CBPF-NF-061/93

Introduction to Quantum Groups

by

M.R-Monteiro

Centro Brasileiro de Pesquisas Físicas — CBPF/CNPq
Rua Dr. Xavier Sigaud, 150
22290-180 – Rio de Janeiro, RJ – Brasil

Abstract

We present an elementary introduction to Quantum Groups. The example of Universal Enveloping Algebra of deformed $SU(2)$ is analysed in detail. We also discuss systems made up off bosonic q-oscillators at finite temperature within the formalism of Thermo-Field Dynamics.

Key-words: Quantum algebras; Deformed systems.

Introduction

Interesting examples of Quantum Groups [1–4], or Quasi-Triangular Hopf Algebras, are deformations of Lie groups or Lie Algebras through a parameter q, real or imaginary, such that one recovers the non-deformed structures in the limit $q \to 1$. In the past few years they have attracted considerable interest and have found applications in several areas of physics [4–10] such as: inverse scattering method, vertex models, anisotropic spin chain Hamiltonians, knot theory, conformal field theory, quantum field theory, heuristic phenomenology of deformed molecules and nuclei, non-commutative approach to quantum gravity and anyon physics.

In this talk we shall concentrate on some introductory aspects of Quantum Groups. In section I we introduce the mathematical elements in order to give a consistent definition of Quantum Groups. In section II we discuss the elements introduced in the previous section for the simple case of $SU_q(2)$. In section III we discuss two different realizations of $SU_q(2)$. In the first part we consider the realization à la Schwinger with q-oscillators and in the second part of this section the anyonic realization recently developed. In section IV we give a brief introduction of deformed systems at finite temperature using the formalism of Thermo-Field Dynamics (TFD) [11–13].

1 Definition of Quantum Group

Let us consider an associative algebra $\mathcal{A}$ with a unit element $\mathbf{1}$. One can define two maps over this algebra; the coproduct $\Delta : A \to A \otimes A$, and the counit map $\varepsilon : \mathcal{A} \to k$, with k the field.

The coproduct is required to be coassociative which means:

$$(\Delta \otimes id) \circ \Delta = (id \otimes \Delta) \circ \Delta \ , \tag{1.1}$$

and the counit must satisfy $(\varepsilon \otimes id) \circ \Delta(a) = a = (id \otimes \varepsilon) \circ \Delta(a)$, $\forall a \in \mathcal{A}$. By definition the algebra $\mathcal{A}$ with the mappings Δ and ε is called a coalgebra $\mathcal{C}$.

If the mapping above defined has the compatibility properties

$$\Delta(ab) = \Delta(a)\Delta(b) \quad , \quad \Delta(\mathbf{1}) = \mathbf{1} \otimes \mathbf{1} \quad , \quad \varepsilon(ab) = \varepsilon(a)\varepsilon(b) \quad , \quad \varepsilon(\mathbf{1}) = 1 \ , \tag{1.2}$$

the coalgebra $\mathcal{C}$ is called a bialgebra $\mathcal{B}$.

A Hopf algebra $\mathcal{H}$ over k is bialgera over k equipped with an antipode map $S : \mathcal{H} \to \mathcal{H}$ obeying

$$m(s \otimes id) \circ \Delta(h) = m(id \otimes s) \circ \Delta(h) = \mathbf{1}\varepsilon(h) \ , \tag{1.3}$$

$\forall h \in \mathcal{H}$, with m the product $m : (\mathcal{H} \otimes \mathcal{H}) \to \mathcal{H}$.

The coproduct can always be written in the form:

$$\Delta(h) = \sum_i h_i^{(1)} \otimes h_i^{(2)} \tag{1.4}$$

where the right-hand side is a formal sum denoting an element of $\mathcal{H} \otimes \mathcal{H}$.

The twist map $\tau : \mathcal{H}_1 \otimes \mathcal{H}_2 \to \mathcal{H}_2 \otimes \mathcal{H}_1$ is defined by $\tau(h_2 \otimes h_1) = h_1 \otimes h_2$ with $h_1, h_2 \in \mathcal{H}$, and a Hopf algebra it is called cocommutative if, $\tau \circ \Delta = \Delta$.

Finally a Quantum Group (or a Quasi-Triangular Hopf Algebra) is Hopf algebra with an invertible matrix $\mathcal{R} \in \mathcal{H} \otimes \mathcal{H}$ satisfying

$$\begin{aligned} (\Delta \otimes id)\mathcal{R} &= \mathcal{R}_{13}\mathcal{R}_{23} \quad , \quad (id \otimes \Delta)\mathcal{R} = \mathcal{R}_{13}\mathcal{R}_{12} && (5.a) \\ \tau \circ \Delta h &= \mathcal{R}(\Delta h)\mathcal{R}^{-1} \quad , \quad \forall h \in \mathcal{H} && (5.b) \end{aligned}$$

with $\mathcal{R} = \sum_i a_i \otimes b_i$, $\mathcal{R}_{13} = \sum_i a_i \otimes \mathbf{1} \otimes b_i$, $\mathcal{R}_{23} = \sum_i \mathbf{1} \otimes a_i \otimes b_1$ and $\mathcal{R}_{12} = \sum_i a_i \otimes b_i \otimes \mathbf{1}$, where $a, b \in \mathcal{H}$.

