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$A_n^{(1)}$ Toda Solitons and the Dressing Symmetry

by

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Abstract

We present an elementary derivation of the soliton–like solutions in the $A_n^{(1)}$ Toda models which is alternative to the previously used Hirota method. The solutions of the underlying linear problem corresponding to the N–solitons are calculated. This enables us to obtain explicit expression for the element which by dressing group action, produces a generic soliton solution. In the particular example of monosolitons we suggest a relation to the vertex operator formalism, previously used by Olive, Turok and Underwood. Our results can also be considered as generalization of the approach to the sine–Gordon solitons, proposed by Babelon and Bernard.

Key-words: Solitons; Integrable field theories.

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1 Introduction

The solitons appear in various topics of the modern mathematical and elementary particle physics [1]. Generically, the solitons represent field configurations which are topologically non–trivial. Therefore they turn to be a natural object in studying nonperturbative effects in the quantized theory [1], [2]. There is an another (non–equivalent) approach to the solitons which treats them as solutions of integrable non–linear evolution equations [3], [4], [5]. The crucial point is that the equations of motion appear as a zero curvature condition of a certain Lax connection. As a consequence of this, one can map, in a manner explained in [3]–[5], the original dynamical variables into a set of scattering data, which due to the integrability, satisfy linear evolution equations. This transformation, called direct spectral transformation, yields the action–angle variables of the system. To get back the original dynamical variables one has to perform the so–called inverse scattering transformation. This approach, known in the literature as the Inverse Scattering Method (ISM) [3]–[5] provides an elegant procedure to solve integrable non–linear evolution equations.

Within the ISM, the solitons arise after imposing the vanishing of the reflection coefficient of the corresponding Lax operator. Due to the last condition, the inverse scattering equations reduce to a linear algebraic system, which only depends on the scattering data related to the discrete spectrum of the Lax operator.

The Toda equations [6] admit a Lax representation [7], and due to the existence of a classical r–matrix, are integrable. In [7] another important idea has been advanced: to use generalized Cartan matrices in order to obtain integrable field equations. In particular, the affine Toda theories correspond to the extended Cartan matrices of simple Lie algebras. The last are derived from the simple Lie algebra Cartan matrices by adding the extended root which is minus the highest root. There is an alternative approach [8] which consists in the study of the properties of a Lax operator which has a special form. The ideas developed in [8] has been further explored in [9] to get generalized Drinfeld–Sokolov integrable hierarchies. The crucial step within this approach is to expand, after a suitable gauge transformation, the components of the Lax connection into certain maximally commuting subalgebra of the underlying affine Lie algebra. This ensures the existence of infinite number of conserved quantities. Similar procedure applied to the affine Toda models appeared also in [10].

The dressing group is a special symmetry of the integrable evolution equations [11]. It admits an elegant interpretation within the tau–function approach [12]. Dressing transformations act on the components of the corresponding Lax connection without changing its form [13]. Therefore, the dressing group turns out to be a symmetry of the phase space. To ensure the covariance of the symplectic structure under dressing transformations, one has to introduce a specific Poisson bracket on the dressing group [13], [14]. In this paper we are not going to analyze this problem since it presents difficulties even for the dressing group elements which generate monosolitons from the vacuum in the sine–Gordon theory [15]. The dressing symmetry has been exploited recently [16] to treat a huge class of integrable hierarchies which admit a vacuum solution.

The soliton solutions in the $A_n^{(1)}$ Toda model are found by Hollowood [17] who used the Hirota method. Later it became clear that the Hirota method can be applied to get solitons in arbitrary affine Toda models [18]. The relation between the soliton solutions in the affine Toda theories and special elements in the affine (or Kac–Moody) Lie algebras, called vertex operators is clarified in [19]. An intriguing property of the formalism, developed in these papers is that it permits generalization for the Toda theories based on twisted affine Lie algebras [20].

The actual interest to the affine Toda solitons is due to their relation to the N–body integrable systems [21]. It turns out that the solitons are related to certain (relativistically invariant) integrable systems with finite degrees of freedom. Another interesting property of the affine Toda solitons is that they are closely related to the Seiberg–Witten duality [22].

We outline the content of this paper. In Sec. 2 we generalize a method to get solitons introduced by Date [23] which is complementary both to the Hirota method [17], [18] and to the ISM [24]. To illustrate the generality of our approach we fix the coupling constant to be real. This wants to say that we are working with algebraic instead of the usually treated physical solitons which only exist for imaginary values of the coupling constant. As a result we obtain the N–solitons as they were expressed in [21]. In Sec. 3 we evaluate the dressing group element which generates solitons from the vacuum in the defining representation of $sl(n+1)$. Our result appears to be a generalization of the element calculated by Babelon and Bernard [25]. Sec. 4 is devoted to the relation to the vertex operator construction of solitons advanced by Olive, Turok and Underwood [19].

2 Soliton solutions in the $A_n^{(1)}$ Toda theories.

The problem of finding soliton solutions in the affine Toda models was previously approached in the literature by using the Hirota method [17], [18] and with the help of group theoretical methods [19]. Soliton solutions only exist for imaginary values of the coupling constant. Moreover, it was clarified that despite the equations of motion and the lagrangian density are complex, the solitons carry *real* momentum and energy. The properties of the affine Toda systems, both for real and imaginary coupling constants are studied within the ISM [24]. The standard scheme of the ISM meets certain obstructions when applied to the affine Toda models based on arbitrary simple Lie algebras. In particular, it turns out that the Jost solutions [5], and therefore the elements of the transition matrix, loose their nice analiticity properties as functions on the spectral parameter. In this Section we generalize an elegant method [23] to obtain the $A_n^{(1)}$ Toda solitons. It exploites two important features of the soliton solutions: first, due to the vanishing of the reflection coefficient of the auxiliary linear problem, the corresponding Jost solutions are single valued functions on the spectral parameter λ; second, the soliton Jost solutions are uniquely determined by the scattering data related to the discrete spectrum of the underlying Lax operator. Applying the ideas developed in [23] to the $A_n^{(1)}$ Toda models, we recover the equations describing the discrete spectrum of the corresponding linear problem, by using a finite order inner automorphism σ of the simple Lie algebra A_n. The last has order $n+1$ and introduces the so called *principal gradation* in the affine Lie algebra $A_n^{(1)}$ [26].

The $A_n^{(1)}$ Toda equations are equivalent to a zero–curvature condition of a connection, the components of which belong to the loop Lie algebra $\widetilde{sl}(n+1)$ in the principal gradation. We will start by introducing certain basic facts concerning the Lie algebras $sl(n+1)$, $\widetilde{sl}(n+1)$ and the notion of gradation [26], [27]. As it is well known, $sl(n+1)$ is the Lie algebra of the traceless $(n+1)\times(n+1)$ matrices. Denote by E^{ij} the elementary matrices $E^{ij} = |i><j|, \;\; i,j = 1\ldots n+1$ which satisfy the commutation relations of the Lie algebra $gl(n+1)$

$$\left[E^{ij}, E^{kl}\right] = \delta^{jk}E^{il} - \delta^{il}E^{kj} \tag{2.1}$$

The Cartan subalgebra $\mathcal{H}$ is generated by the traceless combinations of the diagonal matrices E^{ii} $H_\xi = \sum_{i=1}^{n+1}\xi_i E^{ii}, \;\; \sum_i \xi_i = 0$. The rank of $sl(n+1)$ is n. To describe the root system we fix an orthonormalized basis $\{e_i\}$ in the $n+1$–dimensional Euclidean space. Then the roots are exhausted by $\alpha_{ij} = e_i - e_j, \;\; i \neq j$. The corresponding step operators are $E^{\alpha_{ij}} = E^{ij}$. As simple roots one can choose the elements $\pm\alpha_i = \pm(e_i - e_{i+1}), \;\; i = 1,\ldots n$. The step operators satisfy the commutation relations

$$\begin{aligned}\left[H_\xi, E^{\pm\alpha_i}\right] &= \pm\alpha_i.\xi E^{\pm\alpha_i} = \pm(\xi_i - \xi_{i+1})E^{\pm\alpha_i}\\ \left[E^{\alpha_i}, E^{-\alpha_j}\right] &= \delta_{ij}H_{\alpha_i}\end{aligned} \tag{2.2}$$

The rest of the step operators is obtained by taking successive commutators of the step operators E^{α_i} and of their transposed $E^{-\alpha_i}$. The highest root is $\psi = \alpha_1 + \ldots + \alpha_n = e_1 - e_{n+1}$. This can be translated in the language of the step operators: $\left[E^{\psi}, E^{\alpha}\right] = 0$ for any step operator related to a positive root α. Note also that $E^{\psi} = E^{1n+1}$. One also introduces the extended root $\alpha_0 = -\psi$ and its step operator $E^{\alpha_0} = E^{n+11}$.

