A Feynman Path-Integral for Brownian Particles and Fields

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August 28, 2000

Abstract

We formally quantize classical dissipative systems defined by Brownian particles or Brownian Fields by means of the Feynman path integral method.

1 Introduction

One of the most interesting and conceptual problems in Non-relativistic quantum mechanics is the study of the quantum mechanical behavior of a particle moving in an arbitrary potential but coupled to a reservoir at given temperature.

In the historical time honored attempts to handle this problem made by Feynman-Vermon and Schwinger ([1]), the main object analyzed was the formal closed time path-integral representation for the quantum mechanical probability coupled to a thermal reservoir by means of a quantum mechanical density matrix.

We feel that these approaches have *some deep conceptual* problems. The first one is the brute force use of the matrix density apparatus which rigorously must be used only in the Heisenberg quantum mechanics formulation for *operators* (quantum mechanical observables) and not for quantum mechanical probability amplitudes as done in these previously cited frameworks. As a consequence of this fact one obtains in general a non-classical Feynman path integral weight involving fourth-order time derivatives for such bath integrated Feynman Path integral representation (see eq. (18.229) of ref. [1]), which somewhat leads to our second criticism exposed below.

The second problem in these earlier attempts is that one is never able to write directly Schrödinger equations or Feynman Path integrals representations for the quantum mechanical propagation (not probabilities amplitudes!) which is the main non-relativistic quantum mechanical object to be understood in Feynman Path Integral formalism.

In this paper, we intend to propose a Feynman path integral solution for these problems by considering the quantization of a classical particle and a $\lambda\phi^4$-field interacting with a Brownian reservoir. Besides and for completeness, we present in sections 3 and 4 the interaction with a *non*-Brownian reservoir simulated by a classical gas and a thermalized lattice of harmonic oscillators respectively.

2 The Path Integral for Damped Quantum Systems

Let us start our study by considering as a macroscopic reservoir a random one dimensional *classical* vibration field

$$\frac{1}{c^2}\frac{\partial^2}{\partial t^2}\,\phi(x,t) - \frac{\partial^2}{\partial x^2}\phi(x,t) = 0 \tag{1}$$

$$\phi(x,0) = 0 \tag{2}$$

$$\frac{\partial \phi}{\partial t}(x,t)|_{t=0} = f(x) \tag{3}$$

where the initial date $f(x)$ belongs to an ensemble of random initial vibration field velocities in order to produce the environment randomness and is supposed to satisfy the white-noise statistics with strenght $\gamma > 0$

$$\langle f(x)\rangle = 0 \tag{4}$$

$$\langle f(x)f(x')\rangle = \gamma\delta(x-x') \tag{5}$$

From textbooks, the quantum mechanical amplitude is built from the classical action through the following formulae

$$\frac{\partial}{\partial t}S(x,t)+\frac{1}{2M}\left(\frac{\partial S(x,t)}{\partial t}\right)^2=V(x) \tag{6}$$

$$\lim_{|t-t'|\to 0}\langle (x,t)|(x',t')\rangle = A(t,t')\ exp\left(\frac{i}{\hbar}S((x,t;x',t'))\right) \tag{7}$$

$$\text{and}(x_N=x;x_2=x') \tag{8}$$

$$\langle (x,t)|(x',t')\rangle = \lim_{N\to\infty}\left\{\prod_{k=1}^{N-1}\int_{-\infty}^{+\infty} dx_k\langle x_{k+1},t_{k+1}|x_k,t_k\rangle\right\} \tag{9}$$

In order to write the associated (formal) Feynman path integral associated to the interaction of the above quantum particle with the reservoir eq. (1)-eq. (3), one needs on base of above written equations only to know the effective classical action $S(x,t)$ of this particle in order to build the *finite time* Feynman propagator eq. (9). In order to write the explicit expression for this classical damped action, we consider firstly the classical particle trajectory $X(t)$ interacting with the "oscilators reservoirs"

$$M\ \frac{d^2X(t)}{dt^2}=-gk\phi_k(t) \tag{10}$$

$$\frac{d^2\phi_k(t)}{dt^2}=-\frac{k^2}{c^2}\phi_k(t)-gkX(t) \tag{11}$$

Here, the random wave field is considered as a random motion of the harmonic oscillators associated to its plane wave expansion problem, namelly:

$$\phi(x,t)=\int_{\frac{1}{\Lambda}<|k|<\Lambda} dk\ \phi_k(t)e^{ikx} \tag{12}$$

$$\phi_{-k}(t)=\phi_k^*(t) \tag{13}$$

$$\phi_k(0)=0\ ,\ \frac{\partial\phi_k(0)}{\partial t}=\int dkf(k)e^{ikx}\equiv f_k \tag{14}$$

Note that we have considered the usual weak reservoir coupling linear interaction in eq. (10)-eq. (11). ([2]).

At this point, we solve the classical problem eq. (10)–eq. (11) of the particle interacting with the reservoir by means of the Laplace transform and obtain, thus, the following result in the frequency domain

$$Ms^2\tilde{X}(s)=-\frac{g}{\left(\int_{1/\Lambda}^{\Lambda}dk\right)}\left\{\int_{1/\Lambda|k|<\Lambda}dk.k\left[f_k-gk\tilde{X}(s)\right]\ \frac{c^2}{k^2}\left(1-\frac{s^2}{s^2+\left(\frac{k}{c}\right)^2}\right)\right\} \tag{15}$$

The motion equation in the time domain, thus, is given by

$$M\ddot{X}(t)=(c^2g^2)X(t)-\lambda^{(\Lambda)}\frac{dX}{dt}(t)+E(t) \tag{16}$$

where the damping term is given explicitly in terms of the system macroscopic parameters by $\lambda^{(\Lambda)}=\frac{2\pi g^2c}{\Lambda}$. It is worth remarking that the external random force $E(t)$ coming from the randomness of the reservoir oscillators initial velocities satisfies the white-noise Gaussian statistics (see appendix 2)

$$\langle E(t)E(t')\rangle=\frac{g^2}{\Lambda^2}\gamma c^2\int_{1/\Lambda<|k|<\Lambda}dk\ sen\left(\frac{kt}{c}\right)sen\left(\frac{kt'}{c}\right)=\frac{g^2\gamma c^3}{\Lambda^2}\ \delta_{(\Lambda)}(t-t') \tag{17}$$

The interaction, thus, is of the kind of Brownian reservoir ([3]), with a relaxation term and an external random forcing.