The axioms (1.4.a,b) imply the Quantum Yang-Baxter equations

$$\mathcal{R}_{12}\mathcal{R}_{13}\mathcal{R}_{23} = \mathcal{R}_{23}\mathcal{R}_{13}\mathcal{R}_{12} \ . \tag{6}$$

This can be easily proven by using the operation $(\tau \circ \Delta \otimes id)\mathcal{R}$ in two ways, in one way one uses the axiom (1.5.a) and the definition of τ and in the other way by using the axioms (1.5.b) followed by (1.5.a), and comparing the two ways.

It is interesting to notice that the meaning of the axiom (1.5.b) is that though the quasi-triagular Hopf algebra is not usually cocommutative, the lack of cocommutativity is under control, being controlled by $\mathcal{R}$.

2 $SU_q(2)$ as an Example

Let us recall the $SU(2)$ algebra in the Cartan basis

$$[j_0, j_\pm] = \pm j_\pm \quad \text{and} \quad [j_+, j_-] = 2j_0 \tag{1}$$

and one considers the universal enveloping algebra (UEA) of $SU(2)$ as the algebra generated by $\mathbf{1}$ and the elments of $SU(2)$. It is possible to verify that the UEA of $SU(2)$ is endowed with a Hopf structure if one defines

$$\begin{aligned} \Delta(\mathbf{1}) &= \mathbf{1} \otimes \mathbf{1} \ , \ \Delta J_\alpha = J_\alpha \otimes \mathbf{1} + \mathbf{1} \otimes J_\alpha \ , \ \varepsilon(\mathbf{1}) = 1 \ , && (2) \\ S(\mathbf{1}) &= \mathbf{1} \ , \ S(J_\alpha) = -J_\alpha \ . \end{aligned}$$

Notice that as $\tau \circ \Delta = \Delta$, the UE Hopf algebra of $SU(2)$ is not quasi-triangular and the coproduct in this case can be interpreted as the sum of momentum angular operators.

Now, one can deform the above structure in order to get a quasi-triangular structure. To this end one introduces a non-cocommutative coproduct Δ as

$$\Delta q^{J_0} = q^{J_0} \otimes q^{J_0} \quad , \quad \Delta J_\pm = J_\pm \otimes q^{J_0} + q^{-J_0} \otimes J_\pm \tag{3}$$

where the second one in (2.3) is different from (2.2) showing the non-cocommutativity, and q is a general complex number.

It in easy to show that using the compatibility properties (1.2) and the coproducts (2.3) we have

$$\begin{aligned}\Delta([J_o, J_\pm]) &= [J_0, J_\pm] \otimes q^{J_0} + q^{-J_0} \otimes [J_0, J_\pm] &(4.a)\\ \Delta([J_+, J_-]) &= [J_+, J_-] \otimes q^{2J_0} + q^{-2J_0} \otimes [J_+, J_-] &(4.b)\end{aligned}$$

thus from (2.4.a,b) we see that the algebra has to be modified in order to be consistent with the modified coproduct (2.3), it is easy to see that

$$[J_0, J_\pm] = \pm J_\pm \quad , \quad [J_+, J_-] = [2J_0] \tag{5}$$

with $[x] = (q^x - q^{-x})/(q - q^{-1})$, is consistent with the coproduct (2.3), using equations (2.4.a,b). The denominator in the definition of $[2J_0]$ is chosen in ordor to obtain in the limit $q \rightarrow 1$ the non-deformed case, $2J_0$.

One can show that the mappings which endow the UEA of deformed $SU(2)$, or $SU_q(2)$ (2.5), with a Hopf algebra are given by

$$\varepsilon(J_\alpha) = 0 \quad , \quad S(J_\pm) = -q^{\pm 1} J_\pm \quad \text{and} \quad S(q^{\pm J_0}) = q^{\pm J_0} \tag{6}$$

The matrix $\mathcal{R}$ given by

$$\mathcal{R} = q^{2J_0 \otimes J} \sum_{n=0}^{\infty} \frac{(1-q^{-2})^n}{[n]!} \left(q^{nJ_0} (J_+)^n \otimes q^{-nJ_0} (J_-)^n \right) \tag{7}$$

with $[n]! = [n] \cdots [1]$ satisfies the quantum Yang-Baxter equations (1.6) and endow $SU_q(2)$ with a Quantum Group structure.