We proceed by recalling some facts about the finite order inner automorphisms of the simple Lie algebras. There is a general theorem due to Kac [26] (for an introductory review see also [9]) which states that the finite order inner automorphisms of a simple Lie algebra $\mathcal{G}$ are parametrized by $r+1$ relatively prime non-negative integers where r stands for the rank of the Lie algebra. In what follows, we shall need a special inner automorphism σ of $\mathcal{G} = sl(n+1)$, the order of which is $n+1, \;\; \sigma^{n+1} = 1$. To define it, we first recall that the fundamental weights λ_i are dual to the simple roots $2\frac{\alpha_i\cdot\lambda_j}{\alpha_i\cdot\alpha_i} = \delta_{ij}, \;\; i,j = 1, \ldots, r$. Specifying $\mathcal{G} = sl(n+1)$ one gets

$$
\begin{aligned}
\lambda_i &= \sum_{k=1}^{i} e_k - \frac{i}{n+1}\sum_{k=1}^{n+1} e_k \\
i,j &= 1,..,n
\end{aligned}
\tag{2.3}
$$

Consider also the vector in the root space

$$
\rho = \sum_{i=1}^{n} \lambda_i \tag{2.4}
$$

Using the above notations one defines the following inner automorphism $X \to \sigma(X)$ of the Lie algebra $sl(n+1)$ [19], [20], [26]

$$
\begin{aligned}
\sigma(X) &= SXS^{-1} \\
S &= e^{2\pi i \frac{H_\rho}{n+1}}
\end{aligned}
\tag{2.5}
$$

where the diagonal matrix $H_\rho \in \mathcal{H}$ depends linearly on the vector ρ (2.4) $H_\rho = \sum_{k=1}^{n+1} \rho_k |k><k|$; ρ_k are the components of (2.4) in the basis $\{e_i\}$ (2.3). In view of (2.2), σ acts diagonally in the corresponding Cartan–Weyl basis

$$
\begin{aligned}
\sigma(H_\xi) &= H_\xi \\
\sigma(E^{\alpha_{kl}}) &= \omega^{\alpha_{kl}\cdot\rho} E^{\alpha_{kl}} = \omega^{l-k} E^{\alpha_{kl}} \\
\omega &= e^{\frac{2\pi i}{n+1}}
\end{aligned}
\tag{2.6}
$$

from where it becomes clear that σ has order $n+1$. Note that the Lie algebra $\mathcal{G} = sl(n+1)$, equipped by the above automorphism becomes a graded Lie algebra:

$$
\begin{aligned}
\mathcal{G} &= \oplus_{k\in\mathbb{Z}_{n+1}} \mathcal{G}_k \\
\sigma(\mathcal{G}_k) &= \omega^k \mathcal{G}_k \\
[\mathcal{G}_k, \mathcal{G}_l] &\subseteq \mathcal{G}_{k+l}
\end{aligned}
\tag{2.7}
$$

in the above equations and in what follows we adopt the following convention: the summation of indices which take values in the cyclic group $\mathbb{Z}_{n+1}$ is understood modulo $n+1$.

There exists an alternative basis in $sl(n+1)$ closely related to the automorphism (2.5). We shall briefly review its construction (for further details, see [19], [20]). First of all one observes that the generators

$$
\begin{aligned}
\mathcal{E}_i &= \sum_{k=1}^{n+1-i} E^{kk+i} + \sum_{k=1}^{i} E^{n+1+k-ik} = \\
&= \sum_{k\in\mathbb{Z}_{n+1}} E^{kk+i}, \;\; i = 1, \ldots n
\end{aligned}
\tag{2.8}
$$

are mutually commuting. Therefore they span another Cartan subalgebra $\mathcal{H}'$. In the second identity of the above equation the summation index $1 \leq k \leq n+1$ is read modulo $n+1$. Due to (2.6), the elements (2.8) are eigenvectors of the automorphism σ *

$$\sigma(\mathcal{E}_i) \;=\; \omega^i \mathcal{E}_i \tag{2.9}$$

Fixing the defining $n+1$–dimensional representation, it is not difficult to verify that the matrix with entries

$$\begin{aligned} \Omega_{ij} &= \omega^{(i-1)(j-1)} \\ \Omega^{-1}_{ij} &= \frac{1}{n+1}\omega^{-(i-1)(j-1)}, \;\; 1 \leq i,j \leq n+1 \end{aligned} \tag{2.10}$$

diagonalizes $\mathcal{H}'$

$$\Omega^{-1}\mathcal{E}_i\Omega \;=\; \sum_{k=1}^{n+1} \omega^{i(k-1)} E^{kk} \tag{2.11}$$

From the expression

$$S \;=\; \omega^{\frac{n}{2}} \sum_{k=1}^{n+1} \omega^{1-k} E^{kk} \tag{2.12}$$

for the operator S which implements the automorphism (2.5), and taking into account (2.11) we get the commutation relations

$$\Omega^{\pm 1}\mathcal{E}_k\Omega^{\mp 1} \;=\; \omega^{\mp\frac{kn}{2}} S^{\pm k} \tag{2.13}$$

To complete the alternative basis, we need to add the corresponding to the Cartan subalgebra $\mathcal{H}'$ step operators. In view of (2.11) we see that

$$F^{ij} \;=\; \Omega E^{ij}\Omega^{-1}, \;\; i \neq j \tag{2.14}$$

are eigenvectors of the adjoint action of the generators (2.8). Combining the last observation with (2.9) we conclude that the automorphism (2.5) permutes the step operators (2.14). We calculate explicitly its action with the help of (2.13)

$$\begin{aligned} \sigma(F^{ij}) &= S\Omega E^{ij}\Omega^{-1}S^{-1} = \Omega\mathcal{E}_1 E^{ij}\mathcal{E}_n\Omega^{-1} = \\ &= \Omega E^{i-1j-1}\Omega^{-1} = F^{i-1j-1} \\ i,j &= 1, \ldots, n+1 \, mod\, (n+1) \end{aligned} \tag{2.15}$$

*Here we only consider the Lie algebra $sl(n+1)$. For general simple Lie algebras, the eigenvalues of the corresponding automorphism, restricted to the alternative Cartan subalgebra are related to the Betti numbers

It is seen from the above expression that σ acts on the alternative basis as an element of the Weyl group (more precisely, it is a Coxeter element). Moreover, it is clear that the action of σ separates the set of the step operators (2.14) into n non–intersecting orbits, parametrized by the difference $i-j$ (2.15), each containing $n+1$ elements. Within the elements of each σ–orbit we choose the following representatives

$$F^i = \Omega E^{i+11}\Omega^{-1} = F^{i+11}, \;\; i = 1, \dots n \tag{2.16a}$$

$$\left[\mathcal{E}_i, F^j\right] = (\omega^{ij} - 1)F^j \tag{2.16b}$$

Introducing the grade expansions (2.7) of the above generators

$$\begin{aligned} F^i &= \sum_{k \in \mathbb{Z}_{n+1}} F^i_k \\ \sigma(F^i_k) &= \omega^k F^i_k \end{aligned} \tag{2.17}$$

we observe that the rest of the elements in the σ–orbit of F^i (2.16a) are linear combinations of the components F^i_k. Therefore one gets a graded basis which is formed by $\mathcal{E}_i$, $i = 1 \dots n$ (2.8) and F^i_k, $i = 1, \dots n$; $k \in \mathbb{Z}_{n+1}$. Note that due to (2.9), (2.16b) and (2.17) the commutation relations

$$\left[\mathcal{E}_i, F^j_k\right] = (\omega^{ij} - 1)F^j_{k+i} \tag{2.18}$$

are valid.

Starting from a Lie algebra $\mathcal{G}$ one associates to it the loop algebra $\tilde{\mathcal{G}} = \mathbb{C}[\lambda, \lambda^{-1}] \otimes \mathcal{G}$, i. e. it is the set of the Laurent series on a formal parameter λ, which will play the role of a spectral parameter of the auxiliary linear problem, with coefficients belonging to $\mathcal{G}$. In other words, $\tilde{\mathcal{G}}$ is spanned on the elements $X_n = \lambda^n X$ where $n \in \mathbb{Z}$ and $X \in \mathcal{G}$. The commutator is

$$[X_m, Y_n] = [X, Y]_{m+n}$$

Now we introduce the the Lie algebra $\widetilde{sl}(n+1)$ in the principal gradation. It is generated by the expansions $X(\lambda) = \sum_{l \in \mathbb{Z}} \lambda^l X_l$ with $X_l \in sl(n+1)$ together with the restriction

$$X(\omega\lambda) = \sigma(X(\lambda)) \tag{2.19}$$

where the automorphism σ acts on the Laurent coefficients X_n as indicated by (2.5). The operator $d = \lambda\frac{\partial}{\partial\lambda}$ introduces a $\mathbb{Z}$–gradation in $\tilde{\mathcal{G}} = \widetilde{sl}(n+1)$

$$\begin{aligned} \widetilde{\mathcal{G}} &= \oplus_{n \in \mathbb{Z}} \widetilde{\mathcal{G}_n} \\ \left[d, \widetilde{\mathcal{G}}\right] &= n\widetilde{\mathcal{G}_n} \end{aligned} \tag{2.20}$$