At this point we propose to consider the following analogous equation for defining the phase of our quantum damped particle on basis of similarity of eq. (16) *for very short time quantum propagation* ($t \sim 0^+$) (see eq. (7))

$$\frac{\partial S(x,t)}{\partial t} + \frac{1}{2M}\left(\frac{\partial S(x,t)}{\partial x}\right)^2 = -\lambda^{(\Lambda)} S(x,t) + \phi(x,t) \quad (18)$$

instead of the well known Caldirola-Kanai lagrangean ([4]).

Here $\phi(x,t) = E(t)x$ is the stochastic potential responsible for the randomness of the environment

$$\langle E(t)E(t')\rangle = (MkT \cdot \lambda)\delta(t-t') \quad (19)$$

where we have introduzed the temperature dependent phenomenological disorder bath strenght and the phenomenological bath viscosity λ by the relationship $\frac{g^2\gamma c^3}{\Lambda^2} = MkT\lambda$.

For completeness, we are going to discuss the complete three-dimensional case in our discussions from now on, namely:

$$\frac{\partial S(\vec{r},t)}{\partial t} + \frac{1}{2M}|\vec{\nabla} S(\vec{r},t)|^2 = -\lambda S(\vec{r},t) + V(\vec{r},t) + \phi(\vec{r},t') \quad (20)$$

Here $S(\vec{r},t)$ is the 3D-version of the damped particle quantum phase $V(\vec{r},t)$ is the deterministic potential, $-\lambda S(\vec{r},t)$ with $\lambda > 0$ denotes the term which is related to the damping effects on the motion of the particle as showed above and $\phi(\vec{r},t)$ is the (intrinsic) stochastic Gaussian noise potential responsible for the classical stochastic behavior of the Brownian particle ([3]).

Note that "Brownian motion" analitical form of this three-dimensional Gaussian random potential should posseses the form of a homogeneous "Electric potential drift $\phi(\vec{r},t) = \vec{E}(t)\cdot\vec{r}$ and its two point correlation function is given explicitly by (compare with eq. (19) for the one-dimensional case)

$$\begin{aligned} \langle \phi(\vec{r},t)\phi(\vec{r}\,',t')\rangle &= \langle E_i(t)E_j(t')\rangle r_i r_j' \\ &= (MkT\cdot\lambda)\delta_{ij}\delta(t-t')r_i r_j' \end{aligned} \quad (21)$$

In the Feynman path integral formalism, one should define as a quantum transition amplitude for the damped quantum particle, eq. (1), the following sum over quantum trajectories ([4],[7]):

$$G((\vec{r},t);(\vec{r}\,',t')) = \sum_{\{C\}} \left\langle \exp\left(\frac{i}{\hbar}\, S\left[(\vec{r},t);(\vec{r}\,',t')\right]\right)\right\rangle_\phi . \quad (22)$$

Here C is some trajectory of the classical system, $S((\vec{r},t);(\vec{r}\,',t'))$ is the classical action of the system and $< \cdots >_\phi$ denotes the stochastic average over all realization of the environment random potential $\phi(\vec{r},t)$ acting on the particle.

The above formula, Eq. (21), is symbolic, but while in the case of non-damping $\lambda = 0$ and no stochasticity, i.e, $\phi(\vec{r},t) \equiv 0$, we know how to decipher and compute it ([1],[7]). But in the general dissipative case, Eq. (21), such knowledgement is not available presently ([4]). Let us propose a formal solution for this problem.

As a first step, we solve the generalized Hamilton-Jacobi equation, Eq. (20). Its solution is easily seen to be given by

$$S(\vec{r},t) = e^{-\lambda t} S^{(0)}(\vec{r},t) \quad (23)$$

with $S^{(0)}(\vec{r},t)$ satisfying the usual Hamiltonain-Jacobi equation with time dependent parameters, including the mass term, i.e,

$$\frac{\partial}{\partial t}\, S^{(0)}(\vec{r},t) + \frac{1}{2Me^{\lambda t}}\left|\vec{\nabla} S^{(0)}(\vec{r},t)\right|^2 = e^{\lambda t}(\phi(\vec{r},t) + V(\vec{r},t)) . \quad (24)$$

An exact solution of Eq. (24), in terms of the action functionals, is easily given in terms of the Caldirola-Kanai action ([2])

$$S^{(0)}(\vec{r},t) = \int_{t'}^{t} d\sigma e^{\lambda\sigma}\left[\frac{M}{2}\left(\frac{d\vec{r}}{d\sigma}\right)^2 - (\phi(\vec{r}(\sigma),\sigma) + V(\vec{r}(\sigma),\sigma))\right] . \quad (25)$$

which, by its turn, leads to the following expression for our complete phase factor, Eq. (23):

$$S(\vec{r},t) = \int_{t'}^{t} d\sigma e^{\lambda(\sigma-t)} \left[\frac{1}{2}\ M \left(\frac{d\vec{r}}{dt} \right)^2 - (\phi(\vec{r}(\sigma),\sigma) + V(\vec{r}(\sigma),\sigma)) \right] \ . \tag{26}$$

Following now our procedure exposed in Ref. [4], we consider the discreticized version of Eq. (26), i.e.,

$$\hat{S}\left((\vec{x}_{k+1},t_{k+1});(\vec{x}_k,t_k)\right) = e^{\lambda(t_k-t_{k+1})} \left[\frac{1}{2}\ M\ \frac{\vec{x}_{k+1}-\vec{x}_k)^2}{\varepsilon^2} - \varepsilon V(\vec{x}_k,t_k) - \varepsilon\phi(\vec{x}_k,t_k) \right] \ . \tag{27}$$

At this point of our study, we remark that the short-time transition amplitude, in the Feynman path integral and propagator formalism, is given explicitly by the asymptotic result, i.e.,

$$G((\vec{x}_{k+1},t_{k+1});(\vec{x}_k,t_k)) \cong A(t_{k+1},t_k)\exp\left(\frac{i}{\hbar}\ \hat{S}((\vec{x}_{k+1},t_{k+1});(\vec{x}_k,t_k)) \right) \ , \tag{28}$$

where $t_{k+1}-t_k \rightarrow 0$.