3 Two Different Realizations of $SU_q(2)$

One calls bosonic q-oscillators (or deformed Heisenberg algebra) [14–19] the associative algebra generated by the elements α, α^+ and N satisfying the relations

$$\begin{aligned}[N, \alpha^+] &= \alpha^+ \quad , \quad [N, \alpha] = -\alpha &(1)\\ [\alpha, \alpha^+]_\alpha &= f_\alpha(N)\ .\end{aligned}$$

We are going to consider here the following forms of the above algebra (3.1):

$$\begin{aligned}[a, a^+]_a &\equiv aa^+ - qa^+a = q^{-N} &(2.a)\\ [A, A^+]_A &\equiv AA^+ - q^2A^+A = 1 &(2.b)\end{aligned}$$

which are related to each other via

$$A = q^{N/2} a \quad , \quad A^+ = a^+ q^{N/2} \tag{3}$$

in the case of q real.

It is possible to construct the representation of relation (3.2) in the Fock space $\mathcal{F}$ generated by the normalized eigenstates $|n>$ of the number operator N as

$$\begin{aligned} \alpha|0\rangle &= 0 \quad , \quad N|n\rangle = n \qquad n = 0,1,2,\cdots \\ |n\rangle &= \frac{1}{\sqrt{[n]_\alpha !}}(\alpha^+)^n|0\rangle \end{aligned} \tag{4}$$

where $[n]_\alpha! \equiv [n]_\alpha \cdots [1]_\alpha$, $[n]_a = (q^n - q^{-n})/(q-q^{-1})$ and $[n]_A = (q^{2n}-1)/(q^2-1)$.

In $\mathcal{F}$ it is possible to express the deformed oscillators in terms of the standard bosonic ones b, b^+ as

$$\alpha = \left(\frac{[n+1]_\alpha}{N+1}\right)^{1/2} b \quad , \quad \alpha^+ = b^+ \left(\frac{[n+1]_\alpha}{N+1}\right)^{1/2} \tag{5}$$

and it can easily be shown in $\mathcal{F}$ that

$$\alpha\alpha^+ = [N+1]_\alpha \quad , \quad \alpha^+\alpha = [N]_\alpha \ . \tag{6}$$

If we now consider two independent q-oscillators, for instance a_1, a_2, one can realize the $SU_q(2)$ algebra à la Schwinger as

$$\begin{aligned} J_+ &= a_1^+ a_2 \quad , \quad J_- = a_2^+ a_1 \\ J_0 &= \frac{1}{2}(N_1 - N_2) \neq \frac{1}{2}(a_1^+ a_1 - a_2^+ a_2) \ . \end{aligned} \tag{7}$$

Further with

$$n_1 = j + m \qquad n_2 = j - m \tag{8}$$

one can define the related realizations of the $|jm\rangle$ basis of $SU_q(2)$ by means of

$$|jm\rangle = |n_1\rangle|n_2\rangle = \frac{(a_1^+)^{j+m}}{\sqrt{[j+m]_a!}} \ \frac{(a_2^+)^{j-m}}{\sqrt{[j-m]!}}|0\rangle \ . \tag{9}$$

Analogously to the above construction for $SU_q(2)$, all the deformed algebras of type A, B, C e D [17], the quantum superalgebras [16] and the deformed exceptional algebras [18] can be realized à la Schwinger.

Anyons [10–22] are two-dimensional objects with arbitrary statistics interpolating between bosons and fermions. We are going now to describe a different construction of the deformed algebra $SU_q(2)$ by means of anyonic oscillators [8-9,20-25] which are non-local operators defined only on a two-dimensional manifold.

We start by considering a two-dimensional lattice with spacing $a = 1$. On a Lattice it is possible to define an angle function analogously to the continuum case. To each point $\vec{x}$ of the lattice Ω, one defines a cut made of bonds on the dual lattices Ω^* from $-\infty$ to $\vec{x}^* = \vec{x} + \vec{0}^*$, with $\vec{0}^* = (1/2, 1/2)$, parallel to the x-axis. The point $\vec{x}$ and its associated cut we call $\vec{x}_\gamma$.

One can define a function $\theta\gamma_x(\vec{x},\vec{y})$ as the lattice analogue of the angle under which the point $\vec{x}$ is seen from $\vec{y}$. By neglecting lattice features it is possible to show

$$\theta_{\gamma_x}(\vec{x},\vec{y}) - \theta_{\gamma_y}(\vec{y},\vec{x}) = \begin{cases} \pi sgn(x_2 - y_2) & \text{for } x_2 \neq y_2 \\ \pi sgn(x_1 - y_1) & \text{for } x_2 = y_2 \ . \end{cases} \tag{10}$$

which is similar to the continuum case.

