, \frac{e(\mu)}{e(\omega^r \mu)} \end{aligned} \tag{4.1}$$

where μ is the unique rapiditylike parameter, $r = 1, \ldots, n$ is the soliton species and c stands for the constant which appear in the right–hand side of (2.5a). From the above identities it follows that the corresponding tau functions (2.4a), (2.4b) are given by

$$\tau_k = 1 + X_k \tag{4.2}$$

In this section we shall demonstrate that the representations (1.1) and (1.2) for the one–soliton tau functions associated to an arbitrary fundamental representation of the Lie algebra $A_n^{(1)}$ are equivalent. Let $\widetilde{g}_{\pm}(\mu)$ are the dressing group elements which generate the one–soliton solution (4.1) of the $A_n^{(1)}$ Toda model, from the vacuum. Taking into account (2.12) and (2.13a) we get the expression

$$\begin{aligned} \widetilde{g}(\mu) &= \widetilde{g}_-^{-1}(\mu)\widetilde{g}_+(\mu) = e^{-W_-(\Phi)}.e^{W_+(\Phi)} \\ \Phi &= \frac{1}{2}\sum_{k\in\mathbb{Z}_{n+1}} \varphi_k E^{kk} \end{aligned} \tag{4.3}$$

where φ_k are the components of the $A_n^{(1)}$ Toda field and (2.15a) $W_{\pm}(\Phi) = \sum_{k\in\mathbb{Z}_{n+1}} \varphi_k W_{\pm}^k(\mu)$. The elements $W_{\pm}(\mu)$ has been introduced by (2.15b), (2.15c). In what follows we shall use the representations

$$\begin{aligned} W_{\pm}(\Phi) &= \sum_{k=1}^{n} \varphi_k \widetilde{W}_{\pm}^{(k)}(\mu) \\ \widetilde{W}_{\pm}^{(k)}(\mu) &= W_{\pm}^{(k)}(\mu) - W_{\pm}^{n+1}(\mu) \end{aligned} \tag{4.4a}$$

$$\widetilde{W}_{\pm}^{(k)}(\mu) = -\sum_{k=1}^{n} \frac{1-\omega^{-rk}}{1-\omega^{-r}} \oint_{\lambda \gtrless |\mu|} \frac{\mathrm{d}\lambda}{2\pi\mathrm{i}\lambda} \frac{\omega^r\lambda - \mu}{\lambda - \mu} F^r(\lambda) \tag{4.4b}$$

where

$$F^r(\lambda) = \sum_p \frac{F_p^r}{\lambda^p} \quad , \quad F_p^r = \oint \frac{\lambda^p \mathrm{d}\lambda}{2\pi\mathrm{i}\lambda} F^r(\lambda) \tag{4.4c}$$

To calculate the element (4.3) in the fundamental representation (3.11), (3.13) of the affine Lie algebra $A_n^{(1)}$ we first introduce the notation.

$$\begin{aligned} h(\alpha) &= e^{-\alpha W_-(\Phi)}.e^{\alpha W_+(\Phi)} \\ h(\alpha) &= \sum_{l=0}^{\infty} \frac{\alpha^l}{l!}\frac{\mathrm{d}^{\mathrm{l}}}{\mathrm{d}\alpha^{\mathrm{l}}} h(0) \end{aligned} \tag{4.5a}$$

$$\begin{aligned} \frac{\mathrm{d}^{\mathrm{l}}}{\mathrm{d}\alpha^{\mathrm{l}}} h(0) &= -W_-(\Phi)\frac{\mathrm{d}^{\mathrm{l}-1}}{\mathrm{d}\alpha^{\mathrm{l}-1}} h(0) + \frac{\mathrm{d}^{\mathrm{l}-1}}{\mathrm{d}\alpha^{\mathrm{l}-1}} h(0).W_+(0) = \\ &= \sum_{k=0}^{l} (-)^k \begin{pmatrix} l \\ k \end{pmatrix} W_-^k(\Phi) W_+^{l-k}(\Phi) \end{aligned} \tag{4.5b}$$

Using (4.4a), (4.4b) and since the matrix (2.1f) is traceless, one can also write

$$W_{\pm}(\Phi) = \sum_{r=1}^{n} \frac{\widehat{\varphi}_r}{1-\omega^{-r}} \oint_{|\lambda| \gtrless |\mu|} \frac{d\lambda}{2\pi i\lambda} \frac{\omega^r\lambda - \mu}{\lambda - \mu} F^r(\lambda) \tag{4.6}$$

where we have introduced the discrete Fourier transform of φ_k ($\varphi_k = \varphi_{k+n+1}$) according to the expressions

$$\widehat{\varphi}_r = \sum_{k\in\mathbb{Z}_{n+1}} \omega^{-kr}\varphi_k; \quad \varphi_k = \frac{1}{n+1}\sum_{r\in\mathbb{Z}_{n+1}} \omega^{kr}\varphi_r \tag{4.7a}$$

Let us also recall the identity

$$\sum_{k\in\mathbb{Z}_{n+1}} \omega^{k(i-j)} = (n+1)\delta_{i,j}^{(n+1)} \tag{4.7b}$$

where the $\delta_{i,j}^{(n+1)}$ is the Kronecker symbol on the cyclic group: it vanishes for $i-j \neq 0 \ (\mathrm{mod} n+1)$ and $\delta_{i,j}^{(n+1)} = 1$ if $i-j = 0 \ (\mathrm{mod} n+1)$.

We proceed by calculating explicitly the taylor serie (4.5a). First of all, we note that

$$\begin{aligned}
&\frac{dh}{d\alpha}(0) = W_+(\Phi) - W_-(\Phi) = \\
&= -\sum_{r=1}^{n} \frac{\widehat{\varphi}_r}{1-\omega^{-r}} \oint_{C_\mu} \frac{d\lambda}{2\pi i\lambda}\frac{\omega^r\lambda-\mu}{\lambda-\mu}F^r(\lambda) = \\
&= -\sum_{r=1}^{n} \widehat{\varphi}_r\omega^r F^r(\mu)
\end{aligned} \tag{4.8}$$

Note that these identities are in accordance with (2.15a), (2.16). To calculate the higher derivatives of $h(\alpha)$ (4.5a) we first observe that due to (3.6) one gets

$$\begin{aligned}
&W_-(\Phi)V^r(\mu) - V^r(\mu)W_+(\Phi) = \\
&= \frac{1}{n+1}\sum_{s=1}^{n} \frac{\omega^{s(k+1)}\widehat{\varphi}_s}{(\omega^s-1)^2}\left(\oint_{|\lambda|>|\mu|} - \oint_{|\lambda|<|\mu|}\right)\frac{d\lambda}{2\pi i\lambda}\frac{\omega^s\lambda-\mu}{\lambda-\mu}V^s(\lambda)V^r(\mu) = \\
&= \frac{1}{n+1}\sum_{s=1}^{n} \frac{\omega^{s(k+2)}\widehat{\varphi}_s}{(\omega^s-1)^2}\left(\oint_{|\lambda|>|\mu|} - \oint_{|\lambda|<|\mu|}\right)\frac{d\lambda}{2\pi i\lambda}\frac{\lambda-\omega^{r-s}\mu}{\lambda-\omega^r\mu} : e^{i(\theta_s(\lambda)+\theta_r(\mu))} := \\
&= \frac{1}{n+1}\sum_{s=1}^{n} \frac{\omega^{s(k+1)}}{\omega^s-1}\widehat{\varphi}_s V^{r+s}(\mu) \\
&k \in \mathbb{Z}_{n+1}
\end{aligned} \tag{4.9}$$

in the irreducible representation of $A_n^{(1)}$ with a highest–weight $|\Lambda_k>$ (3.11)–(3.13). Using the above identity together with (4.5b), it is easy to prove inductively that

$$\frac{d^l h}{d\alpha^l}(0) = (-)^l \sum_{s_1,\ldots,s_l=1}^{n} \prod_{j=1}^{l} \Psi_{s_j}(\Phi).V^{s_1+\ldots+s_l}(\mu) \tag{4.10a}$$

$$\Psi_s(\Phi) = \frac{1}{n+1}\frac{\omega^{s(k+1)}\widehat{\varphi}_s}{\omega^s-1} \tag{4.10b}$$

where we have omited the dependence on the parameter $k \in \mathbb{Z}_{n+1}$ which label the different fundamental representations of the affine Lie algebra. Substituiting the last result into the Taylor expansion (4.5a), it turns out that (4.3) is given by the following intermediate expression

$$\widetilde{g}(\mu) = h(1) = \sum_{l=0}^{\infty} \frac{(-1)^l}{l!} \sum_{s_1,\ldots,s_l=1}^{n} \prod_{j=1}^{l} \Psi_{s_j}(\Phi).V^{s_1+\ldots+s_l}(\mu) \tag{4.11}$$