The pre-factor in Eq. (28) is easily obtained from the $t \rightarrow 0^+$ condition, i.e.,

$$\lim_{(t_{k+1}-t_k)\rightarrow 0} G((\vec{x}_{k+1},t_{k+1});(\vec{x}_k,t_k)) = \delta^{(D)}(\vec{x}_{k+1}-\vec{x}_k) \ , \tag{29}$$

and leading, thus, to the exact result:

$$A(t_{k+1},t_k) = e^{\frac{D\lambda}{2}(t_k-t_{k+1})} \left[\frac{M}{2\pi\hbar(t_{k+1}-t_k)} \right]^{\frac{D}{2}} \ . \tag{30}$$

As a consequence of the above displayed formulae, we obtain the finite time propagator, i.e.,

$$G((\vec{r},t);(\vec{r}\,',t')) = \lim_{N\rightarrow\infty} \int \left(\prod_{k=0}^{N-1} d\vec{r}_k \right) exp\left\{ \frac{D\lambda}{2} \left[\sum_{k=0}^{N} (t' + \frac{t-t'}{N}k) - (t' + \frac{(t-t')}{N}(k+1)) \right] \right\}$$
$$\prod_{k=1}^{N-1} \left(\frac{M}{2\pi\hbar(t_{k+1}-t_k)} \right)^{\frac{D}{2}} exp\left\{ \frac{i}{\hbar} \prod_{k=0}^{N} \varepsilon e^{\frac{\lambda}{2}(t_k-t_{k+1})} \left(\frac{M}{2}\frac{(\vec{r}_{k+1}-\vec{r}_k)^2}{\varepsilon^2} - V(\vec{r}_k,t_k) - \phi(\vec{r}_k,t_k) \right) \right\} \ . \tag{31}$$

Now it is easy to evaluate the sum in Eq. (30) where D is the space-time dimension:

$$exp\left\{ \frac{D\lambda}{2} \left[\sum_{k=0}^{N-2} (t' + \varepsilon k) - (t' + \varepsilon(k+1)) \right] \right\} = exp^{\left[-\frac{D\lambda}{2}(t-t') \right]} \ , \tag{32}$$

and thus arrive at the following computable Feynman path integral (without making the evaluation of the classical stochastic average over the random potentials at this point of our study), i.e.,

$$G_\phi((\vec{r},t);(\vec{r}\,',t)) = exp - \left(\frac{D\lambda(t-t')}{2}\right) \int_{\vec{r}(t')=\vec{r}\,';\vec{r}(t)=\vec{r}} D^F[\vec{r}(\sigma)]$$
$$exp\left\{\frac{i}{\hbar}\int_{t'}^{t} d\sigma e^{\lambda(\sigma-t)}\left[\frac{1}{2}\,M\left(\frac{d\vec{r}}{d\sigma}\right)^2 - V(\vec{r}(\sigma),\sigma) - \phi(\vec{r}(\sigma),\sigma)\right]\right\}\ . \tag{33}$$

Note that, in constrast to previous studies (Ref. [4] and [5]), the dissipative anomaly in Eq. (32) decays to zero at the equilibrium limit $t \to \infty$, independent of the discretization procedure used to define the Path-integral.

At this point we take the average of Eq. (31) in the ensemble of the classical stochastic potentials $\{\phi(\vec{r},t)\}$ eq. (20) with the result of a harmonic oscillator action with a time dependent imaginary frequency $Mw_0^2 = \{iMkT\lambda\ exp[2(\sigma-t)]\}/\hbar$, namely

$$\left\langle exp\left\{\frac{i}{\hbar}\int_{t'}^{t} d\sigma\ exp(\lambda(\sigma-t))\phi(\vec{r}(\sigma),\sigma)\right\}\right\rangle = exp\left\{-\frac{MkT}{2(\hbar)^2}\int_{t'}^{t} d\sigma \exp(2\lambda(\sigma-t))(\vec{r}(\sigma))^2\right\} \tag{34}$$

It is worth point out that this result is a direct consequence of the form of eq. (19).

The complete Brownian propagator takes, thus, the final form

$$G((\vec{r},t);(\vec{r}\,',t')) = e^{-\frac{D\lambda(t-t')}{2}}$$
$$\int_{\vec{r}(t')=\vec{r}\,';\vec{r}(t)=\vec{r}} D^F[\vec{r}(\sigma)] \exp\left\{\frac{i}{\hbar}\int_{t'}^{t} d\sigma e^{\lambda(\sigma-t)}\left[\frac{1}{2}\,M\left(\frac{d\vec{r}}{d\sigma}\right)^2 - V(\vec{r}(\sigma),\sigma)\right]\right\}$$
$$\exp\left\{-\frac{MkT\lambda\exp(-2t\lambda)}{2(\hbar)^2}\int_{t'}^{t} d\sigma e^{(2\lambda\sigma)}(\vec{r}(\sigma))^2\right\} \tag{35}$$

Note that the path integral weight in the above proposed propagator for the quantum version of a Brownian classical particle described be eq. (20) *differs* from the usual Caldirola-Kanai action by an over all factor $e^{\lambda t}$. At this point we remark that the appearence of this factor makes the problem of writing *local* Schrödinger equations associated to the effective propagator eq. (35), a difficult problem ([5]). It is our opinion that only path integrals techniques are available for analysing these "bath integrated out" particle quantum transitions amplitudes given by the proposed propagator eq. (36), similar to those techniques used in the "Polaron problem" by R.P. Feynman [1] and [7]) and myself ([5]).