Equation (3.10) can be used to define an ordering relation among the points of the lattice which will be important to define later on anyonic operators. Given two distinct points of the lattice with their associated cuts, $\vec{x}_\gamma$ and $\vec{y}_\gamma$, one can postulate

$$\vec{x}_\gamma > \vec{y}_\gamma \Leftrightarrow \theta_{\gamma_x}(\vec{x},\vec{y}) - \theta_{\gamma_y}(\vec{y},\vec{x}) = \pi \ , \tag{11}$$

which is equivalent to

$$\vec{x}_\gamma > \vec{y}_\gamma \Leftrightarrow \begin{cases} x_2 > y_2 \\ x_2 = y_2 \ , \ x_1 > y_1 \end{cases} . \tag{12}$$

If one considers another cut δ made with bonds of the dual lattice Ω^* from $+\infty$ to ${}^*\vec{x} = \vec{x} + {}^*\vec{0}$, with ${}^*\vec{0} = (-1/2, -1/2)$ and defines $\vec{x}_\delta$ as the point $\vec{x}$ with its associated δ-cut, it is possible to define another lattice angle function for these points $\vec{x}_\delta$ which has the following property:

$$\tilde{\theta}_{\delta_x}(\vec{x},\vec{y}) - \tilde{\theta}_{\delta_y}(\vec{y},\vec{x}) = \begin{cases} -\pi sgn(x_2 - y_2) & \text{for} \quad x_2 \neq y_2 \\ -\pi sgn(x_1 - y_2) & \text{for} \quad x_2 = y_2 \end{cases} . \tag{13}$$

Type δ and γ lattice angles can be related, if $\vec{x} \neq \vec{y}$, as

$$\tilde{\theta}_{\delta_x}(\vec{x},\vec{y}) - \theta_{\gamma_x}(\vec{x},\vec{x}) = \begin{cases} -\pi sgn(x_2 - y_2) & \text{for} \quad x_2 \neq y_2 \\ -\pi sgn(x_1 - y_1) & \text{for} \quad x_2 = y_2 \end{cases} \tag{14.a}$$

$$\tilde{\theta_x}(\vec{x},\vec{y}) - \theta_{\gamma_y}(\vec{y},\vec{x}) = 0 \tag{3.14.b}$$

and

$$\tilde{\theta}_{\delta_x}(\vec{x},\vec{x}) - \theta_{\gamma_x}(\vec{x}\vec{x}) = 0 \ . \tag{15}$$

Type-γ anyons can be defined using the lattice angle function θ_γ as:

$$a_i(\vec{x}_\gamma) = K_i(\vec{x}_\gamma) C_i(\vec{x}) \tag{16}$$

with

$$K_i(\vec{x}_\gamma) = e^{i\nu \sum\limits_{\substack{\vec{y}\in\Omega \\ \vec{y}\neq\vec{x}}} \theta_{\gamma_x}(\vec{x},\vec{y}) C_i^+(y) C_i(y)} \tag{17}$$

disorder operators, $C_i(\vec{x})$ are fermionic oscillators defined on the lattice Ω obeying the standard anticommutation relations

$$\{C_i(\vec{x}), C_j^+(\vec{y})\} = \delta_{ij}\delta(\vec{x},\vec{y}) \tag{18}$$

where

$$\delta(\vec{x},\vec{y}) = \begin{cases} 0 & \text{if} \quad \vec{x} \neq \vec{y} \\ 1 & \text{if} \quad \vec{x} = \vec{y} \end{cases} , \tag{19}$$

and ν a real parameter which, as we shall see, represents the statistics.

Using (3.10), (3.16-18) one can show for $\vec{x} > \vec{y}$, which from now on will represent $\vec{x}_\gamma > \vec{y}_\gamma$,

$$a_i(\vec{x}_\gamma) a_i(\vec{y}_\gamma) + q^{-1} a_i(\vec{y}_\gamma) a_i(\vec{x}_\gamma) = 0 \tag{20.a}$$

$$a_i(\vec{x}_\gamma) a_i^+(\vec{y}_\gamma) + q a_i^+(\vec{y}_\gamma) a_i(\vec{x}_\gamma) = 0 \ , \tag{20.b}$$

and their hermitean conjugate, with $q = e^{i\nu\pi}$. For completeness we recall that

$$[a_i(\vec{x}_\gamma)]^2 = [a_i^+(\vec{x}_\gamma)]^2 = 0 \tag{21}$$

and

$$\{a_i(\vec{x}_\gamma), a_j(\vec{y}_\gamma)\} = \{a_i(\vec{x}_\gamma), a_j^+(\vec{y}_\gamma)\} = 0 \tag{22}$$

for $i \neq j$. At the same point one has

$$a_i(\vec{x}_\gamma)a_i^+(\vec{x}_\gamma) + a_i^+(\vec{x}_\gamma)a_i(\vec{x}_\gamma) = 1 \tag{23}$$

without any phase factor, showing that anyonic operators obey the standard anticommutation relations at the same point.

Type-δ anyons can also be defined as

$$a_i(\vec{x}_\gamma) = K_i(x_\delta)C_i(\vec{x}) \tag{24}$$

with

$$K_i(x_\delta) = e^{i\nu \sum\limits_{\substack{\vec{y}\in\Omega \\ \vec{y}\pm\vec{x}}} \tilde{\theta}_x(\vec{x},\vec{y})C_i^+(\vec{y})C_i(\vec{y})} \quad . \tag{25}$$

If one computes now the braiding relations among type-δ anyons one obtains the same braiding relations obeyed by type-γ anyons with $q \leftrightarrow q^{-1}$. One can say that type-γ and type-γ anyons are related by a parity transformation.

