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QUANTUM FIELDS IN COSMOLOGICAL SPACE-TIMES: A SOLUBLE EXAMPLE

by

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Abstract. We solve the Klein-Gordon equation for a massive real scalar field in the Novello-Salim Eternal Universe, i.e., non singular spatial homogeneous and isotropic cosmological background which is tangent to Milne universes in the distant past and future (and hence asymptotically flat) and evolves between these two geometries via a phase of contraction to a point of maximum curvature followed by expansion. This allows a computation of the Bogolyubov coefficients of the scalar field, usually interpreted as the rate of creation of matter by the time varying gravitational field, either when the vacuum is defined at the moment of maximum curvature (the false Big-Bang) or at the far beginning of the cosmic evolution. This new exact solution is compared to the results obtained when the geometry is that of the Milne universe. It is intended to shed some light on the controversial issue of disentangling observer dependent and curvature effects in the process of matter creation within the framework of general relativity.

Key-words: Quantum field theory; Cosmology; Particle creation.

1. Introduction

The theory of quantum fields in cosmological backgrounds and more specifically the possibility of the creation of matter by strong and rapidly varying cosmological gravitational fields have been extensively studied [cf. e.g. Birrel & Davies 1984 for a review]. If indeed the universe started its evolution from a highly curved initial state (the "Big-Bang") the conditions existed for such a mechanism to operate and it is physically relevant to study it.

When however the background is taken to be the standard Friedmann model the evolution of which is driven by classical matter, either dust or pure radiation, the program runs into two specific difficulties: the existence of a curvature singularity, which renders the definition of an initial vacuum state for the quantum fields particularly delicate, and the absence of an asymptotically flat region where the well-established special relativistic theory of quantum fields can serve as a guide to define the particle states hic et nunc. These difficulties must be carefully dealt with but perhaps not from the outset. Indeed the conceptual problem of the very definition of quantum particles within the framework of general relativity, that is the problem of combining a quantum definition of matter, which is global, with a theory based on the principle of equivalence, which is local, is still pending and should perhaps be tackled first.

A number of toy models have hence been designed which cut out these difficulties while retaining the main feature of the problem: a time varying gravitational field. Unfortunately most of these models cannot be justified by some underlying cosmological theory and very few, however simplified they may be, are exactly soluble.

These are our motivations for studying quantum fields in a Friedmann Robertson - Walker universe with negatively curved spatial sections the line element of which, in the spherical coordinates ($r=\sinh\chi, \theta, \phi$) and the "cosmic time" t or the "conformal time" η, is:

$$\begin{aligned} ds^2 &= dt^2 - C(t)\left[\frac{dr^2}{1+r^2} + r^2(d\theta^2+\sin^2\theta d\phi^2)\right] \\ &= C(\eta)\left[d\eta^2 - \frac{dr^2}{1+r^2} - r^2(d\theta^2 + \sin^2\theta d\eta^2)\right] \\ &= C(\eta)\left[d\eta^2 - d\chi^2 - \sinh^2\chi(d\theta^2+\sin^2\theta d\phi^2)\right] \end{aligned} \qquad (1.1)$$

with $C(t) = (a_0{}^2+t^2)$ or equivalently $C(\eta) = a_0{}^2\cosh^2\eta$, a_0 being a positive constant of the order of the Planck length if the model is to be thought of in the context of some unified theory.

This universe is non singular and asymptotically tangent, when t or η tend to $+\infty$ or $-\infty$, to two distinct Milne universes (Milne 1932 and e.g. Bondi 1952). It is hence asymptotically flat. The entire cosmic evolution then consists in an infinite period of contraction to a point of maximum curvature (reached at $t = \eta = 0$ when $R_{\mu\nu}R^{\mu\nu} \equiv 6/a_0{}^2\cosh^2\eta=6/a_0{}^2$) followed by an infinite phase of expansion. This universe is therefore well adapted for our purpose of examining the problem of the definition and creation of matter in cosmological geometries that avoid the additional complications of the standard model. In this universe moreover, the Klein-Gordon equation is, as we shall see, exactly soluble in terms of Mathieu functions. Finally it has the interesting property to tend to a flat space-time described by the Milne coordinates rather than by the standard Minkowskian cartesian coordinates. This feature should allow some insights in the question of disentangling coordinate, observer-dependent and curvature effects in the process of matter creation within the framework of general relativity.