Comparing (2.7) with (2.19) and taking into account the above decomposition one concludes that $\mathcal{G}_k \simeq \tilde{\mathcal{G}}_{k+l(n+1)}$ for $k \in \mathbb{Z}_{n+1}$ and $l \in \mathbb{Z}$. Starting from the alternative basis (2.8), (2.17) on $sl(n+1)$ one gets a basis of $\widetilde{sl}(n+1)$ in the principal gradation. It is formed by the elements $\mathcal{E}_{i+l(n+1)}$ for $i = 1, \ldots n$, $l \in \mathbb{Z}$ and $F^j_{i+l(n+1)}$ for $j = 1, \ldots n$, $i \in \mathbb{Z}_{n+1}$, $l \in \mathbb{Z}$. The subalgebra generated by $\mathcal{E}_k$, $k \neq 0 \; mod\,(n+1)$ is a maximal abelian subalgebra. It is known in the literature as the principal Heisenberg subalgebra†. Introducing the element [19], [20]

$$F^i(\mu) \;\; = \;\; \sum_{l\in\mathbb{Z}} \mu^{-l} F^i_l \tag{2.21}$$

and taking into account the commutator (2.18) and its extension in the loop algebra, one obtains‡

$$[\mathcal{E}_i, F^j(\mu)] \;\; = \;\; (\omega^{ij} - 1)\mu^i F^j(\mu) \tag{2.22}$$

To introduce the $A^{(1)}_n$ Toda equations we first define the following element of the Cartan subalgebra

$$\Phi = \frac{1}{2}\sum_{i=1}^{n+1} \varphi_i E^{ii} \qquad \sum_i \varphi_i = 0 \tag{2.23}$$

Then the $A^{(1)}_n$ affine Toda equations can be written in the following form:

$$\begin{aligned} \partial_+\partial_-\Phi &= m^2 \left[e^{ad\Phi}\mathcal{E}_+, e^{-ad\Phi}\mathcal{E}_-\right] \\ x^\pm &= x \pm t, \qquad \partial_\pm = \frac{\partial}{\partial x^\pm} \\ & e^{adX} Y = e^X.Y.e^{-X} \end{aligned} \tag{2.24}$$

where $\mathcal{E}_\pm$ are grade ± 1 elements of the principal Heisenberg subalgebra of $\widetilde{sl}(n+1)$. More precisely, they are liftings of $\mathcal{E}_1$ and $\mathcal{E}_n$ (2.8) in the loop algebra

$$\mathcal{E}_\pm \;\; = \;\; \lambda^{\pm 1} \sum_{k\in\mathbb{Z}_{n+1}} E^{kk\pm 1} \tag{2.25}$$

Substituting back the above expressions into the equations of motion (2.24) and taking into account the notation (2.23) we end up with the system

$$\begin{aligned} \partial_+\partial_-\varphi_i &= m^2(e^{\varphi_i - \varphi_{i+1}} - e^{\varphi_{i-1}-\varphi_i}) \\ i &\in \mathbb{Z}_{n+1} \end{aligned} \tag{2.26}$$

It is an easy task to check that (2.24) is equivalent to the zero-curvature condition

$$\partial_+ A_- - \partial_- A_+ + [A_+, A_-] = 0 \tag{2.27}$$

†The Heisenberg subalgebras of the loop and the affine Lie algebras play crucial role in constructing integrable hierarchies [8]–[10]

‡Here and in what follows we will perform a slight abuse of notations, namely the lower index will be used to indicate both the discrete $\mathbb{Z}_{n+1}$ and to parametrize the $\mathbb{Z}$ gradation as introduced in (2.20) of a connection whose components belong to the loop algebra $\widetilde{sl}(n+1)$ in the principal gradation:

$$\begin{aligned} A_+(x,\lambda) &= 2\partial_+\Phi(x) + m\mathcal{E}_+ \\ A_-(x,\lambda) &= me^{-2ad\Phi(x)}\mathcal{E}_- \end{aligned} \tag{2.28}$$

where we adopted the abbreviation $x = (x^+, x^-)$. The dependence on the spectral parameter in the above expressions comes from the dependence of the elements of the principal Heisenberg subalgebra $\mathcal{E}_\pm$ on it. The zero curvature condition (2.27) implies that there exists covariantly constant vector $w(x,\lambda)$ with respect to the covariant derivatives $D_\pm = \partial_\pm + A_\pm$:

$$(\partial_\pm + A_\pm(x,\lambda))w(x,\lambda) = 0 \tag{2.29}$$

In what follows we shall assume that the above equation is considered in the defining representation of . Since the components $A_\pm$ are in the principal loop algebra $\widetilde{sl}(n+1)$, they obey the relations (2.19). Performing the rescaling of the spectral parameter $\lambda \to \omega^{-1}\lambda$, one immediately observes that (2.29) remains invariant provided that

$$\begin{aligned} w(x,\lambda) &\to (\mathcal{S}w)(x,\lambda) \\ (\mathcal{S}w)(x,\lambda) &= Sw(x,\omega^{-1}\lambda) \end{aligned} \tag{2.30}$$

where the matrix S implements the automorphism (2.5). The above symmetry of the equation (2.29) allows to get a matrix solution of it, starting from the column $w(x,\lambda)$

$$W(x,\lambda) = \| w(x,\lambda), \omega^{\frac{n}{2}}(\mathcal{S}^{-1}w)(x,\lambda) \ldots \omega^{\frac{n^2}{2}}(\mathcal{S}^{-n}w)(x,\lambda) \| \tag{2.31}$$

Note that the last expression justifies our choise to work with the defining representation. Since the order of the automorphism σ (2.5) is $n+1$ (2.6), it is clear that the $n+1$–th power of the operator S (and therefore of $\mathcal{S}$ (2.30)) which implements this automophism is proportional to the identity operator in any irreducible representation of . This restricts us to look for matrix solutions of (2.29), starting from a vector one, by using the symmetry (2.9) in the defining representation only since it has dimension $n+1$.

To get soliton solutions we shall look for special solutions of the system (2.29) which admit the expansion

$$\begin{aligned} w(x,\lambda) &= \sum_{j=0}^{N} \lambda^j w^{(j)}(x) e(x,-\lambda) \\ e(x,\lambda) &= exp\{m(\lambda x^+ + \frac{x^-}{\lambda})\} \end{aligned} \tag{2.32}$$

where N is non–negative integer which will be identified with the number of solitons and $w^{(j)}(x),\ j = 1, \ldots, N$ are λ–independent vectors. We shall require also that $w^{(N)}$ is the constant vector with unit components

$$\begin{aligned} w^{(N)} &= \sum_{k=1}^{n+1} |k\rangle \\ \mathcal{E}_{\pm} w^{(N)} &= \lambda^{\pm 1} w^{(N)} \end{aligned} \tag{2.33}$$

To fix the rest of the coefficients in the expansion (2.32) we shall impose the following relations on $w(x, \lambda)$ [23]

$$\begin{aligned} (\mathcal{S}^{-r_j} w)(x, \mu_j) &= \omega^{-\frac{r_j n}{2}} c_j w(x, \mu_j) \\ j &= 1, \ldots, N \end{aligned} \tag{2.34}$$

for certain constants c_j and μ_j and discrete parameters, called soliton species, r_j which take *non–vanishing* values in the cyclic group $\mathbb{Z}_{n+1}$. Taking into account (2.12) and (2.30) we conclude that the above equations can be equivalently written as follows

$$w_k(x, \omega^{r_j} \mu_j) = \omega^{(1-k) r_j} c_j w_k(x, \mu_j) \tag{2.35}$$

where w_k are the components of the vector w

$$w = \sum_{k=1}^{n+1} w_k |k>$$

From (2.34) we see that the matrix $W(x, \lambda)$ (2.31) is degenerate for $\lambda = \omega^k \mu_j$ where $k \in \mathbb{Z}_{n+1}$ and $j = 1, \ldots, N$. For these values of the spectral parameter, the columns with numbers $1 - k \bmod (n+1)$ and $r_j + 1 - k \bmod (n+1)$ are proportional.