In order to write the right–hand side of the above identity in a more convenient form, we recall that in view of (3.3b) and (3.5), the vertex operators are periodic functions on the upper case index $V^s(\mu) = V^{s+n+1}(\mu)$ with $V^0(\mu) = 1$. This observation suggests us to introduce the commutative associative algebra $\mathcal{F}$ of (complex) dimension $n+1$. It is generated by the elements $V^s,\ s \in \mathbb{Z}_{n+1}$. The multiplication $\star$ in $\mathcal{F}$ is defined

$$V^r \star V^s = V^{r+s} \tag{4.12}$$

Due to the periodicity condition, the element $V^0 = V^{s+n+1}$ is the unity of $\mathcal{F}$. It is worthwhile to note that (4.12) describe the *"fusion rules"* of a class of Rational Conformal Field Theoies [36]. They correspond to a theory of a free massless scalar field in two dimensions which is compactfied on circle with a rational value of the square of the radius. The algebra $\mathcal{F}$ can be endowed with a symmetric bilinear invariant form

$$\begin{aligned} < a, b >_{\mathcal{F}} &= < b, a >_{\mathcal{F}} \\ < a, b \star c >_{\mathcal{F}} &= < a \star b, c >_{\mathcal{F}} \\ a, b, c &\in \mathcal{F} \end{aligned} \tag{4.13a}$$

which is uniquely fixed by

$$< V^r, V^s >_{\mathcal{F}} = \delta^{n+1}_{r+s} \tag{4.13b}$$

The algebra $\mathcal{F}$ together with the above bilinear form is an example of a Fröbenius algebra. Fröbenius algebras and their deformations are powerful tool in the study of the Topological Field Theory [37]. We go back to the problem to simplify the expression (4.11). Using the multiplication rules (4.12) we can rewrite (4.11) in the following form

$$\widetilde{g} = \mathcal{F} \exp\{-\sum_{s=1}^{n} \Psi_s(\Phi) V^s\} \tag{4.14}$$

where the symbol $\mathcal{F}$ means that the exponential is taken in the Fröbenius algebra (4.12), (4.13a), (4.13b). In the above expression the dependence on μ of the elements $V^s,\ s = 1, \ldots, n$ was skipped. Note that the Fourier transformation (4.7a) diagonalizes the "fusion rules" (4.7a)

$$\begin{aligned} &V^r \star \hat{V}^s = \omega^{pr} \hat{V}^p \\ &\hat{V}^p \star \hat{V}^q = (n+1)\delta^{n+1}_{p,q} \hat{V}^p \end{aligned} \tag{4.15}$$

Substituing the above expression into (4.14) we obtain

$$\widetilde{g} \ = \ \frac{1}{n+1} \sum_{p \in \mathbb{Z}_{n+1}} e^{-\sum_{s=1}^{n} \omega^{ps} \Psi_s(\Phi)} \hat{V}^p =$$

$$= \frac{1}{n+1} \sum_{p,r\in\mathbb{Z}_{n+1}} \omega^{-pr}.e^{\chi_p(\Phi)} V^r \qquad (4.16a)$$

$$\chi_p(\Phi) = -\sum_{s=1}^{n} \omega^{ps}\Psi_s(\Phi)$$

$$p \in \mathbb{Z}_{n+1} \qquad (4.16b)$$

Due to the identity (4.7b), the above quantities satisfy the restriction

$$\sum_{p\in\mathbb{Z}_{n+1}} \chi_p(\Phi) = 0 \qquad (4.17a)$$

Moreover, taking into account (4.10b), we get the recurrence relations

$$\chi_p(\Phi) - \chi_{p+1} = \varphi_{p+k+1} \qquad (4.17b)$$

where $k \in \mathbb{Z}_{n+1}$ is the parameter which labels the $n+1$ unequivalent fundamental representation of $A_n^{(1)}$ (3.11), (3.13). Comparing (4.17a) and (4.17b) with the definition (2.4a)–(2.4c) of the Hirota tau functions, we get

$$e^{\chi_p(\Phi)} = \frac{\tau_{p+k}(\Phi)}{\left(\prod_{l\in\mathbb{Z}_{n+1}} \tau_l(\Phi)\right)^{\frac{1}{n+1}}} \quad , \quad p \in \mathbb{Z}_{n+1} \qquad (4.18)$$

Substituting back the above expression into (4.16a) and taking into account (4.1) as well as the vertex operator realization of the affine Lie algebra $A_n^{(1)}$ (3.11), (3.8), we conclude that

$$\widetilde{g}(\mu) = \frac{1}{n+1} \sum_{p,r'\in\mathbb{Z}_{n+1}} \omega^{-pr'} e^{\chi_p(\Phi)} V^{r'}(\mu) =$$

$$= \frac{1}{(n+1)\prod_{l\in\mathbb{Z}_{(n+1)}} \tau_l^{\frac{1}{n+1}}(\Phi)} \sum_{p,r'\in\mathbb{Z}_{n+1}} \left(\omega^{-pr'} + \omega^{p(r-r')+rk} X_0 F^r(\mu)\right) =$$

$$= \frac{1}{\prod_{l\in\mathbb{Z}_{(n+1)}} \tau_l^{\frac{1}{n+1}}(\Phi)} \left(1 + (n+1)(\omega^r - 1) X_0 F^r(\mu)\right) \qquad (4.19a)$$

From the above identity, (2.10), (2.11) and (2.4c) one derives the relation

$$\frac{\tau_{\Lambda_k}(\Phi)}{\tau_{\Lambda_k}(\Phi_0)} = e^{\frac{\zeta_0-\zeta}{n+1}} \langle \Lambda_k | \widetilde{g}_-(\mu)\widetilde{g}_+(\mu) | \Lambda_k \rangle =$$

$$\frac{e^{\frac{\zeta_0-\zeta}{n+1}}}{\prod_{l\in\mathbb{Z}_{(n+1)}} \tau_l^{\frac{1}{n+1}}(\Phi)} \langle \Lambda_k | (1 + (n+1)(\omega^r - 1) X_0 F^r(\mu)) | \Lambda_k \rangle =$$

$$= \langle \Lambda_k | (1 + (n+1)(\omega^r - 1) X_0 F^r(\mu)) | \Lambda_k \rangle \qquad (4.19b)$$

Therefore, the representations (1.1) and (1.2) are equivalent in the one–soliton case.