We wish now to stress that the metric (1.1) is not just another artificial example designed t test a theory. It arises in the context of primordial cosmologies when the matter fields which drive the evolution are no longer supposed to be minimally coupled to gravity. In the model of Novello and Salim for example (Novello & Salim 1979), the main source of curvature of the universe is a spin-one field $A_\mu(x)$ coupled to gravity in a gauge-breaking way, which can be interpreted as non-linear photon endowed with a mass. The Lagrangian density of that model is

$$L = (1 + \lambda A_\mu A^\mu)R - \frac{1}{2} F_{\mu\nu}F^{\mu\nu}$$

with R the scalar curvature of the manifold, $F_{\mu\nu} \equiv \partial_\mu A_\nu - \partial_\nu A_\mu$ λ a constant, so that the derived field equations admit (1.1) as a solution. The field $A_\mu(x)$ is then given by

$$A_\mu = (A_0,\vec{0}) \quad , \quad A_0(\eta) = \sqrt{\frac{1}{\lambda}(1 + \tanh\eta)} \quad .$$

Another example is the Melnikov-Orlov model (Melnikov & Orlov 1979) where the cosmic evolution is driven by a scalar field $\Phi(x)$ conformally coupled to gravity. It is based on a Lagrangian density often considered in the context of inflationary cosmology, to:

$$L = (1 - \frac{1}{6}\,\Phi^2)R + \partial_\mu\Phi\partial^\mu\Phi - m^2\Phi^2 + \sigma\phi^4 \quad .$$

The metric (1.1) is again a solution of the corresponding field equations with ${a_0}^2 = 1/24\sigma$ and $\Phi(x)$ given by

$$\Phi(\eta) = \left[\sqrt{2\sigma}\; a_0 \;\cosh\eta\right]^{-1}$$

(see also Sthyaprakash et al. 1986).

Finally the field equations derived from the very simple Lagrangian

$$L = (1 - \frac{1}{6}\,\Phi^2)R + \partial_\mu\Phi\partial^\mu\Phi$$

also admit (1.1) as a solution with $\Phi(x)$ given by

$$\Phi(\eta) = \Phi_0 - \frac{A}{{a_0}^2}\tanh\eta$$

where A is an integration constant. In this last model the effective gravitational constant $(1 - \frac{1}{6}\,\Phi^2)^{-1}$ changes sign in the course of the evolution, a property which may be used to test the stability of the theory against the apparition of ghosts (Deruelle & Novello 1988).

The paper is intended to be self-contained and organised as follows. In section 2 we briefly review the second quantisation of a massive real scalar field in a Robertson Walker background in order to specify our framework and fix the notations. In section 3 we apply this formalism to the case when the background is the Milne universe, that is flat space--time in a non cartesian coordinate system which does not cover the whole Minkowski manifold. By means of this thoroughly studied example (see e.g. Birrell and Davies 1984 and references therein) we intend to illustrate the well-known conceptual difficulties encountered when trying to define a quantum particle, even in flat space-time, when the existence of privileged inertial reference frames is denied. Section 4 is the core of the paper where we solve the Klein-Gordon equation for a massive real scalar field in the metric (1.1) in terms of Mathieu functions.

We first define the vacuum state at the beginning of the cosmic evolution, when the geometry is asymptotically flat, and compute the number of quanta present in that state at the end of the whole evolution, after the universe has gone through its phase of contraction and is expanding approaching its final, asymptotically flat state. In the physically relevant limit when the ratio ma_0 of the minimum value of the scale factor of the universe and the Compton wave lenght of the scalar field is small, an explicit expression for the number of created quanta can be given, which is shown to depend crucially on the relationship between ma_0 and the energy of the field. For "resonant" values of the energy, the number of quanta created reaches a maximum value; for "antiresonant" values it vanishes. Then we quantise the field following as closely as possible the procedure used in the Milne case. This leads us to define a vacuum state at the moment of maximum contraction (the false Big-Bang) and compute the number of quanta that are present in this state at the end of the evolution. Section 5 has a pedagogical motivation. It treats the problem in a metric that approximates (1.1) and reproduces the basic features of the preceding results by means of the more familiar Bessel functions. Section 6 draws a few conclusions and attempts to interpret the results in terms of an actual creation of particles. A first appendix gathers the properties of the Mathieu functions that are required to derive the results of section 4. In the second appendix the problem is treated in the WKB approximation.