For quantum fields, the interaction with a Brownian reservoir should be given by the following Caldirola-Kanai Field lagrangean with the random white noise drift eq. (20) as on can see from considering the harmonic oscilator field expansion with each harmonic oscilator coupled to the reservoir ([6]).

$$\mathcal{L}_{dam}(\varphi,\partial_t\varphi,\nabla\varphi) = \int_{-\infty}^{+\infty} d\vec{x}\int_0^{\infty} dt\ e^{\nu t}\left[(\dot{\varphi})^2 - (\vec{\nabla}\varphi)^2\right](\vec{x},t)$$
$$+\int_{-\infty}^{+\infty} d\vec{x}\int_0^{\infty} dt\ e^{\nu t}(E(t)\varphi(\vec{x},t) + \frac{\lambda}{4!}\int_{-\infty}^{+\infty} d\vec{x}\int_0^{\infty} dt\ e^{\nu t}(\varphi(\vec{x},t)^4 \tag{36}$$

the above lagrangean leads, by its turn, to the classical Brownian field equation

$$\left(\frac{\partial^2}{\partial t^2} - \Delta\right)\varphi(\vec{x},t) = -\nu\ \frac{\partial}{\partial t}\ \varphi(\vec{x},t) + E(t)\varphi(\vec{x},t) + \frac{\lambda}{3!}(\varphi(x,t))^3 \tag{37}$$

Let us, thus, write the quantum field generating functional associated to the second quantization of eq. (35) by means of a Feynman path integral averaged over all white-noise drifts $E(t)$, namely

$$Z[J(\vec{x},t)] = \left\langle \frac{1}{Z(0)} \int D^F[\varphi(x,t)] e^{\frac{i}{\hbar}\int_0^\infty dt \int_{-\infty}^{+\infty} d\vec{x}\mathcal{L}_{dam}(\varphi,\partial_t\varphi,\vec{\nabla}\varphi)} e^{\frac{i}{\hbar}\int_0^\infty dt \int_{-\infty}^{+\infty} d\vec{x}(\varphi\cdot J)(\vec{x},t)} \right\rangle_E \tag{38}$$

A Feynman diagrammatic expansion can be easily implemented to eq. (38) as follows.

The free-field causal Feynman infrared regularized by the damping constant ν propagator (in the range $[0,\infty)$) satisfies the motion equation

$$e^{\nu t}\left(\frac{\partial^2}{\partial t^2} + \nu\frac{\partial}{\partial t} - \Delta_x\right) G((\vec{x},t);(\vec{x}',t')) = \delta(\vec{x}-\vec{x}')\delta(t-t') \tag{39}$$

or in the momentum space

$$\left(\frac{d^2}{dt^2} + \nu\frac{d}{dt} + (\vec{k})^2\right)\bar{G}(\vec{k},t,t') = e^{-\nu t}\delta(t-t') \tag{40}$$

with the explicitly solution.

For $(\vec{k})^2 > \nu^2/4$:

$$G(\vec{k},t,t') = \frac{1}{\sqrt{\left((\vec{k})^2 - \frac{\nu^2}{4}\right)}} e^{-\frac{\nu}{2}(t-t')} sen\left(\sqrt{((\vec{k})^2 - \frac{\nu^2}{4})}(t-t')\right)\theta(t-t') \tag{41}$$

For $(\vec{k})^2 < \frac{\nu^2}{4}$:

$$G(\vec{k},t,t') = \frac{1}{2\sqrt{\frac{\nu^2}{4} - (\vec{k})^2}} e^{-\frac{\nu}{2}(t-t')}\theta(t-t') \times\left\{ e^{\frac{\nu}{2}(\sqrt{1-4\frac{(\vec{k})^2}{\nu^2}})(t-t')} - e^{-\frac{\nu}{2}(\sqrt{1-4\frac{(\vec{k})^2}{\nu^2}})(t-t')}\right\} \tag{42}$$

and for $(\vec{k})^2 = \frac{\nu^2}{4}$.

$$G(\vec{k}, t, t') = (t - t')e^{-\frac{\nu}{2}(t - t')} \tag{43}$$

The interaction vextex are the same of the usual massless $\lambda\varphi^4$-Field theory but now with an explicit interaction with the "classical" source field $e^{\nu t}$ and with the white-noise drift $E(t)$, which should be averaged at the end of the diagrammatic computations. At this point is worth remark that is unphysical to consider for the range of time t the whole range $(-\infty, \infty)$ since this system is damped and does not have time invariance $(e^{\nu t} \neq e^{-\nu t}!)$ (see eq. (36)).

A complete study of these diagrammatic computations will be presented elsewhere.

3 The Quantum Propagator of a Particle Interacting with a *Classical Gas*

Let us start this section by considering the grand canonical partition functional of a classical gas in a volume $\Omega \subset R^3$ at temperature $T = 1/k\beta$ with a two body interaction potential $V(\vec{r})$

$$Z(z, \beta, \Omega) = \sum_{n=0}^{\infty} z^n Z_n(\beta; \Omega) \tag{44}$$

where $Z_n(\beta, \Omega)$ denotes the N-body partition functional

$$\begin{aligned} Z_n(\beta, \Omega) &= \frac{1}{(\hbar)^{3n} n!} \prod_{i=1}^{N} \left\{ \int_\Omega d^3\vec{r}_i \int_{-\infty}^{+\infty} d^3\vec{p}_i \exp\left[-\beta \left(\sum_{i=1}^{n} \frac{p_i^2}{2m} + \sum_{i<j}^{n} V(\vec{r}_i - \vec{v}_j) \right) \right] \right\} = \\ &= \left(\frac{2\pi m}{\beta \hbar^2} \right)^{\frac{3n}{2}} \prod_{i=1}^{n} \left\{ \int_\Omega d^3\vec{r}_i \ \exp\left[-\beta \left(\sum_{i<j}^{n} V(\vec{r}_i - \vec{r}_j) \right) \right] \right\} \end{aligned} \tag{45}$$

Let us represent eq. (44) by means of the functional integral of a field theory on the volume Ω (a closed manifold on R^3!)

$$Z(z, \beta, \Omega) = \int D^F[\phi(\vec{r})] \ \exp\left\{ -\int_\Omega d^3\vec{r} \ \phi(\vec{r})(\mathcal{L}\phi)(\vec{r}) \right\} \exp(\alpha \int_\Omega d^3\vec{r} \ \exp(i\gamma\phi(\vec{r}))) \overset{det}{\equiv} \langle 1 \rangle_\phi \tag{46}$$

with the following coupling identifications

$$\begin{aligned} \gamma^2 &= \beta = \frac{1}{kT} \\ \alpha &= z \left(\frac{2\pi m}{\beta \hbar^2} \right)^{3/2} \ \exp\left(\frac{1}{2} \beta \ V(\vec{0}) \right) \ . \end{aligned} \tag{47}$$

The kinetic term of the gas field theory eq. (46) has the two-body potential as the Green function namely

$$\mathcal{L} V(\vec{r} - \vec{r}\,') = \delta^{(3)}(\vec{r} - \vec{r}\,') \tag{48}$$

A proof of eq. (46) is easily obtained by considering the power expansion on α of the Exponential Field interaction and evaluating the resulting gaussian functional integrals obtained on the process. As a result one obtains the validity of eq. (46).