5 N–soliton tau fuctions and vertex operators.

The existence of N-soliton solutions is a peculiar property of the integrable systems. From another point of view [29], generic soliton solutions provide a relation of classes of solutions of integrable nonlinear partial differential equations with the finite dimensional mechanical systems. The last are known as N-body integrable systems (for a review, see [28]). The N-soliton solutions, in the in– and out– limit, become asymptotically a superposition of monosolitons separated in the space. Moreover, the transformation from the in-variables to the out-variables is canonical. The underlying generating function of this transformation is known as the classical S-matrix [6]. An intriguing property of the classical S-matrix is that it is represented as a sum of terms, each of them representing a two-particle scattering. It is well known that the last property admits a generalization valid within the quatum theory: the quatum S-matrix which describes collision of solitons in a given quantum integrable model is a product of factors describing two particle interactions [7]. We recall that the algebraic N-soliton solutions (2.5a), (2.5b) in the $A_n^{(1)}$ Toda Models are generated from the vacuum by the dressing group elements (2.12), (2.13a)- (2.13d), (2.14), (2.15a), (2.15c). Due to the factorized expression (2.12), we can also write

$$\begin{aligned}\widetilde{g} &= \widetilde{g}_-^{-1}\widetilde{g}_+ = \widetilde{g}_-^{-1}(1)\ldots\widetilde{g}_-^{-1}(N).\widetilde{g}_+(N)\ldots\widetilde{g}_+(1) = \\ &= \widetilde{g}(1)\ .\ \mathrm{Ad}\ \widetilde{g}_+^{-1}(1)(\widetilde{g}(2))\ldots\mathrm{Ad}(\widetilde{g}_+^{-1}(1))\ldots\widetilde{g}_+^{-1}(N-1)).(\widetilde{g}(N)) \\ \widetilde{g}(i) &= \widetilde{g}_-^{-1}(i)\widetilde{g}_+(i)\end{aligned} \tag{5.1}$$

In the above expression and in what follows we shall assume that the rapidity like parameters $\mu_i \quad , \quad i = 1, \ldots, N$ corresponding to the factors (2.13a)-(2.13d) are radially ordered $|\mu_1| > |\mu_2| > \ldots > |\mu_N|$. In view of (4.16a), (4.16b) and the vertex operator construction of the affine Lie algebra $A_n^{(1)}$ (3.5), (3.8), (3.11), to calculate (5.1) in the fundamental representations, one first has to obtain an expression for the adjoint action of $\widetilde{g}_+(i)$ on the alternative basis (A.15). To do that we first note that diagonal traceless $(n+1)\times(n+1)$ matrices $D = \sum_{k\in\mathbb{Z}_{n+1}} d_k|k><k|$ are written in the alternative basis (A.13), (A.15) as follows

$$D = \sum_{s\in\mathbb{Z}_{n+1}} \omega^s \hat{d}_s F_0^s \tag{5.2}$$

where $\hat{d}_s,\ s\in\mathbb{Z}_{n+1}$ is the discrete Fourier transformation (4.7a) of the diagonal entries of D. Note that F_0^0 is *not* an element of the Lie algebra A_n. However, since D is traceless, it is clear that $\hat{d}_0 = \sum_k d_k = 0$. The above expression can be considered as an element either of the classical Lie algebra A_n or of the affine algebra $A_n^{(1)}$. Taking into account the commutation relations (A.21) together with (A.23), it is easy to get

$$[F^r(\mu), D] = \frac{1}{n+1}\sum_{k\in\mathbb{Z}_{n+1}} \frac{\omega^s(\omega^{sk}-1)\hat{d}_s}{\mu^k} F_k^{r+s} \tag{5.3a}$$

from where it is easy to get:

$$\begin{aligned} e^{-D}F^r(\mu)e^D &= \sum_{l=0}^{\infty}\frac{(-1)^l}{l!}\mathrm{ad}^l D.F^r(\mu) = \\ &= \sum_{k\in\mathbb{Z}}\frac{1}{\mu^k}\sum_{l=0}^{\infty}\frac{1}{l!}\sum_{s_1,\ldots,s_l\in\mathbb{Z}_{n+1}}\prod_{j=1}^{l}\omega^{s_j}\frac{\omega^{s_j k}-1}{n+1}\ \hat{d}_{s_j}\ F_k^{r+s_1+\ldots+s_l}\end{aligned} \tag{5.3b}$$

The above expression is of the same form as (4.11). This suggests to use the approach developed in Sec.4 to write (5.3b) explicitly as a linear combination of the affine algebra elements. In order to do that we first introduce the following (reducible) representation of the algebra $\mathcal{F}$ (4.12)

$$\begin{aligned} V^s(F^r_k) &= F^{r+s}_k \\ r,s &\in \mathbb{Z}_{n+1} \ , \quad k \in \mathbb{Z} \end{aligned} \tag{5.4}$$

Therefore, in complete analogy with the derivation of the compact expression (4.14), we conclude that

$$e^{-D}F^r(\mu)e^D = \sum_{k\in\mathbb{Z}_{n+1}} \frac{1}{\mu^k}\mathcal{F}\exp(\sum_{s\in\mathbb{Z}_{n+1}} \frac{\omega^s(\omega^{sk}-1)}{n+1}\hat{d}_s V^s)(F^r_k) \tag{5.5}$$

As in Sec. 4, we diagonalize the operators V^s (5.4)

$$\begin{aligned} &V^s(\hat{F}^p_k) = \omega^{ps}\hat{F}^p_k \\ &\hat{F}^p_k = \sum_r \omega^{-pr}\hat{F}^r_k \qquad\qquad F^r_k = \frac{1}{n+1}\sum_p \omega^{pr}\hat{F}^p_k \end{aligned} \tag{5.6}$$

Combining (5.5) with (5.6) we obtain

$$e^{-D}F^r(\mu)e^D = \frac{1}{n+1}\sum_{\substack{k\in\mathbb{Z}\\ p\in\mathbb{Z}_{n+1}}} \frac{\omega^{pr}}{\mu^k}e^{d_{k+p+1}-d_{p+1}}\hat{F}^p_k =$$

$$= \frac{1}{n+1}\sum_{\substack{k\in\mathbb{Z}\\ p,s\in\mathbb{Z}_{n+1}}} \frac{\omega^{p(r-s)}}{\mu^k}e^{d_{k+p+1}-d_{p+1}}.F^s_k = \frac{1}{n+1}\sum_{l,p,s}\omega^{p(r-s)}\frac{e^{d_{l+p+1}}}{e^{d_{p+1}}}\sum_{l\in\mathbb{Z}}\frac{F^s_{k(n+1)+l}}{\mu^{k(n+1)+l}} =$$

$$= \frac{1}{(n+1)^2}\sum_{l,p,q,s}\omega^{p(r-s)+lq}\frac{e^{d_{l+p+1}}}{e^{d_{p+1}}}\ F^s(\omega^q\mu) \tag{5.7}$$

As a consequence of the above identity and (2.17f), one obtains

$$e^{-\mathrm{ad}(K(F_i)+P_i)}F^r(\omega^a\lambda) = \sum_{s,q\in\mathbb{Z}_{n+1}} U^{ra}_{sq}(i)F^s(\omega^q\lambda) \tag{5.8a}$$

where $K(F_i)$ and P_i are diagonal traceless matrices (2.13b) and

$$\begin{aligned} U^{ra}_{sq}(i) &= M^{r+a}_{s+q}(i)N^a_q(i) \\ M^r_s(i) &= \frac{\omega^{s-r}}{n+1}\sum_{p\in\mathbb{Z}_{n+1}}\frac{\omega^{p(r-s)}}{\rho_{pi}(\mu_{i+1})} \\ N^a_q(i) &= \frac{\omega^{a-q}}{n+1}\sum_{p\in\mathbb{Z}_{n+1}}\omega^{p(q-a)}\rho_{pi}(\mu_{i+1}) \end{aligned} \tag{5.8b}$$

In what follows we shall also need the commutators $\left[W^k_+(\mu_i), F^s(\lambda)\right]$ where the affine algebra elements $W^k_+(\mu_i)$ and $F^s(\lambda)$ are given by (2.15b) and (4.4c) respectively. Using (A.21) with $\hat{c}=1$ we get

$$\left[W^k_+(\mu_i), F^s(\lambda)\right] \;=\; -\frac{1}{n+1}\sum_{r=1}^{n}\frac{(1-\omega^{-rk})\mu_i - \omega^s(1-\omega^{r(1-k)})\lambda}{(1-\omega^{-r})(\mu_i-\omega^s\lambda)}\ F^{r+s}(\lambda) +$$

$$+\frac{1}{n+1}\sum_{r=1}^{n}\frac{(1-\omega^{-rk})\mu_i-\omega^{-r}(1-\omega^{r(1-k)})\lambda}{(1-\omega^{-r})(\mu_i-\omega^{-r}\lambda)}\,F^{r+s}(\omega^{-r}\lambda)-$$
$$-\frac{1}{(n+1)^2}\sum_{r=1}^{n}\frac{\omega^{-rk}\lambda\mu_i}{(\mu_i-\omega^{-r}\lambda)^2}\delta^{(n+1)}_{r+s,0},\ \ |\mu_i|>|\lambda| \qquad (5.9a)$$