The relation between type γ and δ anyons can be obtained, one finds

$$\{a_i(\vec{x}_\delta), a_i(\vec{y}_\gamma)\} \;=\; 0 \quad \forall \vec{x},\vec{y} \tag{26.a}$$

$$\{a_i(\vec{x}_\delta), a_i^+(\vec{y}_\gamma)\} \;=\; 0 \quad \forall \vec{x} \neq \vec{y}\ , \tag{26.b}$$

and at the same lattice point one has

$$\{a_i(x_\delta), a_i^+(x_\gamma)\} = q^{-\left[\sum\limits_{\vec{y}<\vec{x}} - \sum\limits_{\vec{y}>\vec{x}}\right] C_i^+(\vec{y})C_i(\vec{y})} \quad . \tag{27}$$

We are now going to realize, in a Schwinger like construction, the $SU_q(2)$ algebra with the anyonic operators we have just defined. If one considers the density of quantum group generators, it is well-known, for instance, for $SU_q(2)$ that [29]

$$J_q^-(\vec{x}) = [J_{q^{-1}}^+(\vec{x})]^+ \tag{28}$$

where $J_q^\pm = \sum\limits_{x\in\Omega} J_q^\pm(x)$ and $J_q^\pm$ being the generator of the quantum group $SU_q(2)$, in the case of $q^* = q^{-1}$. As we know that type γ and δ anyonic operators are related by $q \leftrightarrow q^{-1}$ we are led to assume

$$J_+(\vec{x}) \;=\; a_1^+(\vec{x}_\gamma)a_2(\vec{x}_\gamma) \tag{29.a}$$

$$J_-(\vec{x}) \;=\; a_2^+(\vec{x}_\delta)a_1(\vec{x}_\delta) \tag{29.b}$$

$$\begin{aligned} J_0(\vec{x}) \;&=\; \frac{1}{2}\left(a_1^+(\vec{x}_\gamma)a_1(\vec{x}_\gamma) - a_2^+(\vec{x}_\gamma)a_2(\vec{x}_\gamma)\right) = \\ &=\; \frac{1}{2}\left(a_1^+(\vec{x}_\gamma)a_1(\vec{x}_\gamma) - a_2^+(\vec{x}_\delta)a_2(\vec{x}_\delta)\right)\ , \end{aligned} \tag{29.c}$$

for the density of quantum group generators inspired by the Schwinger construction. The last equality in (3.29c) comes from the cancellation of the disorder operators.

With a straightforward application of (3.20-23), (3.26-27) one gets

$$\begin{aligned} [J_0(\vec{x}), J_\pm(\vec{y})] &= \pm J^\pm(\vec{x})\delta(\vec{x},\vec{y}) \\ [J_+(\vec{x}), J_-(\vec{y})] &= \delta(\vec{x},\vec{y}) \prod_{\vec{z}<\vec{x}} q^{-2J_0(\vec{z})} 2J_0(\vec{x}) \prod_{\vec{\omega}>\vec{x}} q^{2J_0(\vec{\omega})} . \end{aligned} \tag{30}$$

Defining the generators as

$$\begin{aligned} J_\pm &= \sum_{\vec{x}\in\Omega} J_\pm(\vec{x}) \\ J_0 &= \sum_{\vec{x}\in\Omega} J_0(\vec{x}) \end{aligned} \tag{31}$$

one obtains

$$[J_0, J_\pm] = \pm J_\pm \tag{32.a}$$

$$[J_+ J_-] = \sum_{\vec{x}\in\Omega} \left(\prod_{\vec{y}<\vec{x}} q^{-2J_0(\vec{y})} 2J_0(\vec{x}) \prod_{\vec{z}>\vec{x}} q^{2J_0(\vec{z})} \right) \tag{32.b}$$

Recalling that

$$\Delta[2J_0] = [2J_0] \otimes q^{2J_0} + q^{-2J_0} \otimes [2J_0] \tag{33}$$

and as $J_0(\vec{x})$ is in the spin 0 or spin 1/2 representation for any $\vec{x}$, (3.32.b) can be rewritten as

$$[J_+, J_-] = [2J_0] \tag{34}$$

the usual formula found in the quantum group literature.

All the deformed algebras of the type A, B, C, D [9] and the quantum semialgebra $SL_{q,s}(2)$ [30] can be realized with anyonic operators.

4 Deformed Systems at Finite Temperature

In the formalism of TFD [11–13] one constructs a temperature dependent vacuum $|0(\beta)\rangle$, such that the statistical average of $\hat{O}$ coincides with the vacuum expectation value using this new vacuum $|0(\beta)\rangle$. For instance if one uses the canonical ensemble one has

$$\langle \hat{O} \rangle \equiv Z^{-1}(\beta) Tr[e^{-\beta H}\hat{O}] = \langle 0(\beta)|\hat{O}|(\beta)\rangle \tag{1}$$

with $\beta = (k_B T)^{-1}$ and k_B the Boltzmann constant.

