CBPF-NF-059/98

Topological contributions to fermionic correlators and nonperturbative aspects of QCD in two dimensions

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To appear in the Int. J. Mod. Phys. **A** (1998).

Abstract

We analyze the formation of fermionic condensates in two dimensional quantum chromodynamics for matter in the fundamental representation of the gauge group. We show that a topological regular instanton background is crucial in order to obtain nontrivial correlators. We discuss both massless and massive cases.

Key-words: Condensates; QCD; Topological sectors.

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Motivation

Correlation functions are fundamental quantities in the understanding of nonperturbative QCD and hadron physics. The key point is that they can be considered in terms of fundamental QCD fields or in terms of physical intermediate states. For instance,

$$\langle \bar\psi\psi(x)\bar\psi\psi(0)\rangle = \frac{1}{\pi}\int ds\ \mathrm{Im}K(s)D(s^{1/2},x), \tag{1}$$

where Im K is the spectral density, describing the squared matrix elements of the operators $\bar\psi\psi$ between the vacuum and all the hadronic states of mass $s^{1/2}$, and $D(s^{1/2},x)$ is the corresponding propagator. The spectral density is proportional to the normalized cross-section of anihilation processes, hence, using hadronic phenomenological data, one can extract valuable information about the microscopic structure of the theory. For example, the conservation of the quantum numbers of the vacuum implies that a net chiral charge is carried by such correlators, thus, the difference between vector and axial correlators is entirely due to the chiral asymmetry of the QCD vacuum, see e.g. [1].

One evidence of a non-perturbative vacuum structure is given by the Gell-Mann Oakes Renner relation (see e.g. [2])

$$\langle \bar u u + \bar d d\rangle = -\frac{2f_\pi^2 m_\pi^2}{(m_u+m_d)} \tag{2}$$

since such nonzero result suggests the existence of a dynamical mass in the (massless) quark propagator

$$\langle\bar\psi\psi\rangle = -i\mathrm{lim}_{y\to x^+}\mathrm{Tr}S_F^{full}(x,y) \tag{3}$$

$$S_F^{full}(q) = \frac{A(q^2)}{\not{p} - M(q^2)} \tag{4}$$

Within perturbation theory this mass, $M(q^2)$, is zero; therefore, condensates are particularly useful in connection with chiral symmetry breaking coming from *non-perturbative* effects, for instance, due to an underlying *topological structure.*

The strong attractive interaction among soft quarks and the low energy cost of creating a massless pair seem to be responsible for the instability of the Fock vacuum of massless fermions, giving rise to a quark anti-quark condensation. Therefore, in order to get some insight into the complex structure of the QCD ground-state one should consider possible mechanisms for chiral symmetry breaking and condensate formation.

Quantum chromodynamics in two dimensions (QCD_2), is a convenient framework to discuss this kind of phenomena since it presents several basic features suited for a fundamental four-dimensional theory (non-abelian character, chirality properties, etc.) and, furthermore, analytical results can be generally obtained.

We will use a path-integral approach which is very appropriate to handle non-Abelian gauge theories with topological sectors and to perform a series expansion in the fermionic mass.

Decoupling and Fermionic correlators

We work over $SU(N)$ Yang-Mills gauge fields coupled to massless Dirac fermions in the fundamental representation of the gauge group, in Euclidean two-dimensional space-time:

$$L = \bar{\psi}^q (i\partial_\mu \gamma_\mu \delta^{qq'} + A_{\mu,a} t_a^{qq'} \gamma_\mu) \psi^{q'} + \frac{1}{4g^2} F_{\mu\nu}^a F_{\mu\nu}^a, \tag{5}$$

where the meaning of the indices is clear. As it is done with fermions in four dimensions, we will consider a background of gauge fields belonging to nontrivial topological sectors (see next section for a more detailed discussion). In order to proceed to the path-integration it is worth performing a decoupling transformation of gluons from fermions. It is important that the decoupling operation does not change the topological sector in which the Dirac operator is defined. Hence, we decompose every gauge field belonging to the n^{th} topological sector in the form

$$A_\mu^a(x) = A_\mu^{a\,(n)} + a_\mu^a. \tag{6}$$

$A_\mu^{(n)}$ is a fixed, classical configuration of the n^{th} topological class and a_μ is the path-integral variable which takes into account quantum fluctuations and belongs to the trivial topological sector [3]. Thus, the integration measure must be only defined on the $n = 0$ sector while the Dirac operator depends on a regular localized topological background $A_\mu^{(n)}$, as we will see in what follows.