To demonstrate that the expansions (2.32) together with (2.33) and (2.34) satisfy (2.29), we shall make the following observation: suppose that $U(x, \lambda) = P_{N-1}(x, \lambda) e(x, -\lambda)$, where $P_{N-1}(\lambda)$ is a vector valued polynomial on λ of degree $N - 1$, is a solution of (2.34). Then $U(x, \lambda)$ vanishes identically. To see this, note that due to (2.34), the coefficients of the polynomial P_{N-1} satisfy certain homogeneous linear system of N equations. For generic values of μ_j, the determinant of this system is different from zero. Therefore there only exist a vanishing solution, and hence $U(x, \lambda) \equiv 0$. Let us apply this observation to

$$\begin{aligned} U_+(x, \lambda) &= \partial_+ w(x, \lambda) + 2\partial_+ \Phi(x) w(x, \lambda) + m \mathcal{E}_+ w(x, \lambda) \\ U_-(x, \lambda) &= \partial_- w(x, \lambda) + m e^{-2ad\Phi(x)} \mathcal{E}_- w(x, \lambda) \end{aligned} \tag{2.36}$$

As consequence of (2.5), (2.6), (2.9) and since $w(x, \lambda)$ is a solution of (2.34) one checks immediately that the above expressions satisfy (2.34) also. Inserting (2.32) into (2.36) we get:

$$e(x,\lambda)U_+(x,\lambda) = \lambda^N \left(2\partial_+\Phi w^{(N)} - m(1-\frac{1}{\lambda}\mathcal{E}_+)w^{(N-1)}\right) + R_{N-1}(x,\lambda)$$
$$e(x,\lambda)U_-(x,\lambda) = \frac{m}{\lambda}\left(\lambda e^{-2ad\Phi}\mathcal{E}_- - 1\right) w^{(0)}(x) + S_{N-1}(x,\lambda)$$

where R_{N-1} and S_{N-1} are polynomials on λ of degree not greater than $N-1$. To derive the first of the above expansions we have also used (2.33). Therefore $U_\pm$ vanish identically provided that

$$\begin{aligned} 2\partial_+\Phi w^{(N)} &= m\left(1-\frac{1}{\lambda}\mathcal{E}_+\right) w^{(N-1)} \\ \lambda e^{-2ad\Phi}\mathcal{E}_- w^{(0)}(x) &= w^{(0)}(x) \end{aligned} \tag{2.37}$$

This method to construct affine Toda solitons is a straightforward generalization of the approach applied by Date [23] for the sine–Gordon model. Taking into account (2.8) and (2.23) one can write the above expressions as

$$\begin{aligned} \partial_+\varphi_i &= m(w_i^{(N-1)} - w_{i+1}^{(N-1)}) \\ e^{2\alpha_i.\Phi} &= e^{\varphi_i-\varphi_{i+1}} = \frac{w_{i+1}^{(0)}}{w_i^{(0)}},\ \ i\in\mathbb{Z}_{n+1} \end{aligned} \tag{2.38}$$

To find explicit expressions from (2.38) for the fields φ_i one has to solve the linear equations (2.35) which can be written as follows:

$$\begin{aligned} &\sum_{l=1}^{N}\mu_j^{l-1}\left(\omega_j^{l-1}e(-\omega_j\mu_j) - c_j\omega_j^{1-k}e(-\mu_j)\right) w_k^{(l-1)} = \\ &= \mu_j^N(c_j\omega_j^{1-k}e(-\mu_j) - \omega_j^N e(-\omega_j\mu_j)) \\ &\omega_j = \omega^{r_j}, \qquad k=1,...,n+1 \end{aligned} \tag{2.39}$$

where $w_k^{(l-1)}$ are the components of the coefficients $w^{(l-1)}$ which appeared in the expansion (2.32); here and in what follows we shall omit the $x^\pm$ dependence in the exponentials $e(x,\lambda)$ (2.33). By Kramer's formula we get the solution

$$\begin{aligned} w_k^{(0)} &= (-)^N\prod_{j=1}^{N}\mu_j\omega_j\frac{det\ G^{(k+1)}}{det\ G^{(k)}} \\ w_k^{(N-1)} &= \frac{1}{m}\partial_+\ln\ det\ G^{(k)},\ \ k=1,\ldots,n+1 \end{aligned} \tag{2.40}$$

where $G^{(k)}$ is a $N\times N$ matrix with entries

$$G_{jl}^{(k)} = \mu_j^{l-1}(\omega_j^{l-1}e(-\omega_j\mu_j) - c_j\omega_j^{1-k}e(-\mu_j)) \tag{2.41}$$

There is a "canonical" expression for the determinant of $G^{(k)}$ [23]. A way to obtain it is to multiply and divide by the Van der Mond determinant $\det M$ where $M_{jl} = \mu_j^{l-1}$ $(j, l = 1, ..., N)$, and to calculate the matrix elements of product $G^{(k)}M^{-1}$. The final result reads

$$
\begin{aligned}
det\, G^{(k)} &= (-)^N \frac{\prod_{j=1}^N c_j \omega_j^{1-k} e(-\mu_j)}{\prod_{p>q}(\mu_p - \mu_q)} \tau_{k-1} \\
\tau_k &= det\,(1 + \Omega^{\frac{k}{2}}.V.\Omega^{\frac{k}{2}}) \\
\Omega &= diag\,(\omega_1, ..., \omega_N)
\end{aligned}
\tag{2.42}
$$

where the elements of the matrix V are given by:

$$
\begin{aligned}
&V_{jk} = \frac{\sqrt{X_j\ X_k}}{\mu_j^+ - \mu_k^-} \\
&X_j = \frac{1}{c_j}(\mu_j^- - \mu_j^+) \prod_{l \neq j} \frac{\mu_l^- - \mu_j^+}{\mu_l^- - \mu_j^-} exp\{-2im \sin \frac{\pi r_j}{n+1}(\widetilde{\mu_j}\ x^+ - \frac{x^-}{\widetilde{\mu_j}})\} \\
&\widetilde{\mu_j} = \omega_j^{\frac{1}{2}} \mu_j, \qquad \mu_j^{\pm} = \omega_j^{\pm\frac{1}{2}} \widetilde{\mu_j}
\end{aligned}
\tag{2.43}
$$

Substituting back (2.40) into (2.38) we get:

$$
e^{-\varphi_k} = \frac{\tau_k}{\tau_{k-1}} = \frac{det\,(1 + \Omega^{\frac{k}{2}}.V.\Omega^{\frac{k}{2}})}{det\,(1 + \Omega^{\frac{k-1}{2}}.V.\Omega^{\frac{k-1}{2}})}
\tag{2.44}
$$

The above expression for the affine Toda N–solitons permits to establish a relation to the relativistically invariant N–body integrable systems of Calogero–Moser type [21].

From (2.42), (2.43) it is seen that the solutions (2.44) describe propagation of N solitons: the variables $\widetilde{\mu_j}$ are the rapidities while the quantities

$$
a_j = \frac{1}{c_j}(\mu_j^- - \mu_j^+) \prod_{l \neq j} \frac{\mu_l^- - \mu_j^+}{\mu_l^- - \mu_j^-} \quad ,
\tag{2.45}
$$

are related to the positions. Note that more than the continuous parameters a_j and μ_j, the solitons are characterized by the discrete parameters r_j (2.34).

3 The dressing problem for the $A_n^{(1)}$ Toda solitons

The dressing group is a symmetry of the non–linear evolution equations which admit zero–curvature (or Lax) representation. It was shown in [13] that the dressing group appears as a semiclassical limit of the quantum group symmetry. The dressing group acts by gauge transformations on the components of the Lax connection which preserve its form. Therefore, it is a symmetry of the space of solutions of the corresponding integrable model. The aim of this Section is to present a derivation of the dressing group elements which generate N–solitons in the $A_n^{(1)}$ Toda theories. This problem has been already solved in [25] for the sine–Gordon equation which is the $A_1^{(1)}$ Toda model. Expressions for the dressing group elements which create solitons from the vacuum has been conjectured in [16] for a large class of integrable hierarchies.

In order to be able to compare our results with the expressions of Babelon and Bernard [25], it will be convenient to perform a field dependent gauge transformation on the components of the connection (2.28)

$$\begin{aligned} D_\pm &\rightarrow e^{\Phi} D_\pm e^{-\Phi} = \partial_\pm + \mathcal{A}_\pm \\ \mathcal{A}_\pm &= \pm\partial_\pm\Phi + m e^{\pm ad\Phi}\mathcal{E}_\pm \end{aligned} \tag{3.1}$$

It is clear that for the vacuum solution $\Phi = 0$, the components of the above connection take the following form

$$\mathcal{A}_\pm = m\mathcal{E}_\pm \tag{3.2}$$

In accordance with the general definition, the dressing transformations are represented by loop group elements $g(x,\lambda) \in \widetilde{SL}(n+1)$ which act on (3.1) as gauge transformations $\mathcal{A}_\pm \rightarrow \mathcal{A}_\pm^g$

$$\mathcal{A}_\pm^g = -\partial_\pm g g^{-1} + g\mathcal{A}_\pm g^{-1} \tag{3.3}$$

such that the connection $\mathcal{A}_\pm^g$ has the same form as the original one (3.1) with $\Phi \rightarrow \Phi^g$. Since by gauge transformations the curvature transforms as $F_{+-} = [D_+, D_-] \rightarrow gF_{+-}g^{-1}$ we see that the dressing transformations are symmetries of the underlying equations of motion (2.26), (2.27).