Let $\{|n>\}$ be the orthonormal basis of the state vector space $\mathcal{H}$ consisting of eigenstates of the Hamiltonian H

$$\begin{aligned} H|n\rangle &= E_n|n\rangle \\ \langle m|n\rangle &= \delta_{m,n} \end{aligned} \tag{2}$$

In order to construct such a state $|0(\beta)\rangle$ one introduces a fictitions system (tilde system) characterized by the Hamiltonian $\tilde{H}$ and the state vector space $\tilde{\mathcal{H}}$ spanned by $\{|n\rangle\}$ obeying

$$\begin{aligned} \tilde{H}|n\rangle &= E_n|n\rangle \\ \langle n|m\rangle &= \delta_{m,n} \end{aligned} \tag{3}$$

The thermal vacuum $|0(\beta)\rangle$ belongs to tensor product space $\mathcal{H}\otimes\tilde{\mathcal{H}}$ and is given by

$$|0(\beta)\rangle = Z^{-1/2}(\beta)\sum_n e^{-\beta E_n/2}|n\rangle\otimes|n\rangle \equiv Z^{-1/2}(\beta)\sum_n e^{-\beta E_n/2}|n,\tilde{n}\rangle\ . \tag{4}$$

If one uses (4.4) in (4.1) one has

$$\begin{aligned} \langle 0(\beta)|\hat{O}|0(\beta)\rangle &= Z^{-1}(\beta)\sum_{n.m} e^{-\beta E_n/2}e^{-\beta E_m/2}\langle\tilde{n},n|\hat{O}|m,\tilde{n}\rangle \\ &= Z^{-1}(\beta)\sum_n e^{-\beta E_n}\langle n|\hat{O}|n\rangle = \langle\hat{O}\rangle \end{aligned} \tag{5}$$

which is the result claimed in (4.1). This doubling of degrees of freedom has a sensible physical interpretation and is related to the algebraic formulation of Statistical Mechanics developed by Haag, Hugenholtz and Winnink [31].

This formalism can be extended [32] to the case of statistical averages of systems made up of q-oscillators [33–37] considered in the previous section. Let us now consider an ensemble of q-bosons, satisfying the algebras (3.1-2), with Hamiltonian given by

$$H = \omega N \tag{6}$$

with eigenvalue $\omega n (n = 0, 1\cdots)$ on $\mathcal{F}$. We introduce the Hamiltonian of the tilde system as

$$\tilde{H} = \omega\tilde{N} \tag{7}$$

where the tilde q-oscillators we are considering satisfy the following relations

$$\begin{aligned} [\tilde{N},\tilde{\alpha}^+] &= \tilde{\alpha}^+ \quad , \quad [\tilde{N},\tilde{\alpha}] = -\tilde{\alpha} \\ [\tilde{\alpha},\tilde{\alpha}^+] &= f_{\tilde{\alpha}}(\tilde{N}) \end{aligned} \tag{8}$$

where, in the cases we are going to consider here, we have

$$\begin{gathered} [\tilde{a},\tilde{a}^+]_{\tilde{a}} \equiv a\tilde{a}^+ - q\tilde{a}^+\tilde{a} = q^{-\tilde{N}} \\ [\tilde{A},\tilde{A}^+]_A \equiv \tilde{A}\tilde{A}^+ - q^2\tilde{A}^+\tilde{A} = 1 \end{gathered} \tag{9}$$

and $[\alpha,\tilde{\alpha}] = [\alpha,\tilde{\alpha}^+] = 0$. The temperature dependent vacuum $|0(\beta)\rangle$ is thus given by

$$\begin{aligned} |0(\beta)\rangle &= Z^{-1/2}(\beta)\sum_{n=0}^{\infty} e^{-\beta n\omega/2}\frac{1}{[n]_\alpha!}(\alpha^+)^n(\tilde{\alpha}^+)^n|0\rangle = \\ &= (1-e^{-\beta\omega})^{1/2}\exp_{q_\alpha}(e^{-\beta\omega/2}\alpha^+\tilde{\alpha}^+)|0\rangle \end{aligned} \tag{10}$$

with $\exp_{q_\alpha} x = \sum_{n=0}^{\infty} \frac{1}{[n]_\alpha !} x^n$ the q-exponential [38], and $|0\rangle = |0\rangle \otimes |0\rangle$. In the formula (4.10) we used the information that the partition function of q-bosons [37] corresponding to the Hamiltonian (4.6) coincides with the usual one for harmonic oscillators. We can easily see that the non-deformed case is recovered in the $q \rightarrow$ limit.