To compute fermionic correlators containing products of local bilinears $\bar{\psi}\psi(x)$, we decouple fermions from the a_μ field through a chiral rotation within the topologically trivial sector, yielding a Fujikawa jacobian [4]. The choice of an appropriate background like

$$A_+^{(n)} = 0 \tag{7}$$

is important in order to control the zero-mode problem, so we switch to light-cone coordinates [5].

Let us start by introducing group-valued fields to represent $A_\pm^{(n)}$ and $a_\pm$

$$A_-^{(n)} = id\partial_- d^{-1} \tag{8}$$

$$a_+ = iu^{-1}\partial_+ u \qquad a_- = id(v\partial_- v^{-1})d^{-1} \tag{9}$$

and accordingly define ζ by

$$\psi_+ = dvd^{-1}\zeta_+ \quad \psi_- = u^{-1}\zeta_- \tag{10}$$

The Dirac equation then takes the form

$$\not{D}[A^{(n)} + a] \begin{pmatrix} \psi_+ \\ \psi_- \end{pmatrix} = \begin{pmatrix} 0 & u^{-1}i\partial_+ \\ dvd^{-1}D_-[A^{(n)}] & 0 \end{pmatrix} \begin{pmatrix} \zeta_+ \\ \zeta_- \end{pmatrix} \tag{11}$$

Thus, the interaction Lagrangian in the n^{th} flux sector decouples as follows

$$L = \bar{\psi}\; \not{D}[A^{(n)} + a]\psi = \quad \zeta_+^* \; \not{D}_-[A^{(n)}]\zeta_+ + \zeta_-^* i\partial_+\zeta_- \tag{12}$$

In terms of the representation (9-11) the fermionic determinat can be suitably factorized by repeated use of the Polyakov-Wiegmann identity [6], resulting in a gauge independent expression

$$\det \not{D}[A^{(n)} + a] = \mathcal{N} \; \det \not{D}[A^{(n)}] \; \exp -S_{eff}[u, v; A^{(n)}] \tag{13}$$

where

$$\begin{aligned} S_{eff}[u, v; A^{(n)}] &= W[u] + W[v] + \frac{1}{4\pi} tr_c \int d^2x (u^{-1}\partial_+ u)(d\partial_- d^{-1}) \\ &+ \frac{1}{4\pi} tr_c \int d^2x \, (d^{-1}\partial_+ d)(v\partial_- v^{-1}) + \frac{1}{4\pi} tr_c \int d^2x \, (u^{-1}\partial_+ u)\, d\, (v\partial_- v^{-1})\, d^{-1}, \end{aligned} \tag{14}$$

$W[u]$ being the usual Wess-Zumino-Witten action. Notice that the fermionic jacobian associated with eq.(10) is precisely the quotient of the determinants in eq.(13).

Once the determinant has been written in the form (13), one can work with any gauge choice. The partition function shows the following structure

$$\begin{aligned} Z \;\; = \;\; & \sum_n \det(\not{D}[A^{(n)}]) \int \mathcal{D}a_\mu \, \Delta_{FP} \, \delta(F[a]) \\ & \exp\left(-S_{eff}[A^{(n)}, a_\mu] - \frac{1}{4g^2} \int d^2x F_{\mu\nu}^2[A^{(n)}, a_\mu] \right) \end{aligned} \tag{15}$$

where $\Delta_{FP}\, \delta(F[a])$ comes from the gauge fixing.

As it happens in the Abelian case, the partition function of two dimensional quantum chromodynamics only picks the contribution from the trivial sector because $\det(\not{D}[A^{(n)}]) = 0$ for $n \neq 0$ (see eq.(15)). In contrast, various correlation functions become nontrivial precisely for $n \neq 0$ thanks to the zero-mode contributions when Grassman integration is performed.