From the above expression, (2.13b) and (2.15a) we also derive the commutator

$$[W_+(i),F^s(\lambda)] \;=\; \sum_{k\in\mathbb{Z}_{n+1}} f_{ki}\left[W_+^k(\mu_i),F^s(\lambda)\right]=$$
$$=\frac{1}{n+1}\sum_{r=1}^{n}\frac{\hat f_{ri}}{1-\omega^{-r}}\left(\frac{\mu_i-\omega^{r+s}\lambda}{\mu_i-\omega^{s}\lambda}F^{r+s}(\lambda)-\frac{\mu_i-\lambda}{\mu_i-\omega^{-r}\lambda}F^{r+s}(\omega^{-r}\lambda)\right)-$$
$$-\frac{\hat f_{-si}}{(n+1)^2}\frac{\lambda\mu_i}{(\mu_i-\omega^{-s}\lambda)^2}\quad,\quad |\mu_i|>|\lambda|\ ,\quad s\in\mathbb{Z}_{n+1} \qquad (5.9b)$$

where $\hat f_{ri}$ stands for the discrete Fourier transformation (4.7a) of f_{ki} , $k\in\mathbb{Z}_{n+1}$. Using (5.9b) one proves inductively

$$\mathrm{ad}^l W_+(i)F^s(\lambda)=\frac{\mu_i-\lambda}{\mu_i-\omega^s\lambda}\sum_{k=0}^{l}(-)^{l-k}\begin{pmatrix} l\\ k\end{pmatrix}\times$$
$$\times\sum_{r_1,\dots,r_l=1}^{n}\frac{\mu_i-\omega^{r_1+\dots+r_k+s}\lambda}{\mu_i-\omega^{-r_{k+1}-\dots-r_l}\lambda}\prod_{p=1}^{l}\psi_{r_pi}\quad F^{r_1+\dots+r_l+s}(\omega^{-r_{k+1}-\dots-r_l}\lambda)+$$
$$+\frac{\mu_i(\mu_i-\lambda)}{(n+1)^2(\mu_i-\omega^s\lambda)}\sum_{k=0}^{l-1}(-)^{l-k}\begin{pmatrix} l-1\\ k\end{pmatrix}\times$$
$$\sum_{r_1,\dots,r_{l-1}=1}^{n}\frac{\lambda\omega^{-r_{k+1}-\dots-r_{l-1}}}{(\mu_i-\omega^{-r_{k+1}-\dots-r_{l-1}}\lambda)(\mu_i-\omega^{r_1+\dots+r_k+s}\lambda)}\cdot\hat f_{-r_1\dots-r_{l-1}-si}$$
$$\times\prod_{p=1}^{l}\psi_{r_pi}\quad,\quad l\geqslant 1$$
$$\psi_{ri}=\frac{\hat f_{ri}}{(n+1)(1-\omega^{-r})}\quad,\quad r=1,\dots,n \qquad (5.10)$$

Note that after the change $\hat f_{r,i}\to\hat\varphi_i$, the quantities ψ_{ri} coincide up to phase with (4.10b). To write the right-hand side of the above identity in a compact form, we consider the commutative associative algebra $\mathcal{A}=\mathcal{F}\times\mathcal{T}$. We recall that $\mathcal{F}$ is the Frobenius algebra (4.12), (4.13a), (4.13b) while $\mathcal{T}$ is spanned on the elements $\mathbb{T}^r$, $r\in\mathbb{Z}_{n+1}$. The multiplication in $\mathcal{A}$ is introduced by

$$V^r*V^s \;=\; V^{r+s}\quad,\quad \mathbb{T}^r*\mathbb{T}^s=\mathbb{T}^{r+s}$$
$$V^r*\mathbb{T}^s \;=\; \mathbb{T}^s*V^r \qquad (5.11a)$$

we shall also need a specific representation of $\mathcal{A}$. It is defined by the relations

$$V^r(g(\lambda)F^s(\lambda)) \;=\; g(\lambda)V^r(F^s(\lambda))=g(\lambda)F^{r+s}(\lambda)$$
$$\mathbb{T}^r(g(\lambda)F^s(\lambda)) \;=\; g(\omega^{-r}\lambda)\mathbb{T}^r(F^s(\lambda))=g(\omega^{-r}\lambda)F^s(\omega^{-r}\lambda) \qquad (5.11b)$$

for any function g on the complex parameter λ. It is also assumed that the algebra $\mathcal{A}$ acts trivially on μ_i (5.10). Using the notations (5.11a) and (5.11b), it is easy to check that (5.10) can be written as follows

$$
\begin{aligned}
&(-)^l \mathrm{ad}^l W_+(i) F^s(\lambda) = \\
&= \frac{\mu_i-\lambda}{\mu_i-\omega^s\lambda}\left((\sum_{r=1}^{n}\psi_{ri}V^r(\mathbb{T}^r-1))_*^l \frac{\mu_i F^s(\lambda)}{\mu_i-\lambda} - \omega^s (\sum_{r=1}^{n}\omega^r\psi_{ri}V^r(\mathbb{T}^r-1))_*^l \frac{\lambda F^s(\lambda)}{\mu_i-\lambda}\right) + \\
&+\frac{\mu_i(\mu_i-\lambda)}{(n+1)^2(\mu_i-\omega^s\lambda)}\left\langle \sum_{p\in\mathbb{Z}_{n+1}}\hat{f}_{pi}V^p, (\sum_{r=1}^{n}\psi_{r,i}V^r(1\otimes_{\mathcal{T}}\mathbb{T}^r-\mathbb{T}^r\otimes_{\mathcal{T}}1))_*^{l-1}V^s\right\rangle_{\mathcal{F}} \times \\
&\times \frac{1}{\mu_i-\omega^s\lambda}\otimes\frac{\lambda}{\mu_i-\lambda}
\end{aligned}
\tag{5.12}
$$

where $\langle\ ,\ \rangle_{\mathcal{F}}$ is the invariant bilinear form (4.13a), (4.13b) in the Fröbenius algebra $\mathcal{F}$ (4.12)–(4.13b). We have also used the notation

$$
(\mathcal{P})_*^l = \underbrace{\mathcal{P}*\ldots*\mathcal{P}}_{l}
\tag{5.13a}
$$

to indicate that the $l-$th power of $\mathcal{P}\in\mathcal{T}$ is taken with respect to the multiplication (5.11a). In what follows we shall use the operators

$$
\begin{aligned}
\mathcal{T}\exp(\mathcal{P}) &= \sum_{l=0}^{\infty}\frac{(\mathcal{P})_*^l}{l!} \\
\mathcal{T}\frac{\exp(\mathcal{P})-1}{\mathcal{P}} &= \sum_{l=1}^{\infty}\frac{(\mathcal{P})_*^{l-1}}{l!}
\end{aligned}
\tag{5.13b}
$$

