CBPF-NF-060/93

Deformed Systems at Finite Temperature

by

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Abstract

We apply the formalism of Thermo-Field Dynamics to two kinds of deformed Hamiltonians. These Hamiltonians are made up of operators which satisfy deformed Heisenberg algebras. The entropy and the specific heat are computed for small $q-1$, where q is the deformation parameter.

Key-words: Thermal physics; Quantum algebras.

Introduction

There has been an intense activity in the area of deformed algebras and groups by physicists and mathematicians. These mathematical structures, Quasi-Triangular Hopf Algebras [1–4], are sometimes called Quantum Groups and have found applications in several areas of physics [4–8] such as: inverse scattering method, vertex models, anisotropic spin chain Hamiltonians, knot theory, conformal field theory, heuristic phenomenology of deformed molecules and nuclei, non-commutative approach to quantum gravity and anyon physics.

Because of the relevance played by Heisenberg algebra in physics, deformed Heisenberg algebras have attracted a lot of attention in the last few years. Their study is not a new subject [9] and recently their connection with quantum algebras and superalgebras have been established [10–12], together with their derivation from the contraction of quantum algebra [11]. A two-parameter Heisenberg algebra has also recently been obtained [13] by the Schwinger's contraction procedure for the two-parameter quantum semigroup $S\ell_{q,s}(2)$ [14]. The algebra of $S\ell_{q,s}(2)$ is called quantum semigroup because it is not possible to find an antipode function on the bialgebra $S\ell_{q,s}(2)$ to endow it with a Hopf algebra structure [14].

The thermodynamic properties of deformed systems have also started to be investigated [15–21] and many attempts to find a direct physical application have been made [15–21].

On the other hand Termo-Field Dynamics (TFD) [22] is a formalism whereby the usual field theory defined in real space-time can be generalized to the case of finite temperature. In this formalism the Feynman diagram method can be easily formulated by means of the real-time causal Green's function [23], which are expressed in terms of "temperature-dependent vacuum" expectation values, and all the operator relations of $T = 0$ field theory are preserved. Thermo Field Dynamics has been extensively developed and has been applied to problems in condensed matter physics as well as in high-energy physics [23–24].

In this paper we apply the formalism of TFD to two kinds of deformed Hamiltonians. These Hamiltonians are made up of operators which satisfy deformed Heisenberg algebras (q-oscillators). In section 1, we review the subject of deformed oscillators and extend the results of ref. [20] which come out of the application of TFD to a simple deformed Hamiltonian. In section 2 we consider a more complicated deformed Hamiltonian and compute, its entropy and the specific heat. Section 3 is devoted to some final comments.

1 Thermo-Field Dynamics of Deformed Systems

In the formalism of TFD [22–24] one constructs a temperature dependent vacuum $|0(\beta)>$, in which the statistical average of $\hat{O}$ coincides with the vacuum expectation value, using this new vacuum $|0(\beta)>$. For instance if one uses the canonical ensemble one has

$$< \hat{O} > \equiv Z^{-1}(\beta) Tr[e^{-\beta H}\hat{O}] = < 0(\beta)|\hat{O}|0(\beta)> \tag{1.1}$$

with $\beta = (k_B T)^{-1}$ and k_B the Boltzmann constant.

Let $\{|n>\}$ be the orthonormal basis of the state vector space $\mathcal{H}$ consisting of eigenstates of the Hamiltonian H

$$\begin{aligned} H|n> &= E_n|n> \\ <m|n> &= \delta_{m,n} \ . \end{aligned} \tag{1.2}$$

In order to construct such a state $|0(\beta)>$ one introduces a fictitious system (tilde system) characterized by the Hamiltonian $\tilde{H}$ and the state vector space $\tilde{\mathcal{H}}$ spanned by $\{|n>\}$ obeying

$$\begin{aligned} \tilde{H}|n> &= E_n|n> \\ <\tilde{n}|\tilde{m}> &= \delta_{n,m} \ . \end{aligned} \tag{1.3}$$

The thermal vacuum $|0(\beta)>$ belongs to the tensor product space $\mathcal{H}\otimes\tilde{\mathcal{H}}$ and is given by:

$$|0(\beta)>= Z^{-1/2}(\beta)\sum_n e^{-\beta E_n/2}|n>\otimes|n>\equiv Z^{-1/2}(\beta)\sum_n e^{-\beta E_n/2}|n,\tilde{n}> \ . \tag{1.4}$$