In terms of the new fields, the elementary bilinear $\bar{\psi}\psi$ takes the following form

$$\bar{\psi}\psi \;\; = \;\; \zeta_-^* udvd^{-1}\zeta_+ + \zeta_+^*(udvd^{-1})^{-1}\zeta_- \tag{16}$$

and after a lengthy calculation, arbitrary non-Abelian correlators of fundamental fermions are found:

$$
\begin{aligned}
&\langle \bar\psi\psi(x^1)\dots\bar\psi\psi(x^l)\rangle = \sum_n \int \mathcal{D}u\mathcal{D}v\ \Delta_{FP}\ \delta(F[a_\mu])\ \exp[-S^{Bos}_{full}(A^{(n)},u,v)] \\
&\int \mathcal{D}\bar\zeta\mathcal{D}\zeta\ \exp(\zeta^*_+\ \not{D}_-[A^{(n)}]\zeta_+ + \zeta^*_- i\partial_+\zeta_-) \\
&\big(B^{q_1p_1}(x^1)\dots B^{q_lp_l}(x^l)\ \zeta^{*q_1}_-\zeta^{p_1}_+(x^1)\dots\zeta^{*q_l}_-\zeta^{p_l}_+(x^l) + B^{q_1p_1}(x^1)\dots \\
&B^{-1q_lp_l}(x^l)\ \zeta^{*q_1}_-\zeta^{p_1}_+(x^1)\dots\zeta^{*q_l}_+\zeta^{p_l}_-(x^l) + B^{q_1p_1}(x^1)\dots \\
&B^{-1q_{l-1}p_{l-1}}(x^{l-1})B^{-1q_lp_l}(x^l)\ \zeta^{*q_1}_-\zeta^{p_1}_+(x^1)\dots\zeta^{*q_{l-1}}_+\zeta^{p_{l-1}}_-(x^{l-1})\zeta^{*q_l}_+\zeta^{p_l}_-(x^l) \\
&+\dots+B^{-1q_1p_1}(x^1)\dots B^{-1q_lp_l}(x^l)\ \zeta^{*q_1}_+\zeta^{p_1}_-(x^1)\dots\zeta^{*q_l}_+\zeta^{p_l}_-(x^l)\big)\,, \qquad (17)
\end{aligned}
$$

where $B = udvd^{-1}$. This is a general and completely decoupled expression for fermionic correlators, which shows that the simple product one finds in the Abelian case becomes here an involved sum due to color couplings.

The fermionic path-integral can be easily performed, amounting to a sum of products of zero modes of the Dirac operator. Concerning the bosonic sector, the presence of the Maxwell term crucially changes the effective dynamics with respect to that of a pure Wess-Zumino model. One then has to perform approximate calculations to compute the bosonic factor, for example, by linearizing the group transformation [7]; nevertheless, the point relevant to our discussion of obtaining nonzero fermionic correlators is manifest in eq.(17).

The introduction of a flavor index implies additional degrees of freedom which result in N_f independent fermionic field variables. Consequently, the growing number of Grassman (numeric) differentials calls for additional Fourier coeficients in the integrand. On the other hand, dealing with N_f fermions coupled to the gauge field, leads to the fermionic jacobian computed for one flavor to the power N_f, the bosonic measure remaining the same. For a given number of zero modes, more fermion bilinears will be needed in order to obtain a nonzero fermionic path-integral. Since flavor yields a factor N_f on the number of Grassman coeficients, the minimal nonzero product of fermion bilinears requires a factor N_f on the number of insertions.

Remark on Topology

At this point, it is worth making some clarifying remarks concerning topological gauge field configurations in this context. In two Euclidean dimensions, finite action *topologically stable* regular configurations of a gauge theory, necessarily require maximal symmetry breaking, i.e. up to Z_N for the $SU(N)$ case. Unlike in four dimensions, where finite action solutions do exist for pure Yang-Mills theory, true instanton *solutions* in 2d are Nielsen-Olesen type vortices [8] (both in the Abelian and non-Abelian case) thus requiring maximal spontaneous symmetry breaking through Higgs scalars. It is important to note that even with adjoint fermions, where the center of $SU(N)$ is immaterial (thus allowing nontrivial winding gluon fields in the theory) *there is no topologically stable regular solution* as it stands. The best one can do in that case is choosing an Abelian-like vortex which can be taken from an effective bosonic action coming from the Schwinger model, as an ansatz. However, it is a meaningful procedure just for high temperatures; at low temperatures, one has to construct convenient vortex configurations by hand [9].

For fundamental quarks, in turn, we can proceed as explained above, putting fermions in the background of the Z_N-vortices of de Vega and Schaposnik [10] which give a realization in a two-dimensional Euclidean theory of regular gauge field configurations carrying a topological charge. This is ensured by the presence of a Higgs field in the adjoint which maximally breaks the gauge symmetry down to $SU(N)/Z_N$.