Note that the second operator has eigenvalue one when applied on the zero modes of $\mathcal{P}$. Taking into account (5.12) together with (5.13a), (5.13b) we obtain

$$
\begin{aligned}
&e^{-\mathrm{ad}W_+(i)}.F^s(\lambda) = \\
&= \frac{\mu_i-\lambda}{\mu_i-\omega^s\lambda}\left(\mathcal{T}e^{(\sum_{r=1}^{n}\psi_{ri}V^r(\mathbb{T}^r-1))}\frac{\mu_i F^s(\lambda)}{\mu_i-\lambda} - \omega^s\mathcal{T}e^{(\sum_{r=1}^{n}\omega^r\psi_{ri}V^r(\mathbb{T}^r-1))}\frac{\lambda F^s(\lambda)}{\mu_i-\lambda}\right) + \\
&\frac{\mu_i(\mu_i-\lambda)}{(n+1)^2(\mu_i-\omega^s\lambda)}\left\langle\sum_{p\in\mathbb{Z}_{n+1}}\hat{f}_{p_i}V^p, \mathcal{T}\frac{\exp(\sum_{r=1}^{n}\psi_{ri}V^r(1\otimes_{\mathcal{T}}\mathbb{T}^r-\mathbb{T}^{-r}\otimes_{\mathcal{T}}1))-1}{\sum_{r=1}^{n}\psi_{ri}V^r(1\otimes_{\mathcal{T}}\mathbb{T}^r-\mathbb{T}^{-r}\otimes_{\mathcal{T}}1)}V^s\right\rangle_{\mathcal{F}} \times \\
&\times\frac{1}{\mu_i-\omega^s\lambda}\otimes\frac{\lambda}{\mu_i-\lambda}
\end{aligned}
\tag{5.14}
$$

In view of the above equation, we shall need the (reducible) representations (5.11b) of $\mathcal{T}$ (5.11a) which are spanned on the vectors $g(\omega^r\lambda)F^s(\omega^r\lambda)$, $r,s\in\mathbb{Z}_{n+1}$ and g being an arbitrary meromorphic function on λ. It is easy to check that

$$
G^{pq}(\lambda) = \sum_{r,s\in\mathbb{Z}_{n+1}}\omega^{-ps+rq}g(\omega^r\lambda)F^s(\omega^r\lambda)
\tag{5.15a}
$$

are eigenvectors of $\mathcal{T}$

$$
V^kG^{pq}(\lambda) = \omega^{pk}G^{pq}(\lambda),\quad \mathbb{T}^kG^{pq}(\lambda) = \omega^{kq}G^{pq}(\lambda).
\tag{5.15b}
$$

Due to the identity (4.7b), the inverse of (5.15a) is given by

$$g(\omega^r\lambda)F^s(\omega^r\lambda) = \frac{1}{(n+1)^2}\sum_{p,q\in\mathbb{Z}_{n+1}}\omega^{ps-rq}G^{pq}(\lambda) \qquad (5.15c)$$

Note also that

$$\sum_{r=1}^{n}\omega^{rp}\psi_{ri} = \frac{f_{pi}}{2} - K_p(F_i) \qquad (5.16a)$$

where the diagonal traceless matrices F_i and $K(F_i)$ has been defined by (2.13b). To obtain the above identity we have also used the relations (2.13c). Taking into account (2.17b) we can write

$$e^{-\sum_{r=1}^{n}\omega^{rp}\psi_{ri}} = \beta_p(F_i) \qquad (5.16b)$$

substituing back (5.15a)–(5.16b) into (5.14) one gets

$$\begin{aligned}
e^{-adW_+(i)}.F^s(\lambda) = {}& \frac{\mu_i-\lambda}{(n+1)^2(\mu_i-\omega^s\lambda)}\times \\
&\times\sum_{p,q,r',s'\in\mathbb{Z}_{n+1}}\omega^{p(s-s')+r'q}.\frac{\mu_i-\omega^{r'+s'}\lambda}{\mu_i-\omega^{r'}\lambda}\frac{\beta_p(F_i)}{\beta_{p+q}(F_i)}F^{s'}(\omega^{r'}\lambda)+ \\
&+\frac{\lambda\mu_i(\mu_i-\lambda)}{(n+1)^4(\mu_i-\omega^s\lambda)}\sum_{a,q,q'p,p'\in\mathbb{Z}_{n+1}} f_{ai}\frac{\omega^{as+pq+p'(q'+1)}}{(\mu_i-\omega^{p+s}\lambda)(\mu_i-\omega^{p'}\lambda)}\times \\
&\times\frac{\beta_{a-q}(F_i)-\beta_{a+q'}(F_i)}{\beta_{a+q'}(F_i)(\ln\beta_{a-q}(F_i)-\ln)\beta_{a+q'}(F_i)}
\end{aligned} \qquad (5.17)$$

Therefore, the adgoint action of the affine group element $e^{-W_+(i)}$ on the affine Lie algebra, can be written as follows

$$e^{-adW_+(i)}.F^s(\lambda) = \sum_{q,v\in\mathbb{Z}_{n+1}} W^{sc}_{qv}[F_i](i;\lambda)\,F^q(\omega^v\mu_j) + Z^{sc}[F_i](i,\lambda) \qquad (5.18a)$$

Due to (2.15a), $W_+(i)$ are linear in the entries f_{ki} of the diagonal matrices F_i(2.13b). This wants to say that the relations

$$\begin{aligned}
\sum_{q,v\in\mathbb{Z}_{n+1}} W^{sc}_{qv}[F_i](i;\lambda)\,W^{qv}_{kl}[-F_i](i;\lambda) &= \delta^{(n+1)}_{sk}\delta^{(n+1)}_{cl} \\
\sum_{q,v\in\mathbb{Z}_{n+1}} W^{sc}_{qv}[F_i](i;\lambda)\,Z^{qv}[-F_i](i;\lambda) + Z^{sc}[F_i](i;\lambda) &= 0
\end{aligned} \qquad (5.18b)$$

are valid. Comparing (5.17) with (5.18a) we obtain

$$\begin{aligned}
W^{sc}_{qv}[F_i](i;\lambda) &= K^{s+c}_{q+v}[F_i](i;\lambda)L^c_v[F_i](i;\lambda) \\
K^s_q[F_i](i;\lambda) &= \frac{1}{n+1}\frac{\mu_i-\omega^q\lambda}{\mu_i-\omega^s\lambda}\sum_{p\in\mathbb{Z}_{n+1}}\omega^{p(s-q)}\beta_p(F_i) \\
L^c_v[F_i](i;\lambda) &= \frac{1}{n+1}\frac{\mu_i-\omega^c\lambda}{\mu_i-\omega^v\lambda}\sum_{p\in\mathbb{Z}_{n+1}}\omega^{p(v-c)}\beta_p(-F_i)
\end{aligned} \qquad (5.18c)$$

and

$$\begin{aligned} Z^{sc}[F_i](i,\lambda) &= \frac{\omega^c\lambda\mu_i(\mu_i-\omega^c\lambda)}{(n+1)^4(\mu_i-\omega^{s+c}\lambda)}\sum_{\substack{a,b,p\\ v,v'}} f_{pi}\frac{\omega^{sa+(p-a)v+(b+1-p)v'+c(a-b-1)}}{(\mu_i-\omega^v\lambda)(\mu_i-\omega^{v'}\lambda)}\times \\ &\times \frac{\beta_a(F_i)-\beta_b(F_i)}{\beta_b(F_i)(\ln\beta_a(F_i)-\ln\beta_b(F_i))} \end{aligned} \tag{5.18d}$$

Taking into account that the matrices F_i (2.13b) are traceless and the identity

$$\sum_{\substack{p\in\mathbb{Z}_{n+1}\\ 1\le r\le n}} \frac{\omega^{(p-a)r}}{1-\omega^r} f_{pi} = -(n+1)\ln\beta_a(F_i)$$

we conclude that (5.18d) can be alternatively written in the form

$$\begin{aligned} Z^{sc}[F_i](i,\lambda) &= \frac{\lambda(\mu_i-\omega^c\lambda)}{(n+1)^3(\mu_i-\omega^{s+c}\lambda)}\sum_{a,b,v\in\mathbb{Z}_{n+1}} \frac{\omega^{a(s+c+v)-b(v+c)}}{\lambda-\omega^v\mu_i}\frac{\beta_a(F_i)}{\beta_b(F_i)}+ \\ &+ \frac{\lambda\omega^c\delta^{(n+1)}_{s0}}{(n+1)(\mu_i-\omega^{s+c}\lambda)} \end{aligned} \tag{5.19a}$$