If one uses (1.4) in (1.1) one has

$$\begin{aligned} <0(\beta)|\hat{O}|0(\beta)> &= Z^{-1}(\beta)\sum_{n,m} e^{-\beta E_n/2}e^{-\beta E_m/2} <\tilde{n},n|\hat{O}|m,\tilde{m}> \\ &= Z^{-1}(\beta)\sum_n e^{-\beta E_n} <n|\hat{O}|n>=<\hat{O}> \end{aligned} \tag{1.5}$$

which is the result claimed in (1.1). This doubling of degrees of freedom has a sensible physical interpretation [22, 20] and is related to the algebraic formulation of Statistical Mechanics developed by Haag, Hugenholtz and Winnink [25].

This formalism can be extended to the case of statistical averages of systems made up of q-oscillators [20]. One calls bosonic q-oscillators the associative algebra generated by the elements $\mathcal{A}, \mathcal{A}^+$ and N satisfying the relations [10, 11, 26]

$$\begin{aligned} [N,\mathcal{A}^+] &= \mathcal{A}^+ \ , \ [N,\mathcal{A}] = -\mathcal{A} \\ [\mathcal{A},\mathcal{A}^+]_{\mathcal{A}} &= f_{\mathcal{A}}(N) \ . \end{aligned} \tag{1.6}$$

We are going to consider here the following forms of the above algebra (1.6):

$$[a,a^+]_a \equiv aa^+ - qa^+a = q^{-N} \tag{7.a}$$
$$[A,A^+]_A \equiv AA^+ - q^2A^+A = 1 \ , \tag{7.b}$$
$$[\alpha,\alpha^+]_\alpha \equiv \alpha\alpha^+ - s^{-1}q\ \alpha^+\alpha = (sq)^{-N} \tag{7.c}$$

the first two algebras are related to each other via [26]

$$A = q^{N/2}a \quad , \quad A^+ = a^+q^{N/2} \ , \tag{8}$$

and the last one is the two-parameter deformed Heisenberg [13] algebra obtained from the quantum semi-group $S\ell_{q,s}(2)$ [14] by the Schwinger's contraction.

It is possible to construct the representation of the relations (1.7) in the Fock space $\mathcal{F}$ spanned by the normalized eigenstates $|n>$ of the number operator N as

$$\begin{aligned} \mathcal{A}|0> &= 0 \quad , \quad N|n>= n|n> \quad , \quad n = 0,1,2\cdots \\ |n> &= \frac{1}{\sqrt{[n]_{\mathcal{A}}!}}(\mathcal{A}^{+})^{n}|0> , \end{aligned} \tag{9}$$

where $[n]_{\mathcal{A}}! \equiv [n]_{\mathcal{A}}\cdots[1]_{\mathcal{A}}$, $[n]_a = (q^n - q^{-n})/(q - q^{-1})$, $[n]_A = (q^{2n} - 1)/(q^2 - 1)$ and $[n]_\alpha = ((s^{-1}q)^n - (sq)^{-n})/(s^{-1}q - (sq)^{-1})$.

In the Fock space $\mathcal{F}$ it is possible to express the deformed oscillators in terms of the standard bosonic ones b, b^+ as [26]

$$\mathcal{A} = \left(\frac{[N+1]_{\mathcal{A}}}{N+1}\right)^{1/2} b \quad , \quad \mathcal{A}^+ = b^+ \left(\frac{[N+1]_{\mathcal{A}}}{N+1}\right)^{1/2} , \tag{10}$$

and it can easily be shown in $\mathcal{F}$ that

$$\mathcal{A}\mathcal{A}^+ = [N+1]_{\mathcal{A}} \quad , \quad \mathcal{A}^+\mathcal{A} = [N]_{\mathcal{A}} . \tag{11}$$