Then, if one is to consider instantons in two-dimensional non-Abelian gauge theories, the relevant homotopy group is $\pi_1(SU(N)/Z_N)$ for fermions in the adjoint as well as in the fundamental. The homotopy group being Z_N, signals the appearence of N topologically different sectors and instanton effects become apparent. Gauge field configurations lying in the Cartan subalgebra of $SU(N)$ generate nontrivial topological fluxes [11] and the index of the Dirac operator can be computed for fundamental fermions in a model involving the above mentioned adjoint scalar fields [12]. These winding configurations may be used in the nonperturbative analysis of a gauge theory like QCD_2 when coupled to massless fermions, as we have shown. This yields important phenomenological results, especially, concerning chiral symmetry breaking. Once regular gauge field configurations carrying a topological charge are identified, the associated fermionic zero modes can be found [13], and then used to study the formation of fermion condensates.

Large N limit and BKT effect

As a by-product, our approach gives $\langle\bar{\psi}\psi\rangle = 0$ in every flux sector, including $n = 0$, as expected. This result is in agreement with independent analytical calculations based on operator product expansion and dispersion relations [14], and canonical quantization on the light-cone front [15] (for any finite N). However, previous analyses have not taken into account topological sectors.

In any case, despite this zero result, the existence of a nonvanishing elementary condensate is desirable in the model, as it turns out in real QCD. Actually, such an outcome has been put forward for one flavor QCD_2 within the alternative scenarios mentioned above, provided one works with an infinite number of colors, assuming that cluster decomposition holds in order to compute this condensate from a two-point one.

In a sense, this reminds the case of massless two-dimensional QED. In the Abelian case, a multipoint composite receives contributions from different topological sectors. In particular, one can obtain the value of the elementary scalar condensate from a two-point correlator. From cluster decomposition the term $\langle\bar{\psi}_+\psi_+(x)\bar{\psi}_-\psi_-(y)\rangle$ factorizes as $\langle\bar{\psi}_+\psi_+(x)\rangle\cdot\langle\bar{\psi}_-\psi_-(y)\rangle$, the two factors being equal to each other and nonzero [16]. Then, by means of the chiral decomposition $\langle\bar{\psi}\psi(x)\rangle = \langle\bar{\psi}_+\psi_+(x)\rangle + \langle\bar{\psi}_-\psi_-(x)\rangle$ one is able to construct the condensate from the trivial topological sector whereas, in fact, $\langle\bar{\psi}_\pm\psi_\pm(x)\rangle$ come exclusively from flux classes ± 1 respectively. Now, we would like to find analogous features in two-dimensional QCD, in order to emphasize the role played by instanton contributions in the non-Abelian counterpart, as it seems to occur in four dimensions. We will see below that this is possible.

In the non-Abelian theory, one needs the large N limit for cluster decomposition to take place, so that all v.e.v. can be reduced to a product of elementary scalar densities, i.e. $\langle AB\rangle = \langle A\rangle\langle B\rangle + O(1/N)$. On the other hand, for very separated composites factorization also holds.

By means of QCD sum rules, in the weak coupling regime it can be shown that a BKT effect (Berezinskii-Kosterlitz-Thoules [17]) takes place [14]. For very large distances and number of colors:

$$\langle\bar{\psi}_+\psi_+(x)\bar{\psi}_-\psi_-(y)\rangle \sim |x-y|^{-1/N} \tag{18}$$

implying that for large, but finite N, it smears away softly as $|x-y| \to \infty$. Though, when N is infinity, it is nonzero.

In fact, this effect has been found when the $M \to 0$ limit is taken at the end of the calculation in the *massive* theory, provided $M >> g \sim 1/\sqrt{N} \to 0$ ('t Hooft regime).

One could then say that after the chiral weak phase is reached in the massive theory, the symmetry still keeps in a broken phase, but instead, it happens dynamically. In the large N limit the physics changes, the vacuum being dressed non-perturbatively by means of purely planar diagrams. The spectrum is completely different depending on the relations shown above: In the weak phase there is an infinite number of massive mesons while in the strong coupling phase there are just massless baryons.

Outside any of the asymptotic relations above, the actual results, for arbitrary values of both color and relative positions, are given by eq.(17), provided topological sectors are included. As already announced by the chiral jacobian given by (13), the existence of nontrivial correlators signals that a chiral anomaly takes place in the model. This is expected according to the Coleman's theorem, which prohibits the spontaneous breakdown of continuous symmetries in two dimensions [18].