At this point, we consider the interaction of a *quantum* particle of mass M with the gas by the simplest linear interaction of strenght g of the field $\phi(\vec{r})$ with the Feynman particle trajectory current $\vec{r}(\sigma)$ (as it is usually made on the treatment of Feynman path integrals for quantum particles on the presence of a electromagnetic field). Namely, the Effective Particle propagator representing the interaction of our quantum particle with the classical gas as a reservoir is proposed to be given by

$$\overline{G}((\vec{r},t);(\vec{r}\,',t')) = \int_{\vec{r}(t)=\vec{r};\vec{r}(t')=\vec{r}\,'} D^F[\vec{r}(\sigma)]\ \exp\left(\frac{i}{\hbar}\int_{t'}^{t}\frac{M}{2}\left(\frac{d\vec{r}}{d\sigma}\right)^2 d\sigma\right)$$
$$\left\langle exp(\frac{i}{\hbar}g\int_{t'}^{t} d\sigma\ \phi(r(\sigma))\right\rangle_{\phi} \qquad (49)$$

Here the average $\langle\ \rangle_\phi$ is defined by the (euclidean) path integral eq. (46).

A closed expression for our proposed "Bath-interaction" quantum propagator is easily written down by considering again a α-perturbative calculation similar to that one used to show the functional integral representation eq. (46). We, thus, have the following result

$$\overline{G}((\vec{r},t);(\vec{r}\,',t')) = \sum_{n=0}^{\infty}\frac{\alpha^n}{n!}\int_{\Omega} d^3r_1\cdots d^3r_n\left[\int_{\vec{r}(t)=\vec{r};\vec{r}(t')=\vec{r}\,'} D^F[\vec{r}(\sigma)]\right.$$
$$\exp\left(\frac{i}{\hbar}d\sigma\int_{t'}^{t}\frac{M}{2}\left(\frac{d\vec{r}}{d\sigma}\right)^2\right)\times\exp\left(-\gamma^2\sum_{(k,j)}^{n}V(\vec{r}_k-\vec{r}_j)\right)\ \exp\left(-\frac{\gamma}{\hbar}\sum_{k=1}^{n}d\sigma\ V(\vec{r}(\sigma)-\vec{r}_k)\right)$$
$$\left.\exp\left(-\frac{g^2}{2\hbar^2}\int_{t'}^{t}d\sigma\int_{t'}^{t}\ d\sigma' V(\vec{r}(\sigma)-\vec{r}(\sigma'))\right)\right] \qquad (50)$$

In the regime of higher-temperature $\gamma \to 0$, the leading contribution on γ is exactly given by the self-avoiding interaction trajectory path integral

$$\overline{G}((\vec{r},t);(\vec{r}\,',t')) \sim \int_{\vec{r}(t)=\vec{r};\vec{r}(t')=\vec{r}\,'}\ \exp\left(\frac{i}{\hbar}\int_{t'}^{t}d\sigma\left(\frac{M}{2}\right)\left(\frac{d\vec{r}}{d\sigma}\right)^2\right)$$
$$\exp\left(-\frac{g^2}{2\hbar^2}\int_{t'}^{t}d\sigma\int_{t'}^{t}d\sigma'\ V(\vec{r}(\sigma)-\vec{r}(\sigma'))\right) \qquad (51)$$

At this point the reader should compare eq. (51) with the Brownian quantum propagator eq. (35). As a conclusion of this analysis we conclude that the *Bath-reservoir physical nature affects drastically the mathematical structure of the quantum system in interaction with the above mentioned environment.*

4 The Quantum Propagator of a Particle Interacting with a reservoir of thermalized Quantum Oscilators

In this section we consider the motion of a quantum free particle in the presence of a thermalized lattice of harmonic oscillators. Let us, thus, write the effective particle propagator interacting with the environment (the thermolized lattice) by means of a microscopic interaction potential $\tilde{V}(\vec{x})$. The formal expression for the effective quantum propagator is, thus, given by similar expression to eq. (49)

$$\overline{G}((\vec{r},t);(\vec{r}\,',t')) = \prod_{(i\in\text{Lattice})} \left\{ \int_\Omega d\overline{Q}_i \left[\int_{Q_i(0)=Q_i(\beta)=\overline{Q}_i} D^F[\overline{Q}(s)] \right.\right.$$
$$\left.\left. \exp\left(-\int_0^\beta ds \left[\frac{1}{2}\, m \left(\frac{dQ}{ds}\right)^2 + \frac{1}{2} m\Omega^2 Q^2(s)\right]\right)\right] \right.$$
$$\left[\int_{\vec{r}(t)=\vec{r};\vec{r}(t')=\vec{r}\,'} D^F[\vec{r}(\sigma)]\ \exp\left(\frac{i}{\hbar}\int_{t'}^t \frac{M}{2}\left(\frac{dr}{d\sigma}\right)^2\right)\right.$$
$$\left.\left. \exp\left(\frac{i}{\hbar}\int_{t'}^t d\sigma \int_0^\beta ds\ \tilde{V}(\vec{r}(\sigma)-\vec{Q}(s))\right)\right]\right\} \tag{52}$$

In order to have more tractable analytical expression for eq. (52), let us rewrite eq. (52) in the following form taking into account all the correct combinatoric factors of the lattice product on eq. (52)

$$\overline{G}((\vec{r},t);(\vec{r}\,',t')) = \int_{\vec{r}(t)=\vec{r};\vec{r}(t')=\vec{r}\,'} \exp\left(\frac{i}{\hbar}\int_{t'}^t d\sigma\ \frac{M}{2}\left(\frac{dr}{d\sigma}\right)^2\right)\ \exp[+W(\vec{r}(\sigma)] \tag{53}$$

Here $W(\vec{r}(\sigma))$ is the effective piece of the path-integral weight coming from the interaction with the thermolized lattice and given by following path-integral

$$W(\vec{r}(\sigma)) = \rho \int_{\overline{Q}(0)=\overline{Q}(\beta)=\vec{0}} D^F[\overline{Q}(s)]\ \exp\left(-\int_0^\beta ds\left[\frac{m}{2}\dot{Q}^2(s) + m\Omega^2 Q^2(\Omega)\right]\right)$$
$$exp\left(\frac{i}{\hbar}\int_{t'}^t d\sigma\int_0^\beta ds\ \tilde{V}(\vec{r}(\sigma)-\vec{Q}(s))\right) \tag{54}$$

where $\rho = N/V$ is the particle density of the lattice.