Let us now consider an ensemble of q-bosons with Hamiltonian given by

$$H = \omega N \tag{12}$$

with eigenvalue $\omega n(n = 0,1,2,\cdots)$ on $\mathcal{F}$. We introduce the Hamiltonian of the tilde system as

$$\tilde{H} = \omega\tilde{N} , \tag{13}$$

where the tilde q-operators we are considering satisfy the following relations

$$\begin{aligned} [\tilde{N},\tilde{\mathcal{A}}^+] &= \tilde{\mathcal{A}}^+ \quad , \quad [\tilde{N},\tilde{\mathcal{A}}] = -\tilde{\mathcal{A}} \\ [\tilde{\mathcal{A}},\tilde{\mathcal{A}}^+]_{\mathcal{A}} &= f_{\tilde{A}}(\tilde{N}) . \end{aligned} \tag{14}$$

In the cases we are going to consider here, we have

$$\begin{aligned} [\tilde{a},\tilde{a}^+]_{\tilde{a}} &\equiv \tilde{a}\tilde{a}^+ - q\tilde{a}^+\tilde{a} = q^{-\tilde{N}} \\ [\tilde{A},\tilde{A}^+]_{\tilde{A}} &\equiv \tilde{A}\tilde{A}^+ - q^2\tilde{A}^+\tilde{A} = 1 \\ [\tilde{\alpha},\tilde{\alpha}^+]_{\tilde{\alpha}} &\equiv \tilde{\alpha}\tilde{\alpha}^+ - s^{-1}q\ \tilde{\alpha} + \tilde{\alpha} = (sq)^{-N} \end{aligned} \tag{15}$$

and $[\mathcal{A},\tilde{\mathcal{A}}] = [\mathcal{A},\tilde{\mathcal{A}}^+] = 0$. The temperature dependent vacuum $|0(\beta)>$ is thus given by

$$\begin{aligned} |0(\beta)> &= Z^{-1/2}(\beta)\sum_{n=0}^{\infty} e^{-\beta n\omega/2}\frac{1}{[n]_{\mathcal{A}}!}(\mathcal{A}^+)^n(\tilde{\mathcal{A}}^+)^n|0>= \\ &= (1-e^{-\beta\omega})^{1/2}\exp_{q_{\mathcal{A}}}(e^{-\beta\omega/2}\mathcal{A}^+\tilde{\mathcal{A}}^+)|0> \end{aligned} \tag{16}$$

with $\exp_{q_{\mathcal{A}}} x = \sum_{n=0}^{\infty}\frac{1}{[n]_{\mathcal{A}}!}x^n$ the q-exponential [27], and $|0>= |0>\otimes|0>$. In the formula (1.16) we used the information that the partition function of q-bosons [19] corresponding to the Hamiltonian (1.12) coincides with the usual one for harmonic oscillators. We can easily see that the non-deformed case is recovered in the $q \to 1$ limit.

We denote

$$\begin{aligned} u_\beta &= (1-e^{-\beta w})^{-1/2} \\ v_\beta &= (e^{\beta\omega}-1)^{-1/2} \\ G_B &= -i\theta_\beta\left[\left(\frac{\tilde{N}+1}{[\tilde{N}+1]_{\mathcal{A}}}\right)^{1/2}\left(\frac{N+1}{[N+1]_{\mathcal{A}}}\right)^{1/2}\tilde{\mathcal{A}}\mathcal{A}\right. \\ &- \left.\left(\frac{\tilde{N}}{[\tilde{N}]_{\mathcal{A}}}\right)^{1/2}\left(\frac{N}{[N]_{\mathcal{A}}}\right)^{1/2}\tilde{\mathcal{A}}^+\mathcal{A}^+\right] \end{aligned} \tag{17}$$

where $\cosh\theta_\beta \equiv u_\beta$. With these definitions we can rewrite the temperature dependent vacuum $|0(\beta)>$, (1.16), as

$$|0(\beta)>= \exp(-iG_B)|0>\equiv B|0> \ . \tag{18}$$

Let us now define the temperature dependent operators; they are given by:

$$\begin{aligned} \mathcal{A}_\beta &\equiv \exp(-iG_B)\mathcal{A}\exp(iG_B) \\ \tilde{\mathcal{A}}_\beta &\equiv \exp(iG_B)\tilde{\mathcal{A}}\exp(iG_B)\ . \end{aligned} \tag{19}$$

It is interesting to observe that this transformation preserves the q-Heisenberg algebra (1.6,7), i.e.

$$\begin{aligned} a_\beta a_\beta^+ - q a_\beta^+ a_\beta &= q^{-N_\beta} \\ A_\beta A_\beta^+ - q^2 A_\beta^+ A_\beta &= 1 \\ \alpha_\beta\alpha_\beta^+ - s^{-1}q\alpha_\beta^+\alpha_\beta &= (sq)^{-N_\beta}\ . \end{aligned} \tag{20}$$