The thermal vacuum, $|0(\beta)\rangle$, can be related to the usual one, $|0\rangle$, by means of a unitary transformation which resembles a Bogoliubov transformation. This transformation can be used to define the temperature dependent operators $\alpha_\beta(\tilde{\alpha}_\beta)$, $\alpha^+_\beta(\tilde{\alpha}^+_\beta)$, $N_\beta(\tilde{N}_\beta)$, and the "thermal" Fock space can be constructed by applying this transformation on (3.4) leading to

$$|n, \tilde{m}\rangle_\beta = \frac{1}{\sqrt{[n]_\alpha !}} \frac{1}{\sqrt{[m]_\alpha !}} (\alpha^+_\beta)^n (\tilde{\alpha}^+_\beta)^m |0(\beta)\rangle \tag{11}$$

for $n = 0, 1, \cdots$.

We are going now to sketch the computation of the average of $\alpha^+\alpha$ [32] which, as we are going to see, depends on the deformation considered. In the TFD approach this average is given by

$$\langle \alpha^+ \alpha \rangle = \langle 0(\beta)|\alpha^+\alpha|0(\beta)\rangle = \langle 0(\beta)|[N_\alpha]|0(\beta)\rangle \; . \tag{12}$$

In order to perform this calculation we go to the basis of the non-deformed bosonic operators. In this basis expressing the number operator in terms of the temperature dependent operator we have

$$N = (u^2_\beta + v^2_\beta) j_0 + u_\beta v_\beta (j_+ + j_-) + \frac{1}{2}\, C \tag{13}$$

where

$$\begin{aligned} u_\beta &= (1 - e^{-\beta\omega})^{-1/2} \\ v_\beta &= (e^{\beta\omega} - 1)^{-1/2} \end{aligned} \tag{14}$$

$j_\pm, j_0$ are the generators of hidden $SU(1,1)$ algebra

$$j_+ = b^+_\beta \tilde{b}^+_\beta \quad , \quad j_- = \tilde{b}_\beta b_\beta \quad , \quad j_0 = \frac{1}{2}(N_\beta + \tilde{N}_\beta + 1) \tag{15}$$

and

$$C = N_\beta - \tilde{N}_\beta - 1 \; . \tag{16}$$

Notice that C commutes with the first two terms of the right-hand side of (4.13).

Using now (4.13) and taking $q^m = \exp \lambda$, the relevant terms in the calculation of (4.12) have the form

$$\langle 0(\beta)|e^{\lambda N}|0(\beta)\rangle = \langle 0(\beta)|e^{\lambda[(u^2_\beta + v^2_\beta) j_0 + u_\beta v_\beta (j_+ + j_-) + \frac{1}{2} C]}|0(\beta)\rangle \; . \tag{17}$$

This last expression can be computed by means of the Backer-Campbell- Hausdorf (BCH) formula, which can be derived for the $SU(1,1)$ algebra [39]. The BCH formula for this case yields

$$e^{\lambda[(u^2_\beta + v^2_\beta) j_0 + u_\beta v_\beta (j_+ j_-)]} = e^{\rho j_+} e^{\gamma j_0} e^{\rho j_-} \tag{18}$$

with

$$\begin{aligned} \rho &= 2v_\beta \sinh(\lambda/2)/[\cosh(\lambda/2) - (u_\beta^2 + v_\beta^2)\sinh(\lambda/2)] \\ \gamma &= -2\ln[\cosh(\lambda/2) - (u_\beta^2 + v_\beta^2)\sinh(\lambda/2)] \ . \end{aligned} \tag{19}$$

The above procedure amounts to normal ordering eq. (4.17). Using (4.18-19) in (4.17) we have

$$\langle 0(\beta)|q^{mN}|0(\beta)\rangle = \frac{2q^{m/2}}{q^{m/2} + q^{m/2} - (u_\beta^2 + v_\beta^2)(q^{m/2} - q^{-m/2})} \ , \tag{20}$$

and finally using (4.12) and (4.20) we can easily see that

$$\begin{aligned} \langle 0(\beta)|a^+ a|0(\beta)\rangle &= \frac{e^{\beta\omega} - 1}{e^{2\beta\omega} - (q + q^{-1})e^{\beta\omega} + 1} \\ \langle 0(\beta)|A^+ A|0(\beta)\rangle &= \frac{1}{e^{\beta\omega} - q^2} \end{aligned} \tag{21}$$

The entropy of deformed systems with more complicated Hamiltonians than the one given by (4.6) can be computed in general only for q close to one [32], this is understood in the formalism of TFD by observing that the transformation from the non-deformed thermal vacuum to the defomed one is highly non-trivial [32], being simple only when the deformed and non-deformed Hamiltonians have the same spectrum.