After certain trivial algebraic manipulations involving the first term of the above expression and using (2.18), (2.17d) we arrive at the result

$$\begin{aligned} Z^{sc}[-F_i](i,\lambda) &= \frac{\lambda}{\gamma_i(n+1)^2(\omega^{s+c}-\mu_i)}\sum_{a,b\in\mathbb{Z}_{n+1}} \omega^{(s+c)a-bc}\mathcal{D}^{(i\,i-1)}_{ab}(\lambda)\frac{\rho_{bi-1}(\mu_i)}{\beta_a(F_i)}+ \\ &+ \frac{\lambda\omega^c\delta^{(n+1)}_{s0}}{(n+1)(\mu_i-\omega^{s+c}\lambda)} \end{aligned} \tag{5.19b}$$

The equations (5.8a), (5.8b) and (5.18a) provide an expression for the adjoint action of the element (2.13a) on the affine Lie algebra

$$\begin{aligned} \widetilde{g}_+^{-1}(i)F^s(\omega^c\lambda)\widetilde{g}_+(i) &= \sum_{r,v\in\mathbb{Z}_{n+1}} (R^{sc}_{rv}(i;\lambda)F^r(\omega^v\lambda)+U^{sc}_{rv}(i)Z^{rv}[F_i](i;\lambda)) = \\ &= \sum_{r,v\in\mathbb{Z}_{n+1}} R^{sc}_{rv}(i;\lambda)\,(F^r(\omega^v\lambda)-Z^{rv}[-F_i](i;\lambda)) \\ |\mu_j| &> |\mu_i| \end{aligned} \tag{5.20a}$$

where $R^{sc}_{rv}(i;\lambda)$ are the elements of the matrix $R(j;\mu_i)=U(j)W(j;\mu_i)$ which acts on thensor product $\mathbb{C}^{n+1}\otimes\mathbb{C}^{n+1}$

$$\begin{aligned} R^{sc}_{rv}(i;\lambda) &= \sum_{p,q\in\mathbb{Z}_{n+1}} U^{sc}_{pq}(i)\,W^{pq}_{rv}(i;\lambda) = P^c_v(i;\lambda)\,Q^{s+c}_{r+v}(i;\lambda) \\ P^c_v(i;\lambda) &= \frac{1}{n+1}\sum_{p\in\mathbb{Z}_{n+1}} \omega^{(p-1)(v-c)}\frac{\lambda-\omega^{-v}e^{-f_{pi}}\mu_i}{\lambda-\omega^{-v}\mu_i}\frac{\rho_{pi}(\mu_{i+1})}{\beta_p(F_i)} \\ Q^s_r(i;\lambda) &= \frac{\mu_i-\omega^r\lambda}{(n+1)^2}\sum_{a,b\in\mathbb{Z}_{n+1}} \omega^{(a-1)s-br}\frac{\beta_b(F_i)}{\rho_{ai}(\mu_{i+1})}\sum_{p\in\mathbb{Z}_{n+1}}\frac{\omega^{(b+1-a)p}}{\mu_i-\omega^p\lambda} \end{aligned} \tag{5.20b}$$

Note that the second identity (5.20a) is a consequence of (5.18b). Comparing the above equations with (2.17d) and taking into account (2.18) we obtain the following alternative expressions

$$P_v^c(i;\lambda) = \frac{\gamma_i}{n+1}\sum_{a,b\in\mathbb{Z}_{n+1}} \omega^{(a-1)v-(b-1)c}\frac{1}{\rho_{ai-1}(\mu_i)}\mathcal{D}_{ab}^{(i-1\,i)}(\lambda)\rho_{bi}(\mu_{i+1})$$
$$Q_r^s(i;\lambda) = \frac{1}{(n+1)\gamma_i}\sum_{a,b\in\mathbb{Z}_{n+1}} \omega^{(a-1)s-(b-1)r}\frac{1}{\rho_{ai}(\mu_{i+1})}\mathcal{D}_{ab}^{(i\,i-1)}(\lambda)\rho_{bi-1}(\mu_i) \qquad (5.21)$$

which together with (4.7b) yield

$$(P(j;\lambda)\dots P(k;\lambda))_v^c = \frac{\gamma_j\cdots\gamma_k}{n+1}\sum_{a,b\in\mathbb{Z}_{n+1}} \omega^{(a-1)v-(b-1)c}\frac{1}{\rho_{ak-1}(\mu_k)}\mathcal{D}_{ab}^{(k-1\,j)}(\lambda)\rho_{bj}(\mu_{j+1})$$
$$(Q(j;\lambda)\dots Q(k;\lambda))_v^c = \frac{1}{(n+1)\gamma_j\dots\gamma_k}\sum_{a,b\in\mathbb{Z}_{n+1}} \omega^{(a-1)s-(b-1)r}\frac{1}{\rho_{aj}(\mu_{j+1})}\mathcal{D}_{ab}^{j\,k-1)}(\lambda)\rho_{bk-1}(\mu_k)$$
$$\text{for} \quad k\le j \qquad (5.22\text{a})$$

Therefore, as a consequence of (5.20b), (2.21c), (5.18c) and the above identities we get

$$(R(j;\mu_{j+1})\dots R(k;\mu_{j=1}))_{rv}^{p0} = \frac{1}{(n+1)^2}\sum_{a,a',b'\in\mathbb{Z}_{n+1}} \omega^{(a-1)v+(a'-1)p-(b'-1)(r+v)}\times$$
$$\times\frac{\rho_{ak-1}(\mu_{j+1})}{\rho_{ak-1}(\mu_k)}\mathcal{D}_{a'b'}^{(j\,k-1)}(\mu_{j+1})\frac{\rho_{b'k-1}(\mu_k)}{\rho_{a'j}(\mu_{j+1})} \qquad (5.22\text{b})$$

On the other hand, using (2.17d) and (2.21c), it is not difficult to show that

$$\mathcal{D}_{ab}^{(j\,k)}(\omega^r\lambda) \;=\; \omega^{r(a-b)}\mathcal{D}^{(j\,k})_{ab}(\lambda)$$

Inserting this identity into (5.22b) and taking into account (2.17e) one obtains

$$(R(j;\mu_{j+1})\dots R(1;\mu_{j+1}))_{rv}^{p0} = \frac{\delta_{v0}^{(n+1)}}{n+1}\sum_{a\in\mathbb{Z}_{n+1}} \omega^{(a-1)(p-r)}\frac{\rho_{aj}(\omega^r\mu_{j+1})}{\rho_{aj}(\mu_{j+1})}$$

and therefore

$$\sum_{p\in\mathbb{Z}_{n+1}}\omega^{p(1-k)}\left(R(j;\mu_{j+1})\dots R(1;\mu_{j+1})\right)_{rv}^{p0} = \omega^{r(1-k)}\delta_{v0}^{(n+1)}\frac{\rho_{kj}(\omega^r\mu_{j+1})}{\rho_{kj}(\mu_{j+1})} \qquad (5.23\text{a})$$

Using similar arguments and (2.17b)–(2.22) we also prove the identity

$$\sum_{\substack{1\le l\le j\\ p\in\mathbb{Z}_{n+1}}}\omega^{p(1-k)}\left(R(j;\mu_{j+1})\dots R(l;\mu_{j+1})\,Z[-F_l](l;\mu_{j+1})\right)^{p0} = -\frac{\mu_{j+1}}{n+1}\,\frac{d\ln\rho_{kj}}{d\lambda}(\mu_{j+1}) \qquad (5.23\text{b})$$

where the functions $Z^{sc}[\pm F_i]$ has been defined by (5.18a), (5.18b).