This can be seen by the use of relation (1.10), thus the B transformation resembles a q-Bogoliubov transformation. Obviously

$$\mathcal{A}_\beta|0(\beta)>= \tilde{\mathcal{A}}_\beta|0(\beta)>=0 \tag{21}$$

and the Fock space can be constructed by applying the B-transformation (1.18) on (1.9), i.e.

$$|n,\tilde{m}>_\beta= \frac{1}{\sqrt{[n]_{\mathcal{A}}!}}\frac{1}{\sqrt{[m]_{\mathcal{A}}!}}(\mathcal{A}_\beta^+)^n(\tilde{\mathcal{A}}_\beta^+)^m|0(\beta)> \tag{22}$$

for $n = 0, 1, \cdots$.

Let us now compute the average of $\mathcal{A}^+\mathcal{A}$ which, as we are going to see, depends on the deformation considered. In the TFD approach this average is given by:

$$<\mathcal{A}^+\mathcal{A}>=<0(\beta)|\mathcal{A}^+\mathcal{A}|0(\beta)>=<0(\beta)|[N]_{\mathcal{A}}|0(\beta)> \ . \tag{23}$$

In order to perform this calculation we go to the basis of non-deformed bosonic operators. In the non-deformed case one has [22]

$$\begin{aligned} b_\beta &= \exp(-iG_\beta)b\exp(iG_\beta) = u_\beta b - v_\beta\tilde{b}^+ \\ \tilde{b}_\beta &= \exp(-iG_B)\tilde{b}\exp(iG_B) = u_\beta\tilde{b} - v_\beta b^+ \end{aligned} \tag{24}$$

with the inverse given by

$$\begin{aligned} b &= u_\beta b_\beta + v_\beta \tilde{b}^+ \\ \tilde{b} &= u_\beta \tilde{b}_\beta + v_\beta b_\beta^+ \end{aligned} \tag{25}$$

where

$$G_B = -i\theta_\beta(\tilde{b}b - \tilde{b}^+ b^+) \ . \tag{26}$$

Now using (1.25) one finds the following expression for the number operator N

$$N = v_\beta^2(\tilde{N}_\beta + 1) + u_\beta^2 N_\beta + u_\beta v_\beta (b_\beta^+ \tilde{b}_\beta^+ + \tilde{b}_\beta b_\beta) \ . \tag{27}$$

Here we recall that the generators of $SU(1,1)$ algebra, $J_\pm, J_0$, can be realized à la Schwinger [28] as

$$J_+ = b_\beta^+ \tilde{b}_\beta^+ \ \ , \ \ J_- = \tilde{b}_\beta b_\beta \ \ , \ \ J_0 = \frac{1}{2}(N_\beta + \tilde{N}_\beta + 1) \ . \tag{28}$$

Thus using (1.28) we can rewrite (1.27) as

$$N = (u_\beta^2 + v_\beta^2) J_0 + u_\beta v_\beta (J_+ + J_-) + \frac{1}{2} C \tag{29}$$

with

$$C = N_\beta - \tilde{N}_\beta - 1 \ . \tag{30}$$

Notice that C commutes with the first two terms of the right-hand side of expression (1.29).

Using now (1.29) and taking $q^m = \exp\lambda$, the relevant terms in the calculations of (1.23) have the form

$$< 0(\beta)|e^{\lambda N}|0(\beta) >=< 0(\beta)|e^{\lambda[(u_\beta^2+v_\beta^2)J_0 + u_\beta v_\beta (J_+ + J_-) + \frac{1}{2}C]}|0(\beta) > \ . \tag{31}$$

This last expression can be computed by means of the Baker-Campbell-Hausdorff (BCH) formula, which can be derived for the $SU(1,1)$ algebra [29]. The BCH formula for the case we are considering is given by:

$$e^{\lambda[(u_\beta^2+v_\beta^2)J_0 + u_\beta v_\beta (J_+ + J_-)]} = e^{\rho J_+} e^{\gamma J_0} e^{\rho J_-} \tag{32}$$

where

$$\begin{aligned} \rho &= \frac{2 v_\beta \sinh(\lambda/2)}{\cosh(\lambda/2) - (u_\beta^2 + v_\beta^2)\sinh(\lambda/2)} \\ \gamma &= -2\ln[\cosh(\lambda/2) - (u_\beta^2 + v_\beta^2)\sinh(\lambda/2)] \ . \end{aligned} \tag{33}$$