In view of (3.1), it is clear that

$$\mathcal{T}(x,\lambda) = e^{\Phi(x)} W(x,\lambda) \tag{3.4}$$

where the matrix W was introduced by (2.31), is a solution of the linear problem

$$(\partial_\pm + \mathcal{A}_\pm)\,\mathcal{T}(x,\lambda) = 0 \tag{3.5}$$

Due to the expansion (2.32), it is obvious that the components w_k of the ($n+1$–dimensional) vector w admit the following representation

$$w_k(x,\lambda) = \prod_{j=1}^{N}(\lambda + \epsilon_{kj}(x))e(x,-\lambda), \quad 1 \leq k \leq n+1 \tag{3.6}$$

The dependence of the variables ϵ_{kj} on x^+ and x^- is fixed by (2.35) which, taking into account the above expression, reads

$$\prod_{l=1}^{N} \frac{\epsilon_{kl} + \omega^{r_j}\mu_j}{\epsilon_{kl} + \mu_j} \;=\; c_j \omega^{r_j(1-k)} \frac{e(\omega^{r_j}\mu_j)}{e(\mu_j)} \tag{3.7}$$

In view of (3.6) we can express the matrix (2.31) as follows

$$W(x,\lambda) \;=\; U(x,\lambda)E(x,\lambda) \tag{3.8a}$$

$$U_{kl}(x,\lambda) \;=\; \omega^{(k-1)(l-1)} \prod_{j=1}^{N} (\epsilon_{kj} + \omega^{l-1}\lambda) \tag{3.8b}$$

$$E_{kl}(x,\lambda) \;=\; \delta_{kl} e(-\omega^{k-1}\lambda), \qquad k,l = 1,\ldots,n+1 \tag{3.8c}$$

As a next step, we shall calculate the determinant of (2.31). First of all we note that due to the above equations, the exponential singularities of the matrix elements of W disappear in its determinant. Therefore, $det\, W$ is a meromorphic function on the Riemann sphere $\mathbb{CP}^1$. Further, due to (2.35) $det\, W$ vanishes whenever $\lambda^{n+1} = \mu_j^{n+1}$ for $j = 1,\ldots,N$. This wants to say that $det\, W$ has at least $N(n+1)$ zeroes. This number is exact since one has the expansion

$$det\, W(x,\lambda) \;=\; (-)^{nN} \lambda^{(n+1)N} det\, \Omega \left(1 + O(\frac{1}{\lambda})\right), \quad \lambda \to \infty \tag{3.9}$$

where the matrix Ω is given by (2.10). It is clear that $det\, W$ has no other poles, and therefore, due to the Cauchy's theorem we end up with the result

$$det\, W(x,\lambda) \;=\; (-)^{nN} det\, \Omega \prod_{j=1}^{N} (\lambda^{n+1} - \mu_j^{n+1}) \tag{3.10}$$

In what follows it will be necessary to express the $A_n^{(1)}$ Toda fields (2.23), (2.26) in terms of the variables* ϵ_{kl} (3.6). To do that, it suffices to compare (2.32) with (3.6). The result is

$$w_k^{(0)} \;=\; \prod_{j=1}^{N} \epsilon_{kj} \tag{3.11}$$

which together with (2.38) and (2.40) yields the expression

$$e^{-\varphi_k} \;=\; (-)^N \prod_{j=1}^{N} \frac{\epsilon_{kj}}{\mu_j}, \qquad k = 1,\ldots,n+1 \tag{3.12}$$

*These variables appeared in the study of the periodic solutions of the KdV equation and of the periodic Toda chain[28]

Since the field Φ (2.23) belongs to the Cartan subalgebra of , the following restriction

$$\prod_{k=1}^{n+1}\prod_{j=1}^{N}\epsilon_{kj} \quad = \quad (-)^{N(n+1)}\prod_{j=1}^{N}\mu_j^{n+1} \tag{3.13}$$

takes place.

Turning back to the dressing problem, we define the normalized transport matrix[†]

$$T(x,\lambda) \quad = \quad \mathcal{T}(x,\lambda)\mathcal{T}^{-1}(0,\lambda) \tag{3.14}$$

It is obvious that the above matrix is unimodular. Moreover, due to (3.5), it belongs to the loop group $\widetilde{SL}(n+1)$ in the principal gradation. Let $\mathcal{T}$ (3.4) and T (3.14) be the transport matrices associated to certain N–soliton solution (2.44), (3.12) and $\mathcal{T}_0$, T_0 be the corresponding vacuum transport matrices. Therefore one can write

$$\begin{aligned} T(x,\lambda) &= f(x,\lambda)T_0(x,\lambda)f^{-1}(0,\lambda) && (3.15a)\\ f(x,\lambda) &= \mathcal{T}(x,\lambda)\mathcal{T}_0^{-1}(x,\lambda) = e^{\Phi(x)}W(x,\lambda)W_0^{-1}(x,\lambda) = \\ &= e^{\Phi(x)}U(x,\lambda)U_0^{-1}(x,\lambda) && (3.15b)\\ T_0(x,\lambda) &= \mathcal{T}_0(x,\lambda)\mathcal{T}_0^{-1}(0,\lambda) = e^{-m(\mathcal{E}_+x^+ + \mathcal{E}_-x^-)} && (3.15c) \end{aligned}$$

where $W(x,\lambda)$ and $W_0(x,\lambda)$ are the matices (2.31) corresponding to a generic N–soliton solution (2.44) and to the vacuum respectively; the matrix U in the last equation (3.15b) is given by (3.8a)–(3.8c) while U_0 stands for its vacuum solution analogue. Due to (3.8b) it turns out that $U_0 = \Omega$ (2.10). Taking into account these remarks we conclude that $f(x,\lambda) = e^{\Phi(x)}U(x,\lambda)\Omega^{-1}$. In view of (3.15a), $f(x,\lambda)$ is a gauge transformation which transforms the vacuum solution transport matrix T_0 into the transport matrix T, related to a N–soliton solution. Therefore, it generates a dressing transformation. Note also that the element $f(x,\lambda)$ is in the principal gradation

$$f(x,\omega\lambda) \quad = \quad Sf(x,\lambda)S^{-1} \tag{3.16}$$

To get the above equation we observe that the matrix (3.8b) satisfies the equation $U(x,\omega\lambda) = \lambda\omega^{-\frac{n}{2}}U(x,\lambda)\mathcal{E}_-$ which combined with the commutation relations (2.13) produces (3.16). Due to (2.19) we see that $f(x,\lambda)$ belongs to the loop group $\widetilde{GL}(n+1)$ in the principal gradation. From (3.10) it is seen that

$$det\, f(x,\lambda) \quad = \quad (-)^{nN}\prod_{j=1}^{N}(\lambda^{n+1}-\mu_j^{n+1}) \tag{3.17}$$

Note that the solution of the dressing problem (3.15a) is not unique. The reason is that there exist $x^{\pm}$–independent matrices $\theta(\lambda)$ which are not proportional to the identity and commute with T_0 (3.15c). Therefore the element

$$g(x,\lambda) \quad = \quad f(x,\lambda)\theta(\lambda) \tag{3.18}$$

[†]As a reference point we choose those with light–cone coordinates $x^+ = x^- = 0$

is the general solution of the dressing problem; in the above equation $f(x,\lambda)$ (3.15b) is a particular solution of it. To fix the unknown matrix $\theta(\lambda)$ we impose a set of restrictions

$$
\begin{aligned}
\theta(\lambda)\mathcal{E}_\pm\theta^{-1}(\lambda) &= \mathcal{E}_\pm && (3.19a)\\
\theta(\omega\lambda) &= S\theta(\lambda)S^{-1} && (3.19b)\\
det\theta(\lambda) &= \frac{(-)^N}{\prod_{j=1}^N(\lambda^{n+1}-\mu_j^{n+1})} && (3.19c)\\
& && (3.19d)\\
g(x,\lambda) &= \begin{cases} e^{-\Phi}\left(1+O(\lambda)\right) & for \quad \lambda\to 0\\ e^{\Phi}\left(1+O(\frac{1}{\lambda})\right) & for \quad \lambda\to\infty\end{cases} && (3.19e)
\end{aligned}
$$

To justify the above requirements we note that (3.19a) ensures the commutativity of $\theta(\lambda)$ with T_0 (3.15c); taking into account (3.16), we see that (3.19b) guarantees that the dressing group element $g(x,\lambda)$ is in the principal gradation; (3.19c) comes from the requirement that $g(x,\lambda)$ should be unimodular; finally, (3.19e) is a consequence of a grade analysis applied to (3.3) (for details, see [13], [25]). It is easy to check that the general solution (3.19b) is given by

$$
\theta(\lambda) = \theta_0(\lambda)+\sum_{k=1}^{n}\theta_k(\lambda)\mathcal{E}_k \qquad (3.20)
$$

where the generators $\mathcal{E}_k$ of the alternative Cartan subalgebra $\mathcal{H}'$ were introduced by (2.8). Inserting the above expansion into (3.19b) and taking into account (2.9) we end up with

$$
\theta_k(\omega\lambda) = \omega^k\theta_k(\lambda), \quad k=0,\ldots,n \qquad (3.21)
$$