2. Quantisation of a scalar field in a Robertson-Walker background: a compendium

(For more detailed treatment cf. e.g. Birrel & Davis 1984).

Consider a 4-dimensional space-time V possessing a maximally symmetric 3-dimensional subspace. Let the coordinate system (η, r, θ, ϕ), where (r, θ, ϕ) are the spherical coordinates and η, $\eta_1 < \eta < \eta_2$, is the "conformal" time, cover a part $\overline{V}$ of V. The line element in $\overline{V}$ then is:

$$ds^2 = C(\eta)\left[d\eta^2 - \frac{dr^2}{1-Kr^2} - r^2(d\theta^2+\sin^2\theta d\phi^2)\right] \qquad (2.1)$$

with $K = +1, 0$ or -1 and where $C(\eta)$ solves the dynamical equations (such as Einstein's) that determine the geometry of V; $C(\eta)$ is a positive function that may go to zero only when $\eta \to \eta_1$, or $\eta \to \eta_2$ but that we shall leave otherwise unspecified. The cosmic time t is defined up to a constant by

$$t = \int^{\eta} d\eta' \sqrt{C(\eta')}$$

It is uniformly increasing function of η varying in $]t_1, t_2[$ when $\eta \in]\eta_1, \eta_2[$. It is the proper time of a comoving observer at constant (r, θ, ϕ). We shall also make use of the radial coordinate χ, $\chi \in [0, \infty[$, defined as: $r = \sinh\chi$.

Consider now a real scalar field Φ of mass m few aching this space-time without perturbing its geometry. It satisfies the Klein-Gordon equation:

$$\Box\Phi + (m^2 + \frac{\xi}{6} R)\Phi = 0 \qquad (2.2)$$

where $\Box$ is the d'Alembertian operator in the metric (2.1) and R the scalar curvature of $\overline{V}$; we shall here restrict ourselves to the case of conformal coupling: $\xi = 1$. (Note however that the Milne and the eternal universes that we shall consider on sections 3,4,5 all have a vanishing scalar curvature so that the nature of the coupling, that is the value of ξ , will be irrelevant.)

In order to solve (2.2) let

$$\Phi_{\vec{k}}(x) = C^{-1/2}\, Y_{\vec{k}}(\vec{x})\chi_k(\eta) \qquad (2.3)$$

where $Y_{\vec{k}}(\vec{x})$ is a solution of

$$\Delta^{(3)}\, Y_{\vec{k}}(\vec{x}) = -(k^2-\varepsilon)Y_{\vec{k}}(\vec{x})$$

with $k \in]0,+\infty[$ if $\varepsilon = -1$, $k = |\vec{k}|$, $\vec{k} \in \mathbb{R}^3$ if $\varepsilon = 0$ and $k \in \mathbb{N}$ if $\varepsilon = +1$, $\Delta^{(3)}$ being the Laplacian operator associated with the spatial part (signature +++) of the metric (2.1). See e.g. Birrel and Davies 1984 for the explicit expression of $Y_{\vec{k}}(x)$ in terms of spherical harmonics. As for $\chi_k(\eta)$ it is a solution of

$$\frac{d^2\chi_k}{d\eta^2} + \chi_k\,[k^2+m^2C(\eta)] = 0 \qquad (2.4)$$

The modes (2.3) are normalised to unity:

$$(\Phi_{\vec{k}},\Phi_{\vec{k}'}) = -i\int_\Sigma (\Phi_{\vec{k}}\overleftrightarrow{\partial}_\mu\,\Phi^*_{\vec{k}'})\,n^\mu d\Sigma = \delta_{kk'} \qquad (2.5)$$

where n^μ is the unit vector orthogonal to an arbitrary space--like hypersurface Σ, $d\Sigma$ being its volume element, and where

$$f\overleftrightarrow{\partial}_\mu g = f\frac{\partial g}{\partial x^\mu} - g\frac{\partial f}{\partial x^\mu} \quad .$$