The above procedure amounts to normal ordering eq. (1.31). Using this we obtain

$$< 0(\beta)|q^{mN}|0(\beta) >= \frac{2q^{-m/2}}{q^{m/2} + q^{-m/2} - (u_\beta^2 + v_\beta^2)(q^{m/2} - q^{-m/2})} \ . \tag{34}$$

Finally using (1.23) and (1.34) we can see that

$$
\begin{aligned}
< 0(\beta)|a^+a|0(\beta) > &= \frac{e^{\beta\omega}-1}{e^{2\beta\omega}-(q+q^{-1})e^{\beta\omega}+1} &(35.a)\\
< 0(\beta)|A^+A|0(\beta) > &= \frac{1}{e^{\beta\omega}-q^2} &(35.b)\\
< 0(\beta)|\alpha^+\alpha|0(\beta) > &= 4(e^{\beta\omega}-1)\times[(s^{-2}+s^{-1}(q+q^{-1})+1)(e^{\beta\omega}-1)^2+\\
&- 2(s^{-2}-1)(e^{2\beta\omega}-1)+\\
&+ (s^{-2}-s^{-1}(q+q^{-1})+1)(e^{\beta\omega}+1)^2]^{-1} &(35.c)
\end{aligned}
$$

where (1.35a,b) agree with the results given in ref. [19].

2 Entropy of Deformed Systems

In this section we are going to consider, using the formalism of TFD, the entropy of a deformed system. As we have noticed in the previous section the partition function corresponding to the Hamiltonian (1.12) coincides with the usual one for standard harmonic oscillators and as we are not changing the statistical averages the thermodynamic functions are not modified, therefore we take another Hamiltonian given by

$$H = \varepsilon[N]_{\mathcal{A}} - \mu N \; . \tag{1}$$

The form given by (2.1) is a general deformed Hamiltonian that describes in the $q \to 1$ limit the usual harmonic oscillator. Note that both $[N]$ and N are deformed but, as we shall see later, their statistical properties are different because only $[N]$ affects the extensivity. This choice permits to follow where the deformation modifies the thermodynamic functions (e.g. when $\varepsilon = 0$ one recovers the non-deformed quantities). This is indeed the case as one can easily check later. For a collection of oscillators, such as q-boson gas, the second term in (2.1) corresponds to the chemical potential.

For free systems, in the TFD approach, one can define an entropy operator, $\hat{K}$, as follows. Let us consider the thermal vacuum $|0(\beta)>$ which, as we have seen, is

$$
\begin{aligned}
|0(\beta)> &= Z^{-1/2}(\beta)\sum_n e^{-\beta E_n/2}|n,\tilde{n}>=\\
&= e^{\ell n Z^{-1/2}(\beta)-\beta H}\sum_n |n,\tilde{n}>\equiv e^{-\hat{K}/2}\sum_n |n,\tilde{n}> &(2)
\end{aligned}
$$

with

$$\hat{K} = \ln Z(\beta) + \beta H \; , \tag{3}$$

where the entropy, K, is obtained from $\hat{K}$ as

$$K = k_B < 0(\beta)|\hat{K}|0(\beta) > \; . \tag{4}$$

One can see that if one writes [22]

$$|0(\beta)>=\sum_n \sqrt{w_n}|n,\tilde{n}> \tag{5}$$

with $w_n = e^{-\beta E_n}/Z(\beta)$ we have

$$< 0(\beta)|\hat{K}|0(\beta) >= \sum_n w_n[\ln Z(\beta) + \beta E_n] = -\sum_n w_n \ln w_n \tag{6}$$

where $\sum_n w_n = 1$ by construction. Thus K given by (2.4) recovers the usual Shannon entropy.