To calculate the determinant of $\theta(\lambda)$ we shall use (2.11)

$$
\begin{aligned}
det\,\theta(\lambda) = det\,\Omega^{-1}\theta(\lambda)\Omega = \prod_{k=1}^{n+1}\left(\sum_{l=1}^{n+1}\Omega_{kl}\theta_{l-1}(\lambda)\right) = \\
= \prod_{k=1}^{n+1}\left(\sum_{l=1}^{n+1}\theta_{l-1}(\omega^{k-1}\lambda)\right) \qquad (3.22)
\end{aligned}
$$

More than (3.19a)–(3.19e), we shall require that the entries of the matrix (3.18) are meromorphic functions on $\mathbb{CP}^1$ with simple poles located at the points $\lambda=\omega^p\mu_j$ for $p=0,\ldots,n$ and $j=1,\ldots,N$. In view of the last restriction, only a finite number of solutions survive. Among them we choose that which satisfies the system

$$
\sum_{l=1}^{n+1}\Omega_{kl}\theta_{l-1}(\lambda) = \frac{1}{\prod_{j=1}^N(\omega^{k-1}\lambda-\mu_j)}, \quad k=1,\ldots,n+1 \qquad (3.23)
$$

It is clear that (3.22) together with the above equation guarantee the the validity of (3.19c). Note also that (2.23) is compatible with (3.21). Inserting back (3.20), (3.23) into (3.18) we obtain

$$g^{(N)}(\Phi,\{\mu\},\lambda) \;=\; e^{\Phi}\Gamma^{(N)}(\Phi,\{\mu\},\lambda)\Omega^{-1} \tag{3.24a}$$

where the upper index indicates the number of solitons and $\Gamma^{(N)}$ is a $(n+1)\times(n+1)$ matrix with entries

$$\Gamma^{(N)}_{kl}(\Phi,\{\mu\},\lambda) \;=\; \omega^{(k-1)(l-1)}\prod_{j=1}^{N}\frac{\lambda+\omega^{1-l}\epsilon_{kj}(x)}{\lambda-\omega^{1-l}\mu_j} \tag{3.24b}$$

Note that the dependence on the space–time coordinates is dictated by (3.7) and (3.12). The expansion (3.19e) is satisfied as a consequence from (3.12). We note that the method presented in this Section was previously used in [29] to solve the dressing problem for the algebraic–geometrical solutions in the sine–Gordon model.

4 The factorization problem and the relation to the vertex operator approach

There exists a general scheme to construct solitons in the affine Toda theories [19]. Substantially, it is based on the group–algebraic approach to the integrable systems, developed by Leznov and Saveliev [30]. To apply the Leznov–Saveliev analysis to the affine Toda equations one first considers the Conformal affine Toda (CaT) equations [31],[32]. The last appear as a zero–curvature condition of a connection of the form (2.28) the components of which belong to the affine Lie algebra $\hat{\mathcal{G}}$. It is the central extension of the corresponding loop algebra $\tilde{\mathcal{G}}$. The necessity to introduce central extension of the loop algebra is due to the fact that the Leznov–Saveliev analysis applies to Lie algebras which admit (non–trivial) highest weight representations. Such representations only exist if the central charge is different from zero. In the case of the CaT models, the group–algebraic approach yields the general solution of the equations of motion, parametrized by a free massless field and a group element which belongs to the affine Lie group $\hat{G}$. It was suggested in [19] that solitons arise when the group element factorizes in a product of special elements of the affine Lie group which are closely related to the vertex operators. These elements are exponentials of the loop algebra elements which diagonalize the adjoint action of the principal Heisenberg subalgebra. Within the formalism of [19], insertion of one such element results in a creation of a single soliton. The group–algebraic approach to the solitons in the affine Toda theory was further developed in [25]. In the last paper, on the example of the conformally extended sinh–Gordon model, it was shown that the solitons can be obtained from the vacuum via specific dressing transformations. The explicit form of the corresponding dressing group elements has been used to establish a relation to the vertex operator formalism [19]. Moreover, it was demonstrated that the solution of the dressing problem in the affine group differ from those in the loop group by a factor which is in the center *. In the present Section we extend the results of [25] for the $A_n^{(1)}$ Toda models, i. e. starting from (3.24a), (3.24b) we first show that one can factorize a generic dressing group elements into a product of "monosoliton" factors; second, we analyze our expressions (3.24a), (3.24b) for $N = 1$ and obtain the relation to the vertex operator construction of the soliton solutions [19].

We start by writing the element (3.24a), (3.24b) in a slightly different form

$$g^{(N)}(\Phi, \{\mu\}, \lambda) = \frac{1}{n+1} \sum_{r \in \mathbb{Z}_{n+1}} S^r |v_\Phi(\{\mu\}, \omega^{-r}\lambda) >< v_0| S^{-r} \tag{4.1a}$$

$$\begin{aligned} |v_\Phi(\{\mu\}, \lambda) > &= \sum_{i=1}^{n+1} e^{\frac{\varphi_i}{2}} \prod_{a=1}^{N} \frac{\lambda + \epsilon_{ia}}{\lambda - \mu_a} |i> \\ |v_0> &= \sum_{i=1}^{n+1} |i> \end{aligned} \tag{4.1b}$$

where the operator S was introduced by (2.5), (2.12). It is clear that the property (3.16) is manifestly satisfied by the expression (4.1a). Note that the vector $|v_0>$ already appeared in a different context (2.33). We proceed by advancing the hypothesis that (4.1a) admits the representation

$$g^{(N)}(\Phi, \{\mu\}, \lambda) = e^{\mathcal{P}_N} g^{(1)}(F_N, \mu_N, \lambda) \cdot \ \ldots \ \cdot e^{\mathcal{P}_1} g^{(1)}(F_1, \mu_1, \lambda) \tag{4.2}$$

where

$$\begin{aligned} &\mathcal{P}_l = \frac{1}{2} \sum_{k=1}^{n+1} p_{kl} E^{kk}, \qquad F_l = \frac{1}{2} \sum_{k=1}^{n+1} f_{kl} E^{kk} \\ &\sum_{k=1}^{n+1} p_{kl} = \sum_{k=1}^{n+1} f_{kl} = 0, \qquad l = 1, \ldots, N \end{aligned} \tag{4.3}$$

are certain elements of the Cartan subalgebra of ; $g^{(1)}(F_l, \mu_l, \lambda)$ are monosolitonic factors. They have the same form as (4.1a), (4.1b) with $|v_\Phi>$ substituted by

$$|v_{F_l}(\mu_l, \lambda)> = e^{F_l} \sum_{i=1}^{n+1} \frac{\lambda - \mu_l e^{-f_{il}}}{\lambda - \mu_l} |i> \tag{4.4}$$

Note that substituting back the above expression into (4.1a) and taking into account (3.12) we reproduce (3.24a), (3.24b) with $N = 1$. It is not difficult to calculate the

*The arguments used in [25] can be easily generalized to apply to an arbitrary affine Toda theory inverse element

$$\left(g^{(1)}\right)^{-1}(F,\mu,\lambda) = e^{K(F)}g^{(1)}(-F,\mu,\lambda)e^{-K(F)}$$
$$K(F) = \frac{1}{2}\sum_{i=1}^{n+1}(H_{\lambda_i}+H_{\lambda_{i-1}})f_i \qquad (4.5)$$

where λ_i, $i=1,\ldots,n$ are the fundamental weights (2.3) of ; $\lambda_0=\lambda_{n+1}=0$ and $\lambda\rightarrow H_\lambda$ is the natural identification of the $n+1$–dimensional Euclidean space with the $(n+1)\times(n+1)$ diagonal matrices $H_\lambda=\sum_i\lambda_iE^{ii}$. Therefore, the last of the above equations can be equivalently written as

$$K_i(F)-K_{i+1}(F) = \frac{f_i+f_{i+1}}{2} \qquad (4.6)$$

Note also that $K_i(F)=K_{i+n+1}(F)$ since $\sum_i f_i=0$.