The massive case

Now, in order to find out the above nonzero outcome for the elementary condensate in our approach, we will extend our procedure to massive QCD_2 as follows. The partition function now is

$$Z_M = \int \mathcal{D}\bar{\psi}\mathcal{D}\psi\mathcal{D}A_\mu \exp(-\int d^2x\, L_{M=0})\; e^{-M\int d^2x\,\bar{\psi}\psi} \tag{19}$$

so that, it is apparent that a series expansion may be workable [19]. Actually, the analytical solution to this model has not been found and such a challenge might be a hopeless effort. Nevertheless, for our purposes only small masses need to be examined so we may perform a perturbative expansion in terms of the quark mass. Then, the minimal condensate now reads

$$\langle\bar{\psi}\psi(\omega)\rangle_M = \sum_n \langle\bar{\psi}\psi(\omega)\rangle^{(n)}_{M=0} + M\,\sum_n \int d^2x\,\langle\bar{\psi}\psi(\omega)\bar{\psi}\psi(x)\rangle^{(n)}_{M=0} + \frac{1}{2}M^2\,\sum_n \int d^2x\,d^2y\,\langle\bar{\psi}\psi(\omega)\bar{\psi}\psi(x)\bar{\psi}\psi(y)\rangle^{(n)}_{M=0} + \dots \tag{20}$$

In this fashion, it is apparent that for $M \neq 0$ this elementary condensate receives contributions from every correlator coming from the massless theory. As we have seen in the first section, this is a pleasant result in order to mimic real four-dimensional strong interactions among quarks, where an instanton vacuum seems to be responsible for the chiral symmetry breaking.

Since the fermionic sector has been completely decoupled, the counting of the nonzero pieces simply follows from that of the Abelian case [20, 21] because the topological structure here can be red out from the torus of the gauge group.

In the compactified space, the existence of nN normalizable zero modes in topological sector n [13, 12] implies the vanishing of the first summatory in eq.(20) $\forall n$. However, for higher powers of M it is clear that certain nonzero contributions come into play. As we have explained, the existence of zero modes with a definite chirality (positive for $n > 0$, negative for $n < 0$) set the (Grassman) integration rules to compute v.e.v's.

By using the chiral decomposition $\langle\bar{\psi}\psi(x)\rangle = \langle\bar{\psi}_+\psi_+(x)\rangle + \langle\bar{\psi}_-\psi_-(x)\rangle$ one can see that the first N powers of M ($j = 1 \dots N$) receive an input from the trivial topological sector alone. Then, for $j \geq N$, the $n = 1$ sector starts contributing together with $n = 0$. For powers $j \geq 2N$ the contribution of topological sector $n = 2$ starts, and the counting follows so on in this way.

Now, it can be easily seen that the number of contributions grows together with the number of colors. As we let N go to infinity the elementary 'massive' condensate does so. On the other hand, since within each nontrivial topological sector the number of zero modes grows also to infinity, one has divergent v.e.v. everywhere in the series expansion of eq.(20). Accordingly, high order terms could also produce a nonzero outcome in the chiral limit. Therefore, it is clear that the limit $M \rightarrow 0$ becomes matter of a careful analysis; namely, combined with a large number of colors, eq.(20) leaves place enough for a nontrivial elementary condensate even in the chiral limit.

Summary

General fermionic condensates have been computed in non-Abelian gauge theories in order to discuss chiral symmetry breaking and vacuum properties of QCD in two dimensions.

We have shown that correlation functions can be completely determined by the topological structure underlying the theory. As we have explained, regular gauge fields lying in the Cartan subalgebra of $SU(N)$ have to be taken into account to find a significant outcome for general fermionic correlators.

As it was discussed at the beginning of this paper, this is a welcome result, inasmuch our scheme enables a chiral symmetry breaking mechanism at the vacuum level, by means of instanton contributions. This is clearly reflected in the quantum calculation, which exhibits the formation of nonvanishing vacuum expectation values of fermionic bilinears.

Finally, our analysis of the massive theory put forward possible roots to the BKT phenomenon coming from topological considerations, by means of a path-integral approach. In this way, although an exact result has not been given so far, both the zero outcome in the massless theory and the nontrivial value in the massive one, have gained contact with the results emerging from alternative scenarios [14, 15, 22, 23].

Acknowledgments

I would like to thank T. Hatsuda, F. Lenz, F. Schaposnik, E. Shuryak, J. Stephany and I. Zahed for enlightening comments and suggestions. This work was supported by CBPF and CLAF-CNPq, Brazil.

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