For the deformed system we are considering, (2.1), one can be convinced that the entropy is given by the Shannon entropy with

$$w_n = \bar{e}^{\beta(\varepsilon[n]_A - \mu n)}/Z(\beta) \tag{7}$$

where

$$Z(\beta) = \sum_{n=0}^{\infty} e^{-\beta(\varepsilon[n]_A - \mu n)} \ . \tag{8}$$

As the partition function cannot be computed exactly we take $q = e^{\alpha_1}, \ s = e^{\alpha_2}$ and consider the case where α is small. Defining $\hat{\beta} = \beta(\varepsilon - \mu)$ we may rewrite $Z(\beta)$ in terms of the non-deformed partition function as

$$Z(\beta) = \bar{e}^{\beta\varepsilon([-\hat{\partial}]_A + \hat{\partial})} Z_0(\hat{\beta}) \tag{9}$$

where $\hat{\partial} = \partial/\partial\hat{\beta}$, and

$$Z_0(\hat{\beta}) = \sum_{n=0}^{\infty} e^{-\hat{\beta}n} \tag{10}$$

the non-deformed partition function.

Performing an expansion in α and taking into account only the lowest order in α we have for $Z(\beta)$

$$Z(\beta)^{-1} = Z_0^{-1}(\hat{\beta}) \left[1 - \alpha_1^2 \frac{\beta\varepsilon}{3!}(Q^{(3)} - Q^{(1)}) + O(\alpha_1^4) \right] \tag{11.a}$$

$$Z(\beta)^{-1} = Z_0^{-1}(\hat{\beta})[1 + \alpha_1 \beta\varepsilon(Q^{(2)} + Q^{(1)}) + O(\alpha_1^2)] \tag{11.b}$$

$$\begin{aligned} Z(\beta)^{-1} &= Z_0^{-1}(\hat{\beta}) \Big\{ 1 - \alpha_2 \beta\varepsilon Q^{(1)} - \alpha_1^2 \frac{\beta\varepsilon}{3!}(Q^{(3)} - Q^{(1)}) + \\ &+ \alpha_2^2 \beta\varepsilon \left[\beta\varepsilon Q^{(1)^2} - \frac{1}{2}(Q^{(1)} + 2Q^{(2)} + Q^{(3)} + \beta\varepsilon Q^{(2)}) \right] + \\ &+ O(\alpha_1^4, \alpha_2^4) \Big\} \end{aligned} \tag{11.c}$$

for the algebras (1.6,1.7a) (1.6,1.7b) and (1.6,1.7c) respectively, where

$$Q^{(i)} = Z_0(\hat{\beta})^{-1} \partial_{\hat{\beta}}^i Z_0(\hat{\beta}) \ . \tag{12}$$

A useful expression for the calculation with the $Q^{(i)}$'s is

$$Q^{(i)} = \sum_{j=0}^{i} (-1)^j \ a_j^{(i)} Z_0^j$$

where the coefficient $a_j^{(i)}$ are found by the recursion law:

$$\begin{aligned} a_j^{(i)} &= j a_{j-1}^{(i-1)} + (j+1) a_j^{(i-1)} \\ a_o^{(i)} &= 1 \ ; \ a_{i+1}^{(i)} = 0 \end{aligned}$$

For the exponential in (2.7) we have

$$e^{-\beta(\varepsilon[n]_a - \mu n)} = e^{-\hat{\beta} n} \left[1 - \frac{\alpha_1^2}{3!} \beta \varepsilon n (n^2 - 1) + O(\alpha_1^4) \right] \tag{13.a}$$

$$e^{-\beta(\varepsilon[n]_A - \mu n)} = e^{-\hat{\beta} n} [1 - \alpha_1 \beta \varepsilon n (n-1) + O(\alpha_1^2)] \tag{13.b}$$

$$\begin{aligned} e^{-\beta(\varepsilon[n]_\alpha - \mu n)} &= e^{-\hat{\beta} n} \left\{ 1 - \beta \varepsilon \left[\alpha_2 n + \frac{\alpha_1^2}{3!} n (n^2 - 1) + \right.\right. \\ &+ \left.\left. \frac{1}{2} \alpha_2^2 n ((n-1)^2 - \beta \varepsilon n) \right] + O(\alpha_1^4, \alpha_2^4) \right\} \ . \end{aligned} \tag{13.c}$$