When the functions $Y_{\vec{k}}(\vec{x})$ are normalised to unity (2.5) amounts to imposing:

$$\chi_k \overleftrightarrow{\partial}_\eta \chi_{k'} = i$$

Now denote χ_k^{in} the solution of (2.4) which behaves like a positive frequency wave when $t \to t_1$ (or $\eta \to \eta_1$) that is which is proportional to $\exp(-iE_k^{in} z)$, E_k^{in} being some positive number and z a uniformly increasing function of t such as t itself or η. In Minkowski space-time in cartesian coordinates, $E_k^{in} = \sqrt{\vec{k}^2+m^2}$ and can be interpreted as the energy of a particle associated to the wave. In curved space--time or in Minkowski space-time in non-cartesian coordinates such an interpretation holds in general by analogy only. In any case however the set $\Phi_{\vec{k}}^{in}$ and $\Phi_{\vec{k}}^{in*}$ constructed from χ_k^{in} forms a basis of the Hilbert space of Φ and any real solution of (2.2) can be written as

$$\Phi(x) = \int d\mu(\vec{k}) \; [a_{\vec{k}}^{in}\Phi_{\vec{k}}^{in}(x) + a_{\vec{k}}^{in\dagger}\Phi_{\vec{k}}^{in*}(x)]$$

where $a_{\vec{k}}^{in}$ and $a_{\vec{k}}^{in\dagger}$ are conjugate complex numbers and $d\mu(\vec{k})$ is the measure associated with the functions $Y_{\vec{k}}(\vec{x})$.

Similarly we denote χ_p^{out} the solution of (2.4) which behaves like a positive frequency wave when $t \to t_2$ (or $\eta \to \eta_2$), so that $\Phi(x)$ can also be written as:

$$\Phi(x) = \int d\mu(\vec{p}) \; [a_{\vec{p}}^{out}\Phi_{\vec{p}}^{out}(x) + a_{\vec{p}}^{out\dagger}\Phi_{\vec{p}}^{out*}(x)] \quad .$$

As the vectors of the in-basis can be expanded in the out-basis we has:

$$\phi^{in}_{\vec{k}}(x) = \int d\mu(\vec{p}) \; [\alpha_{\vec{p}\vec{k}} \; \phi^{out}_{\vec{p}} + \beta_{\vec{p}\vec{k}} \; \phi^{out*}_{\vec{p}}]$$

where the Bogoliubov coefficients $\alpha_{\vec{p}\vec{k}}$ and $\beta_{\vec{p}\vec{k}}$ are given by

$$\alpha_{\vec{k}\,\vec{p}} = (\phi^{in}_{\vec{k}}, \phi^{out}_{\vec{p}}) \quad ; \quad \beta_{\vec{k}\vec{p}} = -(\phi^{in}_{\vec{k}}, \phi^{out*}_{\vec{p}}) \quad , \qquad (2.6)$$

The field Φ can now be quantised by inverting $a^{in}_{\vec{k}}$ and $a^{in\dagger}_{\vec{k}}$ into "annihilation" and "creation" conjugate hermitian operators satisfying the commutation rules:

$$[a^{in}_{\vec{k}}, a^{in}_{\vec{k}'}] = [a^{in\dagger}_{\vec{k}}, a^{in\dagger}_{\vec{k}'}] = 0 \quad ;$$
$$[a^{in}_{\vec{k}}, a^{in\dagger}_{\vec{k}'}] = \delta(\vec{k},\vec{k}') \quad . \qquad (2.7)$$