To demonstrate the validity of the factorized expression (4.2) we first introduce the notation

$$g_l^{(N)}(\Phi,\{\mu\},\lambda) = g^{(N)}(\Phi,\{\mu\},\lambda)\cdot\left(g^{(1)}\right)^{-1}(F_1,\mu_1,\lambda)e^{-\mathcal{P}_1}\cdot\ \ldots\ \cdot\left(g^{(1)}\right)^{-1}(F_l,\mu_l,\lambda)e^{-\mathcal{P}_l}$$
$$l = 0,1,\ldots,N$$
$$g_0^{(N)}(\Phi,\{\mu\},\lambda) = g^{(N)}(\Phi,\{\mu\},\lambda) \qquad (4.7)$$

Taking into account (4.1a), (4.1b) and (4.5) we observe that the above element can be alternatively expressed as

$$g_l^{(N)}(\Phi,\{\mu\},\lambda) = \sum_{r\in\mathbb{Z}_{n+1}} S^r|v_\Phi(\{\mu\},\omega^{-r}\lambda)><\rho_l(\omega^{-r}\lambda)|S^{-r}\cdot e^{-K(F_l)-\mathcal{P}_l}$$
$$<\rho_l(\lambda)| = \sum_{j=1}^{n+1}<j|\rho_{jl}(\lambda) \qquad (4.8)$$

where the coefficients ρ_{jl} satisfy the following recursion relations

$$\rho_{jl+1}(\lambda) = \frac{1}{n+1}\sum_{k=1}^{n+1}e^{L_{kl+1}}\rho_{kl}(\lambda)\sum_{s\in\mathbb{Z}_{n+1}}\omega^{(j-k)s}\frac{\lambda-\omega^s\mu_{l+1}e^{f_{kl+1}}}{\lambda-\omega^s\mu_{l+1}}$$
$$L_{kl+1} = K_k(F_{l+1})-K_k(F_l)-\frac{f_{kl+1}+p_{kl}}{2} \qquad (4.9)$$

together with the initial conditions (see (4.1a), (4.1b))

$$\rho_{j0} = \frac{1}{n+1} \qquad (4.10)$$

To fix recursively the unknown abelian factors F_l and $\mathcal{P}_l$ we shall impose the following conditions: first, the element $g_l^{(N)}$ (4.8) has no poles at the points $\lambda=\mu_1,\ \ldots,\ \mu_l$

$$res_{\mu_i}g_l^{(N)} = 0, \qquad 1\le i\le l \qquad (4.11)$$

and second, we require that the multipliers ρ_{jl} (4.8), (4.9) are meromorphic functions on λ which have simple poles at $\lambda = \omega^{r_i}\mu_i$ for $i = 1, \ldots, l$ and are homolomorphic elsewhere. Due to this restriction we can write

$$res_{\omega^r \mu_i}\rho_{jl} \;=\; 0, \qquad r \neq r_i,\;\; 1 \le i \le l \tag{4.12}$$

The discrete parameters $r_i = 1, \ldots, n$ appeared naturally in the description of the soliton solutions (2.34), (2.35), (3.7). We stress that due to (3.16), the requirement (4.11) guarantees that the matrix $g_l^{(N)}$ (4.7) has no singularities for $\lambda^{n+1} = \mu_i^{n+1}$, $i = 1, \ldots, l$. The significance of (4.12) will become clear in what follows. Taking into account (4.12), we see that (4.11), written in terms of the entries of the matrix (4.7), (4.8), reads

$$\begin{aligned} res_{\mu_l}\left(g^{(N)}_{l+1}\right)_{ij} &= 0 \\ &\Updownarrow \\ res_{\omega^{r_{l+1}}\mu_{l+1}}\rho_{jl+1} &= \mu_{l+1}(1-\omega^{r_{l+1}})\omega^{(j+1-i)r_{l+1}} \times \\ \times \prod_{a\neq l+1}\frac{\omega^{r_{l+1}}\mu_{l+1}-\mu_a}{\mu_{l+1}-\mu_a} \quad &\cdot \quad \prod_a \frac{\mu_{l+1}+\epsilon_{ia}}{\omega^{r_{l+1}}\mu_{l+1}+\epsilon_{ia}}\rho_{jl+1}(\mu_{l+1}) \end{aligned} \tag{4.13}$$

The last equation is consistent since due to (3.7), its r. h. s. does not depend on i. Therefore, we conclude that the soliton dynamics, encoded in (3.7) is crucial in solving the factorization problem (4.2). This observation also explains the reason to impose the condition (4.12). On the other hand, in view of (4.9) we get

$$\begin{aligned} res_{\omega^r\mu_{l+1}}\rho_{jl+1} &= \frac{\omega^{jr}\mu_{l+1}}{n+1}\sum_{k=1}^{n+1}\omega^{r(1-k)}e^{L_{kl+1}}\rho_{kl}(\omega^r\mu_{l+1})(1-e^{f_{kl+1}}) \\ r &\in \mathbb{Z}_{n+1} \end{aligned} \tag{4.14}$$

where the factors L_{kl+1} were defined by (4.9). Inserting the above expression into (4.13) we get another consistency condition

$$\begin{aligned} \rho_{jl+1}(\mu_{l+1}) &= \rho_{kl+1}(\mu_{l+1}) \\ j,k &= 1, \ldots, n+1 \end{aligned} \tag{4.15}$$

To calculate the above quantities we recall the general identity

$$\begin{aligned} \frac{1}{n+1}\sum_{r\in\mathbb{Z}_{n+1}}\frac{\omega^{rk}}{\lambda-\omega^{-r}\mu} &= \frac{\lambda^{n-k}\mu^k}{\lambda^{n+1}-\mu^{n+1}} \\ 0 \le \quad k &\quad \le n \end{aligned} \tag{4.16a}$$

from which in the limit $\frac{\lambda}{\mu} \to 1$ one gets

$$\sum_{r=1}^{n}\frac{\omega^{rk}}{1-\omega^r} = \begin{cases} \frac{n}{2}+k & -n \le k \le 0 \\ -\frac{n}{2}-1+k & 1 \le k \le n+1 \end{cases} \tag{4.16b}$$

Setting $\lambda = \mu_{l+1}$ in (4.9) and using (4.12) for $r = 0$, we obtain with the help of (4.16b) the following expressions

$$\begin{aligned} \rho_{jl+1}(\mu_{l+1}) &= e^{L_{jl+1}}\rho_{jl}(\mu_{l+1}) + \frac{\mu_{l+1}}{n+1}\sum_{k=1}^{n+1} e^{L_{kl+1}}(1-e^{f_{kl+1}})\rho'_{kl}(\mu_{l+1}) - \\ &- \frac{1}{n+1}\left(\sum_{k=1}^{n+1}(k-j) + (n+1)\sum_{k=1}^{j}\right) e^{L_{kl+1}}\rho_{kl}(\mu_{l+1})(1-e^{f_{kl+1}}) \end{aligned} \tag{4.17}$$

Inserting this identity in (4.15) and taking into account the second equation (4.9) we obtain

$$e^{\frac{p_{jl}}{2}} = \frac{\rho_{jl}(\mu_{l+1})}{\left(\prod_k \rho_{kl}(\mu_{l+1})\right)^{\frac{1}{n+1}}} e^{-K_j(F_l)} \tag{4.18}$$

The j–independent factor in the denominator is fixed by (4.3). Therefore, the algebraic system (4.12), (4.13) reduces to

$$\begin{aligned} &\sum_{k\in\mathbb{Z}_{n+1}} \omega^{r(1-k)}\left(\frac{\rho_{kl}(\omega^r\mu_{l+1})}{\rho_{kl}(\mu_{l+1})} - \omega^{-r}\frac{\rho_{k+1l}(\omega^r\mu_{l+1})}{\rho_{k+1l}(\mu_{l+1})}\right) e^{K_k(F_{l+1})-\frac{f_{kl+1}}{2}} = \delta_{r,r_{l+1}}(1-\omega^r)\times \\ &\times \prod_{a\neq l+1}\frac{\omega^r\mu_{l+1}-\mu_a}{\mu_{l+1}-\mu_a}\prod_a\frac{\mu_{l+1}+\epsilon_{1a}}{\omega^r\mu_{l+1}+\epsilon_{1a}}\times \\ &\times \sum_{k\in\mathbb{Z}_{n+1}}\left(1+\mu_{l+1}\frac{d}{d\lambda}\ln\frac{\rho_{kl}}{\rho_{k+1l}}(\mu_{l+1})\right) e^{K_k(F_{l+1})-\frac{f_{kl+1}}{2}} \end{aligned} \tag{4.19}$$

Note that for $r = 0$ the above equation is satisfied identically. We recall also that $r_j \neq 0\, mod\,(n+1)$. Due to this and since $K(F_{l+1})$ and F_{l+1} are traceless, we conclude that (4.19) determines uniquely F_{l+1} as a function of F_l. Continuing this procedure, we finally arrive at the element $g_N^{(N)}$ (4.7). From (4.11) we conclude that it is a *holomorphic* function on the spectral parameter. This wants to say that $g_N^{(N)}$ does not depend on λ. Due to the fact that this element satisfies (3.16), it is clear that its off–diagonal elements vanish identically. The unique element which remains undetermined is $\mathcal{P}_N$ (4.3). One can use this ambiguity to set $g_N^{(N)} = 1$. This completes the factorization procedure.