References

[1] V.G. Drinfeld, Sov. Math. Dokl. **32** (1985) 254;

[2] M. Jimbo, Lett. Math. Phys. **10** (1985) 63; **11** (1986) 247;

[3] L.D. Faddeev, N. Yu Reshetikhin and L.A. Takhtadzhyan, Algebra and Analysis **1** (1987) 178;

[4] For reviews see for instance: S. Majid, Int. J. Mod. Phys. **A5** (1990) 1; P. Aschieri and L. Castellani, Int. J. Mod. Phys. **A8** (1993) 1667;

[5] C. Zachos, "Paradigms of Quantum Algebras", Preprint ANL-HEP-PR-90-61;

[6] J.L. Matheus-Valle and M.R-Monteiro, Mod. Phys. Lett. **A7** (1992) 3023; Phys. Lett. **B66** (1993) 330;

[7] L. Castellani and M.R-Monteiro, Phys. Lett. **B314** (1993) 25;

[8] A. Lerda and S. Sciuto, Nucl. Phys. **B401** (1993) 613;

[9] R. Caracciolo and M.R- Monteiro, Phys. Lett. **B308** (1993) 58; M. Frau, M.R-Monteiro and S. Sciuto, "q-Deformed Lie Algebras and Their Anyonic Realization", Preprint DFTT 16/93, CBPF-NF-026/93;

[10] M. Chaichian, R. Gonzalez Felipe and C. Montonen, "On the Class of Possible Non-Local Anyon-Like Operators and Quantum Groups", to appear in J. Phys. **A**;

[11] Y. Takahashi and H. Umezawa, Collective Phenomena **2** (1975) 55;

[12] H. Umezawa, H. Matsumoto and M. Tachiki, "Thermo-Field Dynamics and Condensed States", North-Holland, Amsterdam, 1982;

[13] N.P. Landsmann and Ch. G. van Weert, Phys. **145** (1987) 141;

[14] V.V. Kuryshkin, Ann. Found. L. de Braglie **5** (1980) 111;

[15] A.J. Macfarlane, J. Phys. **A22** (1989) 4581; L.C. Biedenharn, J. Phys. **A22** (1989) L873;

[16] M. Chaichian and P. Kulish, Phys. Lett. **B234** (1990) 72;

[17] T. Hayashi, Comm. Math. Phys. **127** (1990) 129; C.P. Sun and H.C. Fu, J. Phys. **A22** (198) L983;

[18] L. Frappat, P. Sorba and A. Sciarrino, J. Phys. **A24** (1991) L179;

[19] P. Kulish and E. Damaskinsky, J. Phys. **A23** (1990) L415; A. Polychronakos, Mod. Phys. Lett. **A5** (1990) 2325;

[20] J.M. Leinaas and J. Myrheim, Nuov. Cim. **B37** (1977) 1;

[21] F. Wilczek, Phys. Rev. Lett. **48** (1982) 114;

[22] For reviews see for instance: F. Wilczek, in "Fractional Statistics and Anyon Superconductivity", edited by F. Wilczek, World Scientific Publishing Co., Singapore 1990; A. Lerda, "Anyons: Quantum Mechanics of Particles with Fractional Statistics", Springer-Verlarg, Berlin, Germany 1992;

[23] E. Fradkin, Phys. Rev. Lett. **63** (1989) 322;

[24] M. Lüscher, Nucl. Phys. **B326** (1989) 557;

[25] V.F. Müller, Z. Phys. **C47** (1990) 301;

[26] D. Eliezer and G. Semenoff, Phys. Lett. **B266** (1991) 375;

[27] D. Eliezer, G. Semenoff and S. Wu, Mod. Phys. Lett. **A7** (1992) 513;

[28] D. Eliezer and G. Semenoff, Ann. Phys. **217** (1992) 66;

[29] V. Pasquier and H. Saleur, Nucl. Phys. **B330** (1990) 523;

[30] J.L. Matheus-Valle and M.R-Monteiro, work in progress;

[31] R. Haag, N. Hugenholtz and M. Winnink, Comm. Math. Phys. **5** (1967) 215;

[32] M.R-Monteiro and I. Roditi, "Thermo-Field Dynamics of Deformed Systems", preprint CBPP-NF-037/93; "Deformed Systems at Finite Temperature", preprint CBPF-NF-060/93;

[33] M. Martin-Delgado, J. Phys. **A24** (1991) 1285;

[34] P. Neškovic and B. Urosševic, Int. J. of Mod. Phys. **7** (1992) 3379;

[35] V. Man'ko, G. Marmo, S. Solimeno and F. Zaccaria, "Physical Non-Linear Aspects of Classical and Quantum q-Oscillators", preprint DSF-T-92/25;

[36] V. Man'ko, G. Marmo, S. Solimeno and F. Zaccaria, "Correlation Function of Quantum Oscillators", preprint DSF-T-93/06;

[37] M. Chaichian, R. Gonzalez Felipe and C. Montonen, "Statistics of q-Oscillators, Quons and Relations to Fractional Statistics", preprint HU-TFT-93-23, to appear in J. Phys **A**;

[38] H. Exton, "q-Hypergeometric Functions and Applications", Horwood, Chichester, 1983;

[39] R. Gilmore, "Lie Groups, Lei Algebras, and Some of Their Applications", John Wiley & Sons, New York, 1974.