It is worth remark that the path-integral weight on eq. (52) can be exactly evaluated if one has the interaction of the harmonic oscillator type among the lattice particles and the particle: $\tilde{V}(\vec{x}) = \frac{1}{2}k|\vec{x}|^2 = \frac{1}{2}kx^2$.

In the case of $\rho << 1$ (low density lattice), one obtain the leading result

$$\overline{G}\,((\vec{r},t);(\vec{r}\,',t')) \sim \int_{\vec{r}(t)=\vec{r};\vec{r}(t')=\vec{r}\,'} D^F[\vec{r}(\sigma)]\ \exp\left(\frac{i}{\hbar}\int_{t'}^t dr\ \frac{m}{2}\left(\frac{dr}{d\sigma}\right)^2\right)$$
$$\left\{1+\rho\left[\int_{\overline{Q}(0)=\overline{Q}(\beta)=0} D^F[\overline{Q}(s)]\ \exp\left(-\int_0^\beta ds\left[\frac{M}{2}\left(\frac{dQ}{ds}\right)^2 + M\Omega^2 Q^2(s)\right]\right)\right.\right.$$
$$\exp\left(-\frac{i}{\hbar}k\int_0^\beta ds\ Q^2(s)(t-t')\right)$$
$$\left.\left.\exp\left(-\frac{i}{\hbar}k\beta\int_{t'}^t d\sigma\ r^2(\sigma)\right)\ \exp\left(+\frac{2i}{\hbar}k\int_0^\beta ds\ Q(s)\left(\int_{t'}^t d\sigma\ r(\sigma)\right)\right)\right]\right\} + O(\rho^2) \tag{55}$$

Exactly evaluation of eq. (55) is a straightforward procedure ([7]) and left to the reader and should be compared again with eq. (35) and eq. (51) given on the text. Another time we see that the Open quantum particle propagator depends heavily on the physical nature of the environment.

Acknowledgement: Luiz C.L. Botelho is thankful to CNPq – Brazil for financial support and to Professor José Helayël-Neto from CBPF/UCP for warm hospitality.

Appendix 1 – The damped classical phase

In the appendix we make an alternative and more invariant deduction for the basic equation (18) of our paper.

Let me, thus, consider the usual Schrödinger wave equation in its polar form and associated to the Feynman (initial value) Propagator, namely

$$\psi(\vec{x},t) = \rho(\vec{x},t) exp\left\{\frac{i}{\hbar}S(\vec{x},t)\right\} \tag{A.1}$$

$$\frac{\partial}{\partial t}S(\vec{x},t) + \frac{1}{2m}\left|\nabla_{\vec{x}}S(\vec{x},t)\right|^2 + \frac{\hbar^2}{2m}\frac{\Delta\rho(\vec{x},t)}{\rho(\vec{x},t)} = +\text{grad } V(\vec{x}) \tag{A.2}$$

$$S(\vec{x},0) = 0 \tag{A.3}$$

$$\frac{\partial}{\partial t}\rho(\vec{x},t) + div(\rho(\vec{x},t)\vec{\nabla}S(\vec{x},t)) = 0 \tag{A.4}$$

$$\rho(\vec{x},t) = \delta^{(3)}(\vec{x}-\vec{x}') \tag{A.5}$$

I postulate that the interaction of our quantum particle with the bath is of the Brownian form, namely, I added to the right-hand side of eq. (A-2) relaxation terms coming from the damping and the randomness as a result of the interaction with the reservoir of our quantum particle

$$\frac{\partial}{\partial t}\vec{v}(x,t) + \frac{1}{2m}(\vec{v}\cdot\vec{\nabla})\vec{v}(x,t) = -\frac{\hbar^2}{2m}\vec{\nabla}\cdot\left\{\frac{\Delta\rho(\vec{x},t)}{\rho(\vec{x},t)}\right\} + \vec{\nabla}\cdot V(\vec{x}) - \nu\ \vec{v}(\vec{x},t) + \vec{F}(\vec{x},t) \tag{A.6}$$

where $\vec{v}(\vec{x},t)$ is the gradient of the wave function, phase $(\vec{v}(\vec{x},t) = \vec{\nabla}\cdot S(\vec{x},t))$. Note that I have preserved all others equations (A-1), (A-3), (A-4) and (A-5). Besides, I postulate that the bath random force $\vec{F}(\vec{x},t)$ is a white-noise gaussian process with strenght $D > 0$ and in analogy with turbulence theory (ref. [8]) in order to have local expressions in my exposition

$$\langle F_i(\vec{x},t)F_j(\vec{x}',t')\rangle = D\delta^{(3)}(\vec{x}-\vec{x}')\delta(t-t') \tag{A.7}$$

It is well-known that it is impossible to write local Schrödinger wave equation for the set eq. (A-5), eq. (A-6). However it is possible, in principle, to write a path integral expression for the Feynman propagator defined in the text by eq. (9). In order to exhibit this new result, I should firstly write the Feynman propagator in the usual small step form of eq. (6)-eq. (9) as it is given in the text

$$\langle (x,t)|(x',t')\rangle = \lim_{N\to\infty}\int\left(\prod_{k=1}^{N-1}dx_k\right) f(t_{k+1},t_k)\ exp\left\{\frac{i}{\hbar}(S(x_{k+1},t_{k+1}) - S(x_k,t_k))\right\} \tag{A.8}$$

which may be written in the following form for the damped case (*remember that* $\vec{v}(\vec{x},t) = \vec{\nabla} \cdot S(\vec{x},t)$*!)*

$$\langle (x,t)|(x',t')\rangle_{\rm dam} = F(t-t') \int_{\substack{\vec{x}(t') = \vec{x}' \\ \vec{x}(t) = \vec{x}}} D^F[\vec{x}(\sigma)] \left\langle \exp\left\{ \frac{i}{\hbar} \int_{\vec{x}(\sigma)} \vec{v}(\vec{x}(\sigma),\sigma) d\vec{x}(\sigma) \right\} \right\rangle_F \qquad (A.9)$$

where $\langle \quad \rangle_F$ denotes the average of all realizations of the random external source eq. (A-6) through the "hydrodynamical equations" eq. (A-5), eq. (A-3), eq. (A-4) and eq. (A-5).