Substituting (2.7) and (2.10-12) in the entropy we get after a straightforward calculation

$$S_a = S_0 - \alpha_1^2 k_B \beta \varepsilon \hat{\beta} e^{-2\hat{\beta}} (1 - e^{-\hat{\beta}})^{-4} (e^{-\hat{\beta}} + 2) + O(\alpha_1^4) \tag{14.a}$$

$$S_A = S_0 - 4 \alpha_1 k_B \beta \varepsilon \hat{\beta} e^{-2\hat{\beta}} (1 - e^{-\hat{\beta}})^{-3} + O(\alpha_1^2) \tag{14.b}$$

$$\begin{aligned} S_\alpha &= S_0 - \alpha_2 k_B \beta \varepsilon \hat{\beta} e^{-\hat{\beta}} (1 - e^{-\hat{\beta}})^{-2} - \alpha_1^2 k_B \beta \varepsilon \hat{\beta} e^{-2\hat{\beta}} (e^{-\hat{\beta}} + 2)(1 - e^{-\hat{\beta}})^{-4} + \\ &- \alpha_2^2 k_B \beta \varepsilon e^{-\hat{\beta}} \{ \frac{\beta \varepsilon}{2} (1 - e^{-\hat{\beta}})^{-2} + \frac{\hat{\beta}}{2} [2 e^{-\hat{\beta}} (2 + 7 e^{-\hat{\beta}})(1 - e^{-\hat{\beta}})^{-4} \\ &- \beta \varepsilon (1 + e^{-\hat{\beta}})(1 - e^{-\hat{\beta}})^{-3}] \} \end{aligned} \tag{14.c}$$

with S_0 the entropy of the non-deformed system and S_a, S_A and S_α the entropies corresponding to the algebras (1.6,1.7a,b,c) respectively. It has been suggested [30] that there is an underlying lattice structure with lattice spacing $\alpha = q - 1$ for deformed theories with deformation parameter q. We see from the above formulae that the entropies of deformed systems are lower than the standard ones for $q_i > 1$, i.e. the deformations introduce a correlation which could be interpreted as an underlying lattice structure. When the deformation parameter is lower than one we do not have the lattice interpretation anymore and for the (2.14b,c) cases, due to a linear contribution in α_1 and α_2 respectively, the entropy increases with respect to the non-deformed case. This kind of oscillators (1.7) with $q < 1$ were shown to be related to the Brownian motion [31] which could explain the increase of entropy.

After a long but straightforward calculation the specific heat can be found as:

$$C^{(a)} = C_0 + \alpha_1^2 \, C_{\alpha_1^2}^{(a)} + \cdots \tag{15.a}$$

$$C^{(A)} = C_0 + \alpha_1 \, C_{\alpha_1}^{(A)} + \cdots \tag{15.b}$$

$$C^{(\alpha)} = C_0 + \alpha_2 C_{\alpha_2}^{(\alpha)} + \alpha_1^2 C_{\alpha_1^2}^{(\alpha)} + \alpha_2^2 C_{\alpha_2^2}^{(\alpha)} + \cdots \ , \tag{15.c}$$

with C_0 the standard specific heat and

$$\begin{aligned}
C^{(a)}_{\alpha_1^2} &= C^{(\alpha)}_{\alpha_1^2} = 2k_B\beta^2\varepsilon(\varepsilon-\mu)e^{-2\hat{\beta}}(2+e^{-\hat{\beta}})(1-e^{\hat{\beta}})^{-4} + \\
&- \; k_B\beta^3\varepsilon(\varepsilon-\mu)^2e^{-2\hat{\beta}}(2+e^{-\hat{\beta}})^2(1-e^{-\hat{\beta}})^{-5} \qquad (16.a)\\
C^{(A)}_{\alpha_1} &= 4k_B\beta^2\varepsilon(\varepsilon-\mu)e^{-2\hat{\beta}}(1-e^{\hat{\beta}})^{-3}\\
&\quad [2+e^{-\hat{\beta}}-2\beta(\varepsilon-\mu)(1-e^{-\hat{\beta}})^{-1}(1+e^{-\hat{\beta}})] \qquad (16.b)\\
C^{(\alpha)}_{\alpha_2} &= 2k_B\beta^2\varepsilon(\varepsilon-\mu)e^{-\hat{\beta}}(1-e^{-\hat{\beta}})^{-2} -\\
&\quad k_B\beta^3\varepsilon(\varepsilon-\mu)^2e^{-\hat{\beta}}(1+e^{-\hat{\beta}})(1-e^{-\hat{\beta}})^{-3} \qquad (16.c)\\
C^{(\alpha)}_{\alpha_2^2} &= -2k_B\beta^2\varepsilon(\varepsilon-\mu)e^{-2\hat{\beta}}(2+7e^{-\hat{\beta}})(1-e^{-\hat{\beta}})^{-4} -\\
&\quad k_B\beta^2\varepsilon^2e^{-\hat{\beta}}(1-e^{-\hat{\beta}})^{-2} + 4k_B\beta^3\varepsilon^2(\varepsilon-\mu)e^{-\hat{\beta}}(1+e^{-\hat{\beta}})(1-e^{-\hat{\beta}})^{-3}\\
&\quad -k_B\beta^3\varepsilon(\varepsilon-\mu)^2e^{-\hat{\beta}}(1-e^{-\hat{\beta}})^{-5}(20-64e^{-\hat{\beta}}+35e^{-2\hat{\beta}}-27e^{-3\hat{\beta}})\\
&\quad -\frac{3}{2}k_B\beta^4\varepsilon^2(\varepsilon-\mu)^2e^{-\hat{\beta}}(1-e^{-\hat{\beta}})^{-4}(1+4e^{-\hat{\beta}}+e^{-2\hat{\beta}}). \qquad (16.d)
\end{aligned}$$