Now we are ready to calculate explicitly the product (5.1). Inserting back (5.18a) and (5.18a) into (5.1) we get

$$\mathrm{Ad}\left(\widetilde{g}_+^{-1}(1)\ldots\widetilde{g}_+^{-1}(i-1)\right)F^p(\mu_i)=\sum_{r,v\in\mathbb{Z}_{n+1}}\left(R(i-1;\mu_i)\ldots R(1;\mu_i)\right)^{p0}_{rv}F^r(\omega^v\mu_i)-$$
$$-\sum_{j=1}^{i-1}\left(R(i-1;\mu_i)\ldots R(j;\mu_i)\,Z[-F_j](j;\mu_i)\right)^{p0} \qquad (5.24)$$

On the other hand one can repeat the procedure developed in Sec. 4 and to obtain a result analogous to (4.16a), (4.16b)

$$\widetilde{g}(i)=\widetilde{g}_-^{-1}(i)\,\widetilde{g}_+(i)=\frac{\hat{\beta}_0(F_i)}{n+1}+\sum_{p\in\mathbb{Z}_{n+1}}(\omega^p-1)\hat{\beta}_p(F_i)F^p(\mu_i) \qquad (5.25)$$

where $\hat{\beta}_k(F_i)$, $k\in\mathbb{Z}_{n+1}$ is the discrete Fourier transformation (4.7a) of $\beta_k(F_i)$ (2.17b). Inserting back the above expression into (5.24) and taking into account (5.23a), (5.23b) and (2.17a) we conclude that

$$\mathrm{Ad}\left(\widetilde{g}_+^{-1}(1)\ldots\widetilde{g}_+^{-1}(i-1)\right)\widetilde{g}(i)=Y_i\left(1+X_iF^{r_i}(\mu_i)\right)$$
$$X_i=(n+1)(1-\omega^{r_i})\prod_{a\neq i}\frac{\omega^{r_i}\mu_i-\mu_a}{\mu_{l+1}-\mu_a}\prod_{a=1}^{N}\frac{\mu_i+\epsilon_{1a}}{\omega^{r_i}\mu_i+\epsilon_{1a}}$$
$$Y_i=\frac{1}{n+1}\sum_{k\in\mathbb{Z}_{n+1}}\left(1+\mu_i\frac{d}{d\lambda}\ln\frac{\rho_{ki-1}}{\rho_{k+1i-1}}(\mu_i)\right)\beta_k(F_i) \qquad (5.26a)$$

Note that due to the evolution equations (2.5a), the quantities X_i depend exponentially on the light cone variables x^+ and x^-. Combining (5.26a), (2.10) and (2.11) we deduce the relation

$$\frac{\tau_\Lambda(\Phi)}{\tau_\Lambda(\Phi_0)}=e^{\frac{\zeta_0-\zeta}{n+1}}\prod_{i=1}^{N}Y_i<\Lambda|(1+X_1F^{r_1}(\mu_1))\ldots(1+X_NF^{r_N}(\mu_N))|\Lambda> \qquad (5.26b)$$

for any fundamental representation with highest weight vector $|\Lambda>$ of the affine Lie algebra A^1_{n+1}. Therefore, the representations (1.1) and (1.2) coincide provided that

$$e^{\frac{\zeta-\zeta_0}{n+1}}\;=\;\prod_{i=1}^{N}Y_i \qquad (5.27)$$

We hope to go back to the proof of the above equation elsewhere.

A Appendix

In this Appendix we review the necessary information concerning the Lie algebra $A^{(}_n1)$ in the principal gradation. More detailed exposition can be found in [10, 15, 16]. The Lie algebra $sl(n+1)$, $n\geqslant 1$ is the set of the traceless $(n+1)\times(n+1)$ matrices. Within the Cartan classification they are known as A_n Lie algebras. The

Cartan subalgebra is spanned the (traceless) linear combinations of the diagonal elementary matrices $E^{ii} = |i><i|$ $(i = 1 \ldots n+1)$. The root system can be embedded into the $(n+1)-$dimensional Euclidean space. Fixing an orthonormalized basis e_i, the roots are $\alpha_{ij} = e_i - e_j, \quad i \neq j$ $(i, j = 1 \ldots n+1)$. To each root one associates a step operator $E^{\alpha_{ij}} = E^{ij} = |i><j|$ $(i \neq j)$ which is an eigenvector of the adjoint action of the Cartan subalgebra $\mathcal{H}$. As simple roots one chooses the vectors $\alpha_i = e_i - e_{i+1}, \quad i = 1, ..., n$. The related step operators satisfy the commutation relations

$$\begin{aligned}
\left[H_\xi, E^{\pm\alpha_i}\right] &= \pm\alpha_i.\xi E^{\pm\alpha_i} = \pm(\xi_i - \xi_{i+1})E^{\pm\alpha_i} \\
\left[E^{\alpha_i}, E^{-\alpha_j}\right] &= \delta_{ij} H_{\alpha_i} \\
H_\xi &= \sum_i \xi_i |i><i|
\end{aligned} \tag{A.1}$$

The generic step operators are obtained by successive commutators of E^{α_i} and their transposed $E^{-\alpha_i}$.

In the theory of the Lie algebras it is important to study their finite order inner automorphisms. The general theory has been developed by Kac [10] and reviewed in [12]. In this paper we shall only use a special inner automorphism of the simple Lie algebra A_n σ of order $n+1$ $(\sigma^{n+1} = 1)$. Before introducing it, we recall that the fundamental weights are

$$\begin{aligned}
\lambda_i &= \sum_{k=1}^{i} e_k - \frac{i}{n+1} \sum_{k=1}^{n+1} e_k \\
2\frac{\alpha_i \cdot \lambda_j}{\alpha_i \cdot \alpha_i} &= \delta_{ij}, \quad i, j = 1, \ldots, n
\end{aligned} \tag{A.2}$$

We also set $\rho = \sum_{i=1}^{n} \lambda_i$ and define [10, 11]

$$\begin{aligned}
\sigma(X) &= S X S^{-1} \\
S &= e^{2\pi i \frac{H_\rho}{n+1}}
\end{aligned} \tag{A.3}$$

for an arbitrary element X in the Lie algebra A_n. Note that in the defining representation, the element S which implements the automorphism σ takes the following form

$$\begin{aligned}
S &= \omega^{\frac{n}{2}} \sum_{k=1}^{n+1} \omega^{1-k} E^{kk} \\
\omega &= e^{\frac{2\pi i}{n+1}}
\end{aligned} \tag{A.4}$$

From the commutation relations (A.1) and the above identity, one concludes that σ acts trivially on the elements of $\mathcal{H}$ and as a multiplication by phase on the step operators:

$$\begin{aligned}
\sigma(H_\xi) &= H_\xi \\
\sigma(E^{\alpha_{kl}}) &= \omega^{\alpha_{kl} \cdot \rho} E^{\alpha_{kl}} = \omega^{l-k} E^{\alpha_{kl}}
\end{aligned} \tag{A.5}$$

The Lie algebra $\mathcal{G} = A_n$ together, with automorphism σ, is a graded algebra

$$\begin{aligned}\mathcal{G} &= \oplus_{k\in\mathbb{Z}_{n+1}}\mathcal{G}_k \\ \sigma(\mathcal{G}_k) &= \omega^k\mathcal{G}_k \\ [\mathcal{G}_k,\mathcal{G}_l] &\subseteq \mathcal{G}_{k+l}\end{aligned} \tag{A.6}$$

It is well known that for a given simple Lie algebra $\mathcal{G}$, its Cartan subalgebra is fixed up to a conjugation by elements belonging to the corresponding Lie group. In particular, instead of $\mathcal{H}$, one can introduce an *alternative* Cartan subalgebra $\mathcal{H}^{'}$, spanned on the mutually comutting generators:

$$\begin{aligned}\mathcal{E}_i &= \sum_{k=1}^{n+1-i} E^{kk+i} + \sum_{k=1}^{i} E^{n+1+k-ik} = \\ &= \sum_{k\in\mathbb{Z}_{n+1}} |k><k+i|\end{aligned} \tag{A.7}$$