An explicitly path-integral expression for this random average in terms of a weight defined by the "quantum current" $\vec{v}(\vec{x},t) = \vec{\nabla} \cdot S(\vec{x},t)$ is easily obtained by means of the functional variable shift $\vec{F}(\vec{x},t) \to \vec{v}(\vec{x},t)$ (see ref. [8] – eq. (7))

$$\begin{aligned}
&\langle \quad \rangle_F \overset{def}{\equiv} \int D^F[\vec{F}(\vec{x},t)] \exp\left\{ -\frac{1}{D} \int_0^\infty dt \int_{-\infty}^{+\infty} d^3x |\vec{F}(x,t)|^2 \right\} \\
&\int D^F[\vec{v}(\vec{x},t)] det_F \left[\frac{\delta}{\delta v_k(\vec{x},t)} \left\{ \text{eq. (A-5) without the term } \vec{F}(\vec{x},t) \right\} \right] = \\
&\exp\left\{ -\frac{1}{D} \int_{-\infty}^{\infty} dt \int_{-\infty}^{+\infty} d^3x \left| \frac{\partial}{\partial t}\ \vec{v} + \frac{1}{2n} (\vec{v} \cdot \vec{\nabla}) \vec{v} \left(\frac{\hbar^2}{2n} \vec{\nabla} \left\{ \frac{\Delta\rho}{\rho} \right\} (\vec{v}) \right) - \vec{\nabla} \cdot V + \nu \vec{v} \right|^2 (\vec{x},t) \right\} \\
&\overset{def}{\equiv} \langle \quad \rangle_{\vec{v}}
\end{aligned} \qquad (A.10)$$

where I have, in principle, solved exactly the "quantum fluid hydrodynamical density" equation (A-4) $\rho(\vec{x},t) = F[\vec{v}(\vec{x},t)]$ in terms of the "quantum current" $\vec{v}(\vec{x},t)$, and substituted the resulting functional expression in the quantum potential term $\frac{\hbar^2}{2n}\frac{\Delta\rho(x,t)}{\rho(x,t)} \equiv \frac{\hbar^2}{2n}\left\{\frac{\Delta\rho}{\rho}\right\}(\vec{v}(\vec{x},t))$ and expressed it solely in terms of the "quantum current" $\vec{v}(\vec{x},t)$. Unfortunately, we were not able to solve the first order partial differential equations (A-4) and (A-5) and obtained the exact functional form of the quantum potential in terms of $\vec{v}(\vec{x},t)$ (however see appendix 3 for a path-integral solution for this problem).

Note that the functional determinant (the functional jacobian) in eq. (A-9) is unity, and the "prefactor" term $F(t,t')$ in eq. (A-8) is obtained after the evaluation of the average

$$\left\langle \exp\left\{ \frac{i}{\hbar} \int_{\vec{x}(\sigma)} \vec{v}(\vec{x}(\sigma),\sigma) d\vec{x}(\sigma) \right\} \right\rangle_{\vec{v}} \equiv \exp\left[-S_{dam}(\vec{x}(\sigma),\sigma)\right] \qquad (A.11)$$

by imposing the boundary condition $\lim_{t\to t'} \langle \vec{x},t|\vec{x},t'\rangle = \delta^{(3)}(\vec{x}-\vec{x}')$: in the Feynman propagator eq. (A-8) – eq. (A-9).

At this point we deduce the basic eq. (26) of our study presented in the bulk of our paper. For short time propagation $t_{k+1} - t_k \approx 0$, we can see that the $\langle \quad \rangle_{\vec{v}(\vec{r},t)}$ - average is entirely dominated by the "vanishing" $D \to 0$ randomness limit with the hypothesis of $\rho(\vec{x},t)$ =constant in this small step limit. In this case, the average eq. (A-10) is given by (with $t_k < \sigma < t_{k+1}$!)

$$\exp\left\{\frac{i}{\hbar}\int_{\vec{x}(\sigma)}\vec{v}_{CL}(\vec{x}(\sigma),\sigma)d\vec{x}(\sigma)\right\}\stackrel{def}{\equiv}\exp[-S_{dam}(\vec{x}(\sigma),\sigma)] \qquad (A.11')$$

where $\vec{v}_{CL}(\vec{x},t)$ satisfies the classical motion equation (the functional minimum of the weight on eq. (A-9)) (see refs. [9])

$$\frac{\partial}{\partial t}\,\vec{v}_{CL}(\vec{x},t)+\frac{1}{2m}\left[(\vec{v}_{CL}\cdot\vec{\nabla})\vec{v}_{CL}\right](\vec{x},t)=-\nu\,\vec{v}_{CL}(\vec{x},t)+\vec{\nabla}\cdot V(\vec{x}) \qquad (A.12)$$

$$\vec{v}_{CL}(\vec{x},0)=\vec{0} \qquad (A.13)$$

In order to solve eq. (A-12) and eq. (A-13) under the hypothesis of $\vec{v}(\vec{x},t)=\vec{\nabla}S(\vec{x},t)$, we firstly consider the variable change

$$\vec{v}_{CL}(\vec{x},t)=e^{-\nu t}\;\vec{v}^{(0)}\;(\vec{x},t) \qquad (A.14)$$

yielding the result

$$\frac{\partial}{\partial t}\vec{v}^{(0)}(\vec{x},t)+\frac{1}{2(me^{\nu t})}\left[(\vec{v}^{(0)}\cdot\vec{\nabla})\vec{v}^{(0)}\right](\vec{x},t)=e^{\nu t}(\vec{\nabla}\cdot V(\vec{x})) \qquad (A.15)$$

which, by its turn, has the exact solution (see eq. (24) for the phase $S(\vec{x},t)$)

$$\vec{v}^{(0)}(\vec{x},t)=\vec{\nabla}_{\vec{x}}\left\{\int_0^t d\sigma\;e^{\nu\sigma}\left(\frac{1}{2}m|\dot{\vec{x}}(\sigma)|^2-V(\vec{x}(\sigma))\right)\right\}\;. \qquad (A.16)$$