If we consider the high-temperature approximation, $\beta \to 0$, one can see that when the deformation parameter is bigger than one the specific heat is lower than the standard one, C_o. As before, in this limit, when the contribution of the deformation for the specific heat is linear and the deformation parameter is lower than one the specific heat increases with respect to C_o. It is interesting to notice that the system has the memory of the deformation in the $\beta \to 0$ limit.

The difficulty in computing the exact value of the thermodynamic potentials in the case of the Hamiltonian (2.1) can be understood in the formalism of TFD in the following way: Consider the thermal vacuum for the case of the Hamiltonian (2.1) and call it $|0(\beta)>_q$, where

$$|0(\beta)>_q = Z^{-1/2}(\beta)\sum_n e^{-\frac{\beta}{2}(\varepsilon[n]_A-\mu n)}|n,\tilde{n}> \; , \qquad (17)$$

then it is possible to express (2.15) in terms of the thermal vacuum for the Hamiltonian (1.12) $|0(\hat{\beta})>$

$$|0(\hat{\beta})> = Z_0^{-1/2}(\hat{\beta})\sum_n e^{-\hat{\beta}n}|0> \qquad (18)$$

as

$$|0(\beta)>_q = Z^{-1/2}(\beta)F(\partial_{\hat{\beta}})Z_0^{1/2}(\hat{\beta})|0(\hat{\beta})> \qquad (19)$$

with

$$F(x) = e^{-\beta\varepsilon(\frac{1}{2}[-2x]_A+x)} \; . \qquad (20)$$

Showing that the transformation from the non-deformed thermal vacuum to the deformed one (2.17) is highly non-trivial, being simple only when the deformed and non-deformed Hamiltonians have the same spectrum.

Final Remarks

We have considered in this paper the approach of TFD for two different Hamiltonians, each one made up of a different kinds of operators satisfying deformed Heisenberg algebras. The first Hamiltonian is linear in the number operator N and does not present any difficulty, however the second Hamiltonian generates a partition function and thermodynamic potentials which cannot be exactly computed for arbitrary values of the deformation parameter. Even if it is more complicated the second Hamiltonian is more interesting; it is non-interacting with non-additive energy.

We have also seen that the exact computation of mean values for general deformed Hamiltonians is a rather complicated issue, the reason being the non-trivial relation (2.17). We think that it would be very useful to develop a variational approach to deal with such systems.

The non-additivity of energy seems to be a deep property of deformed systems and its is already present in the deformed Poincaré group [32]. There is still no strong evidence of the presence of the Quantum Group structure in physical systems, we believe that the non-additivity of energy present in deformed systems could give evidence of the presence of such structures as discussed in [32] for the case of deformed Poincaré group. The same issue is analysed in the context of the Generalized Statistical Mechanics [33, 34] in ref. [21].

We also consider that it would be interesting to develop TFD in the context of the Generalized Statistical Mechanics [33, 34], which would permit the analysis of the models we have discussed within this approach.

Acknowledgements

Some of the ideas contained in the present paper has benefited from discussions with C. Tsallis.

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