To show that the above elements actually generate certain Cartan subalgebra, it sufices to note that the matrix with entries

$$\begin{aligned}\Omega_{ij} &= \omega^{(i-1)(j-1)} \\ \Omega^{-1}_{ij} &= \frac{\omega^{-(i-1)(j-1)}}{n+1}\end{aligned} \tag{A.8}$$

diagonalizes $\mathcal{E}_i$ (A.7)

$$\Omega^{-1}\mathcal{E}_i\Omega = \sum_{k=1}^{n+1}\omega^{i(k-1)}E^{kk} \tag{A.9}$$

It is worthwhile to note that the alternative Cartan subalgebra generators are eigenvectors of the inner automorphism σ

$$\sigma(\mathcal{E}_i) = \omega^i\mathcal{E}_i \tag{A.10}$$

For general simple Lie algebras, it is known [10, 15] that the eigenvalues of the corresponding automorphism σ, when restricted to the alternative Cartan algebra, are in correspondence to the Betti numbers. To complete the alternative basis of the Lie algebra A_n, we introduce the generators

$$\begin{aligned}F^i &= \Omega E^{i+1i}\Omega^{-1} \qquad , i = 1,\ldots,n \\ \left[\mathcal{E}_i, F^j\right] &= (\omega^{ij}-1)F^j\end{aligned} \tag{A.11}$$

Due to the grade decomposition (A.6) and the above commutation relations one gets

$$\begin{aligned}F^i &= \sum_{k\in\mathbb{Z}_{n+1}} F^i_k \\ \sigma(F^i_k) &= \omega^k F^i_k \\ \left[\mathcal{E}_i, F^j_k\right] &= (\omega^{ij}-1)F^j_{k+i}\end{aligned} \tag{A.12}$$

We thus end up with a graded basis which is formed by the alternative Cartan generators $\mathcal{E}_i$ (A.7) and $F^i_k,$, $i = 1, \ldots n,$, $k \in \mathbb{Z}_{n+1}$. Due to (A.8) and (A.11) the transformation which gives back the usual Cartan–Weyl basis is

$$
\begin{aligned}
E^{il} &= \frac{\mathcal{E}_{l-i}}{n+1} + \sum_{r=1}^{n} \omega^{r(1-i)} F^r_{l-i}, \quad i < l \\
E^{il} &= \frac{\mathcal{E}_{n+1+l-i}}{n+1} + \sum_{r=1}^{n} \omega^{r(1-i)} F^r_{n+1+l-i}, \quad i > l \\
E^{ii} - E^{n+1\ n+1} &= \sum_{r=1}^{n} \omega^r (\omega^{-ri} - 1) F^r_0
\end{aligned}
\tag{A.13}
$$

The commutation relations in the alternative basis are completed by

$$
[F^r_i, F^s_l] = \frac{\omega^{sk} - \omega^{rl}}{n+1} F^{r+s}_{k+l} \tag{A.14}
$$

Introducing notation

$$
F^r_k \; = \; \begin{cases} F^r_k & r = 1, \ldots, n; \quad k \in \mathbb{Z}_{n+1} \\[2mm] \frac{1}{n+1}\mathcal{E}_k & r = 0 \quad\quad k = 1, \ldots, n \end{cases} \tag{A.15}
$$

we see that the commutation relations in the alternative basis assume the umiform expression (A.14).

The A_n Lie algebras are equiped with a nondegenerated invariant scalar product $(X, Y) = tr(X.Y)$. The trace is taken in the defining representation. In the alternative basis (A.15) this scalar product is given by

$$
(F^r_k, F^s_l) = \frac{\omega^{sk}}{n+1} \delta^{(n+1)}_{k+l,0} \tag{A.16}
$$

where $\delta^{(n+1)_{kl}}$ is the delta function in the cyclic group $\mathbb{Z}_{n+1}$. To treat integrable evolution equations, one has to extend the classical Lie algebras by introducing a spectral (or loop) parameter [11]. This wants to say that the Lax connection belongs to the loop Lie algebra $\widetilde{\mathcal{G}} = \mathbb{C}[\lambda, \lambda^{-1}] \otimes \mathcal{G}$. In other words, the loop algebra is the set of the Laurent series with coefficients in the corresponding (classical) Lie algebra $\mathcal{G}$. Therefore $\widetilde{\mathcal{G}}$ is spanned on the elements $X_k = \lambda^k X,\ k \in \mathbb{Z},\ X \in \mathcal{G}$. The Lie bracket is

$$
[X_k, Y_l] \;=\; [X, Y]_{k+l} \tag{A.17}
$$

Loop algebras $\widetilde{\mathcal{G}}$ possess central extension [10], known as affine (or Kac–Moody) algebras $\hat{\mathcal{G}} = \widetilde{\mathcal{G}} \oplus \mathbb{C}\hat{c} \oplus \mathbb{C}\hat{d}$

$$
\begin{aligned}
&[X_k, Y_l] = [X, Y]_{k+l} + \frac{k}{(n+1)} \hat{c}\, \delta_{k+l,0} (X, Y) \\
&\left[\hat{d}, X_k\right] = k X_k \\
&\left[\hat{c}, \hat{\mathcal{G}}\right] = 0
\end{aligned}
\tag{A.18}
$$

The normalization factor which multiplies the central element $\hat{c}$ is chosen for convenience. The derivation $\hat{d}$ can be used to define a $\mathbb{Z}$ gradation in $\hat{\mathcal{G}}$:

$$\begin{aligned} \hat{\mathcal{G}} &= \oplus_{k\in\mathbb{Z}}\hat{\mathcal{G}}_k \\ \left[\hat{d},\hat{\mathcal{G}}_k\right] &= k\ \hat{\mathcal{G}}_k \end{aligned} \tag{A.19}$$

Since σ is an automorfism of the underlying classical Lie algebra $\mathcal{G} = A_n = sl(n+1)$, it is clear that the commutators (A.18) together with the restriction

$$\begin{aligned} X(\omega\lambda) &= \sigma X(\lambda) \\ X(\lambda) &= \sum_{l\in\mathbb{Z}} \lambda^l X_l \\ \sigma X(\lambda) &= \sum_l \sigma(X_l) \end{aligned} \tag{A.20}$$

still define a Lie algebra. In the literature [10, 12, 15] it is known as the affine Lie algebra $A_n^{(1)}$ in the principle gradation. It is known that this Lie algebra is isomorphic to $A_n^{(1)}$. Taking into account (A.14), (A.16), (A.18), togheter with the above restrictions, we obtain the commutation relations

$$\begin{aligned} [F_k^r, F_l^s] &= \frac{\omega^{sk}-\omega^{rl}}{n+1} F_{k+l}^{r+s} + \frac{k\omega^{rl}}{n+1}\ \hat{c}\ \delta_{r+s,0}^{(n+1)}\ \delta_{k+l,0} \\ r,s\ &\in \mathbb{Z}_{n+1};\quad k,l\in\mathbb{Z} \end{aligned} \tag{A.21}$$

of the $A_n^{(1)}$ Lie algebra in the principal gradation. Note that the elements F_k^0 for $k \neq 0 \bmod (n+1)$ decoupled since they commute with all the generators. Therefore, one can set

$$F_{p(n+1)}^0 \equiv 0\quad ,\ p\in\mathbb{Z} \tag{A.22}$$

The generators F_k^0, $k \neq 0(\mathrm{mod})$ form a basis in the well studied Heisenberg subalgebra in the *principal* gradation [10, 11, 12, 15]. In analogy with (A.17), we will also use the following generators

$$\begin{aligned} &\mathcal{E}_k = (n+1)F_k^0 \\ &[\mathcal{E}_k,\mathcal{E}_l] = (n+1)\ k\ \delta_{k+l,0} \quad k\neq 0\,(\bmod\ n+1) \end{aligned} \tag{A.23}$$

The above commutation relations indicate that the $\mathcal{E}_k$'s are a collection of free bosonic oscillators.

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