After collecting eq. (A-15); eq. (A-14), we see that for $t_k\le\sigma\le t_{k+1}$, we have the functional form of the "damped" classical particle action used in the text eq. (26)

$$S_{damp}(\vec{x}(\sigma),\sigma)=\int_{t'}^{t}d\sigma\;e^{\nu(\sigma-t)}\left(\frac{1}{2}m|\dot{\vec{x}}(\sigma)|^2-V(\vec{x}(\sigma))\right) \qquad (A.17)$$

Appendix 2 – The Bath reaction

In this short appendix we give a proof of eq. (17) (the random piece of the bath reaction on the classical particle).

This term comes from the following expression (see eq. (15)) in the frequency domain s and associated to the random initial conditions eq. (4)–eq. (5)

$$\tilde{E}(s)=-\left\{\frac{g}{\left(\int_{1/\Lambda}^{1}\right)dk}\left(\int_{1/\Lambda<k<\Lambda}dk\cdot\;\frac{kf_k}{s^2+\left(\frac{k}{c}\right)^2}\right)\right\} \qquad (A.18)$$

Note that f_k are random variable satisfying the same white-noise statistics of eq. (5) (the environment randomness)

$$\langle f_k\rangle=0$$
$$\langle f_kf_{k'}\rangle=\gamma\delta^{(1)}(k+k') \qquad (A.19)$$

As a consequence of the above written equations, we have as a straightforward result the eq. (17) given on the text

$$\langle E(t)E(t')\rangle=-\left(\frac{g^2}{\Lambda^2}\right)\gamma\int_{1/\Lambda\langle<(k,k')<\Lambda}dkdk'\;\left(\frac{sen(ckt)}{ck}\right)\cdot\left(\frac{sen(ck't')}{ck'}\right)\delta^{(3)}(k+k')(kk')$$
$$=eq.(17) \qquad (A.20)$$

Appendix 3 – The Path-Integral Solution of a Transport Equation

In this appendix we exhibit a closed analytical path-integral solution for the transport equation

$$\frac{\partial}{\partial t}\,\rho(\vec{x},t) + div(\rho(\vec{x},t)\vec{\nabla}S(\vec{x},t)) = D\Delta\rho(\vec{x},t) \tag{A.21}$$

$$\lim_{t\to t'}\rho(\vec{x},t) = \delta^{(3)}(x-x') \tag{A.22}$$

In order to write a path-integral expression for the Green function $G[(\vec{x},t);(\vec{x}\ ,t')]$ solution of eq. (A.21) and eq. (A.22), we compare it with the analogous problem in Quantum mechanics of a particle interacting with a electromagnetic vector potential field $\vec{A}$ and a potential field force ϕ. In this quantum mechanical case, the associated Schrödinger equation reads

$$i\hbar\frac{\partial}{\partial t}\psi(\vec{x},t) = \left\{-\frac{\hbar^2}{2n}\Delta\psi + \frac{ie\hbar}{mc}\ div(\vec{A}\psi) + \frac{e^2}{2mc^2}(\vec{A}^2)\psi + \phi\psi\right\}(\vec{x},t) \tag{A.23}$$

At this point we make the following analytic continuation on eq. (23)

$$\hbar = -i\ ;\ \vec{A} = -\nabla S\ ;\ \phi = -\frac{1}{4D}(\vec{A})^2$$
$$m = \frac{1}{2D}\ ;\ \frac{e}{c} = \frac{1}{2D}\ ;\ c = 1 \tag{A.24}$$

One can see that eq. (A.23) becomes formally identical to eq. (A.21). In terms of path integrals, we, thus, have the following (Euclidean-Wiener) path integral representation for the above transport Green function

$$G[(\vec{x},t);(\vec{x}\ ',t')] = \int_{\vec{Z}(t')=\vec{x}\ ';\vec{Z}(t)=\vec{x}} D^F[\vec{Z}(\sigma)]\ \exp\left\{-\frac{1}{4D}\left(\int_{t'}^{t}d\sigma\left[\frac{d\vec{Z}}{d\sigma}-(\vec{\nabla}S)(\vec{Z}(\sigma),\sigma)\right]\right)\right\} \tag{A.25}$$

At the limit (singular) $D\to 0$, where equation (A.21)-(A.22) reduces to the first-order equation (A.4), the (Functional) integral is given *exactly* by the following expression

$$G[(\vec{x},t);(\vec{x}\ ',t')] = \delta^{(3)}[\vec{x}-\vec{Z}[t;(\vec{x}\ ',t')]] \tag{A.26}$$

Here $\vec{Z}[t;(\vec{x}\ ',t')]$ satisfies the functional minimum of the positive path integral weight on eq. (A.25), i.e. the trajectories $\vec{Z}[\sigma,(\vec{x}\ ',t')]$ satisfies the following Storm-Liouville problem:

$$\frac{d\vec{Z}[\sigma,(\vec{x}\ 't')]}{d\sigma} = (\vec{\nabla}.S)(\vec{Z}[\sigma,(\vec{x}\ ',t')]d\sigma) \tag{A.27}$$

$$\lim_{\sigma\to t}\vec{Z}[\sigma,(\vec{x}\ ,t')] = \vec{x} \tag{A.28}$$

$$\lim_{\sigma\to t'}\vec{Z}[\sigma,(\vec{x};t')] = \vec{x}' \tag{A.29}$$

As a consequence we obtain the closed analytical expression for eq. (A.4)

$$\rho(\vec{x},t) = \delta^{(3)}(\vec{x}-\vec{Z}[t,(\vec{x};t')]) \tag{A.30}$$

However it is very combersome write the quantum potential term $\frac{\hbar^2}{2m}\left(\frac{\delta\rho}{\rho}\right)(\vec{x},t)$ in terms of an amenable functional on the "current" $\vec{v}(\vec{x},t) = (\nabla.\vec{S})(\vec{x},t)$ as it is needed on eq. (A.10).

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