In what follows we shall concentrate our attention on the dressing group elements which generate monosolitons from the vacuum. Due to the general expressions and in view of the of the one–soliton specification (4.4), one gets

$$\frac{\partial}{\partial\varphi_i}g^{(1)}(\Phi,\mu,\lambda)\cdot\left(g^{(1)}\right)^{-1}(\Phi,\mu,\lambda) = e^{K(\Phi)}\left(B^i(\mu,\lambda) - B^{n+1}(\mu,\lambda)\right)e^{-K(\Phi)} \tag{4.20a}$$
$$i = 1,\ldots,n$$

where Φ (2.23) is an one–soliton solution (2.44), $K(\Phi)$ was introduced in (4.5), (4.6) and

$$\begin{aligned} B^i(\mu,\lambda) &= \sum_{l<i} \frac{\lambda^{n+1+l-i}\mu^{i-l}}{\lambda^{n+1}-\mu^{n+1}} E^{il} + \frac{1}{2}\frac{\lambda^{n+1}+\mu^{n+1}}{\lambda^{n+1}-\mu^{n+1}} E^{ii} + \\ &+ \sum_{l>i} \frac{\lambda^{l-i}\mu^{n+1+i-l}}{\lambda^{n+1}-\mu^{n+1}} E^{il}, \qquad i = 1, \ldots, n+1 \end{aligned} \tag{4.20b}$$

We recall that since Φ is traceless, one of its components can be expressed in terms of the others. In calculating the derivatives in the l. h. s. of (4.20a) we have set $\varphi_{n+1} = -\sum_{i=1}^{n} \varphi_i$. The expressions (4.20b) follow from (4.1a), (4.1b), (4.4) and the summation formula (4.16a). Let us introduce the loop group element

$$h^{(1)}(\Phi,\mu,\lambda) = e^{-K(\Phi)} g^{(1)}(\Phi,\mu,\lambda) \tag{4.21}$$

From (4.20a) it follows that this element satisfies the system

$$\begin{aligned} \frac{\partial}{\partial\varphi_i} h^{(1)}(\Phi,\mu,\lambda) &= J^i(\mu,\lambda) h^{(1)}(\Phi,\mu,\lambda) \\ i &= 1, \ldots, n \end{aligned} \tag{4.22a}$$

$$\begin{aligned} J^i(\mu,\lambda) &= \left(\frac{\partial}{\partial\varphi_{n+1}} - \frac{\partial}{\partial\varphi_i}\right) K(\Phi) + B^i(\mu,\lambda) - B^{n+1}(\mu,\lambda) \\ \varphi_{n+1} &= -\sum_{i=1}^{n} \varphi_i \end{aligned} \tag{4.22b}$$

An intriguing property of the loop algebra elements J^i is that they *do not depend* on the affine Toda fields φ_i for $i = 1, \ldots, n$. Combining this observation with the integrability condition of the linear differential system (4.22a) we conclude that

$$\left[J^i(\mu,\lambda), J^j(\mu,\lambda)\right] = 0 \tag{4.23}$$

and therefore the following representation

$$g^{(1)}(\Phi,\mu,\lambda) = e^{K(\Phi)} e^{\sum_{i=1}^{n} \varphi_i J^i(\mu,\lambda)} \tag{4.24}$$

takes place. Note that for $n = 1$, which corresponds to the sinh–Gordon model, the diagonal prefactor in the r. h. s. of the above equation disappears and we end up with the exponentiated form of the one–soliton dressing group element [25].

We proceed with the following remark: as it was noted in [13], the general dressing problem admits two solutions depending on the analiticity properties for $\lambda \to 0$ and $\lambda \to \infty$. In other words, the solutions (3.24a), (3.24b) in particular, due to the Gauss decomposition in the loop group $\widetilde{SL}(n+1)$, represent two different elements. Skipping the dependence on the field parameters one can write

$$g(\lambda) = \begin{cases} g_+(\lambda) & \lambda \to 0 \\ \\ g_-(\lambda) & \lambda \to \infty \end{cases} \tag{4.25}$$

from where we conclude that the element $g(\lambda)$ we analyzed before is an analytic continuation of two *different* elements of the loop group. As noted in [13], [14], a dressing group element is represented by the pair (g_+, g_-) and there is a canonical diffeomorphism between the dressing group and the underlying loop group †

$$(g_+, g_-) \to g_-^{-1} g_+ \tag{4.26}$$

Denote by $J^i_+(\mu,\lambda)$ and $J^i_-(\mu,\lambda)$ the expansions of the elements (4.22b) around $\lambda \to 0$ and $\lambda \to \infty$ respectively. Let us calculate the value of the map (4.26) for the one–soliton dressing group elements. Expanding (4.20b) around $\lambda \to 0$ and $\lambda \to \infty$ and using (4.22b) we get (4.24)

$$(g_-^{(1)})^{-1}(\Phi,\mu,\lambda) g_+^{(1)}(\Phi,\mu,\lambda) =: e^{\sum_{i=1}^n \varphi_i I^i(\mu)} :$$

$$\begin{aligned} I^i(\mu) &= J^i_+(\mu,\lambda) - J^i_-(\mu,\lambda) = \\ &= -\sum_{l=1}^{n+1} \sum_{k\in\mathbb{Z}} \left(\frac{E^{il}_{l-i+k(n+1)}}{\mu^{k(n+1)+l-i}} - \frac{E^{n+1\, l}_{l+k(n+1)}}{\mu^{k(n+1)+l}} \right), \quad i \in \mathbb{Z}_{n+1} \end{aligned} \tag{4.27}$$

where the normal product : : means writing J^i_- on the left : $J^i_- J^j_+ :=: J^j_+ J^i_- := J^i_- J^j_+$ and the lower indices, as we defined in Sec. 2, count the powers of λ. To be more precise, one should stress that the elements I^i are not well defined in the loop algebra. To avoid this difficulty, one considers the level one representations of the corresponding affine Lie algebra [19], [20], [26], [27]. It is not difficult to calculate the commutators of the above elements with the grade ± 1 elements of the principal Heisenberg subalgebra (2.25). The result is

$$\begin{aligned} \left[I^i(\mu), \mathcal{E}_+ \right] &= \mu \left(I^i(\mu) - I^{i-1}(\mu) + I^n(\mu) \right) \\ \left[I^i(\mu), \mathcal{E}_- \right] &= \frac{1}{\mu} \left(I^i(\mu) - I^{i+1}(\mu) + I^1(\mu) \right) \end{aligned} \tag{4.28}$$

Comparing the expressions (4.27) with (2.16a), (2.16b) and (2.21) we obtain

$$I^i(\mu) = \sum_{l\in\mathbb{Z}_{n+1}} \omega^l (\omega^{-il} - 1) F^l(\mu) \tag{4.29}$$

The above identity together with (2.22), (4.27) and (4.28) suggests a relation to the vertex operator approach to the affine Toda solitons [19], [20].

It is worthwhile to make several remarks and comments. First of all, we recall that there are two related but inequivalent notions of solitons [1]–[5]. From the physical point of view a soliton is a localized solution of the field equations which carries finite

†We stress that this map is not an isomorphism of Lie groups since it does not preserve the multiplication physical quantities, like momentum, energy, etc. There is another concept of solitons, adopted within the ISM, namely solitons arise when the underlying auxiliary linear problem is reflectionless. In the present paper we relax the physical requirements and deal with the solitons as they were treated by the ISM. Our reason to do this is that the formalism developed by us can be repeated without any modification in the physical region of the coupling constant. Due to this, for the sake of brevity, we preferred to work with certain real value of the coupling constant and to enjoy the algebraic beauty of the soliton solutions. Second, in the present paper we restricted ourselves to study $A_n^{(1)}$ Toda solitons only. One reason to do that was our intention to keep the discussion as elementary as it is possible. On the other hand, our approach is based on the observation that the connection (2.28) belongs to the principal gradation. Due to that we exploited the observation that the gradation generating automorphism σ (2.5) yields a symmetry of the linear system (2.29). For general Lie algebras, one can repeat the procedure of Sec. 2 to construct soliton solutions, but in general it is not possible to construct the counterpart of (2.32). The reason is that the order of σ (2.5), except for the A_n and C_n Lie algebras, is always smaller than the dimension of any irreducible representation. Therefore, to generalize the results of Sec. 3 and Sec. 4, we have to look for an additional symmetry of the linear system (2.29). This problem is under investigation. Finally, we note that in contrast to the seminal paper [25], where in the particular example of the sinh–Gordon model the factorization problem (4.7) was treated by using Bäcklund transformations, we preferred the algebraic recursive approach described in the present Section. It remains as an open question to relate these two approaches. The Bäcklund transformations for $A_n^{(1)}$ Toda equations have been studied in [33]. As final comment, we stress that the expressions (4.27)–(4.29) provide the relation to the vertex operator formalism only for one soliton solutions. We hope to go back to this problem for general N–solitons elsewhere.

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