These operators act on a Fock space constructed from a "in-vacuum state" $|0>^{in}$ such that

$$a^{in}_{\vec{k}}|0>^{in} = 0 \quad .$$

A basis of the Fock space is then obtained by iteratively applying the creation operators on $|0>^{in}$. The resulting "one-particle" state or "$\vec{k}$-in quantum" $|1_{\vec{k}}>^{in} \equiv a^{in\dagger}_{\vec{k}}|0>^{in}$ as well as all the other many-particles states such as $|^{k}n_{\vec{k}}, {}^{k'}n_{\vec{k}'}>^{in} \equiv (a^{in\dagger}_{\vec{k}})n_k (a^{in\dagger}_{\vec{k}})\, n_k\, |0>^{in}$ are normalised to unity: ${}^{in}<1_{\vec{k}}|1_{\vec{k}'}>^{in} = \delta(\vec{k},\vec{k}')$ etc. The operator $n^{in}_{\vec{k}} = a^{in\dagger}_{\vec{k}} a^{in}_{\vec{k}}$ can then consistently be called the "number of $\vec{k}$ in quanta" operator since, por example:

$$ {}^{in}\langle {}^{k}n_{\vec{k}}, {}^{k'}n_{\vec{k}'} | n_{\vec{k}}^{in} | {}^{k}n_{\vec{k}}, {}^{k'}n_{\vec{k}'} \rangle^{in} = n_k \quad . $$

A non-necessarily equivalent quantisation of Φ consists in converting $a_{\vec{p}}^{out}$ and $a_{\vec{p}}^{out\dagger}$ into opertors satisfying commutation rules similar to (2.7) and acting on another Fock space constructed from an out-vacuum state $|0\rangle^{out}$. It is then a matter of simple algebra to obtain:

$$ N_{\vec{p}}^{out} \equiv {}^{in}\langle 0 | a_{\vec{p}}^{out\dagger} a_{\vec{p}}^{out} | 0 \rangle^{in} = \int d\mu(\vec{k}) \, |\beta_{\vec{p}\vec{k}}|^2 $$

$$ = \int d\mu(\vec{k}) \, |(\phi_{\vec{k}}^{in}, \phi_{\vec{p}}^{out*})|^2 \quad . \qquad (2.8) $$

This number is consistently interpreted as the "number of $\vec{p}$ out-quanta in the in-vacuum state".

In the framework of special relativity when the field Φ interact with, say, an electric field in a Minkowski background, the states $|0\rangle^{in}$ and $|0\rangle^{out}$ represent the physical vacuum in the distant past and future. A "$\vec{k}$ in-quantum" then represents a physical particle in the state $\vec{k}$ in the distant past and a "$\vec{p}$ out-quantum" a physical particle in the state $\vec{p}$ in the distante future. Therefore $N_{\vec{p}}^{out}$ represents the number of particles in the state $\vec{p}$ created from the vacuum during the evolution. (That is the so-called Klein paradox see e.g. Damour 1977.)

In the case here considered where the field interacts with the geometry of space-time the same interpretation is usually believed to remain sensible although the interpretation that the states $|0>^{in}$ and $|0>^{out}$ represent physical vacua is beclouded with uncertainty since it relies on an arbitrary extension to curved space-times of the natural identification, in Minkowski space-time of the modes χ_k^{in} and χ_p^{out} with particles. It is therefore still a matter of debate to know whether an hypothetical observer, moving with the cosmological fluid and measuring frequencies in terms of his proper time (the cosmic time), having switched on a in-particle detector in the distant past and making a measurement at the end of the cosmic evolution, would or would not detect $N_{\vec{p}}^{out}$ particles, given by (2.8), in the state $\vec{p}$. The issue is even more controversial when $\bar{V} \subset V$ so that the coordinate system (η,r,θ,ϕ) covering $\bar{V}$ can be extended to a system (T,X,Y,Z) that covers the whole manifold V. Indeed, as a direct consequence of the non local character of quantum particles, quantisation in these two systems is not equivalent although the principle of equivalence a priori forbids to endow one or the other with any privileged status.

3. On the concept of particle in the Milne universe: a thorny example

(For details cf. e.g. Birrell & Davies 1984 and references therein.)

Consider a Robertson-Walker space-time with line element (2.1) when $\varepsilon = -1$ (negatively curved spatial sections) and where

$$C(\eta) = e^{2\eta} \quad , \quad \eta \in \,]-\infty,+\infty[\quad . \tag{3.1}$$

This defines the Milne universe (Milne 1932 and e.g. Bondi 1952). A real scalar field in this background geometry can be quantised according to the procedure of section 2. We are therefore led to solve eq. (2.4) which reads:

$$\frac{d^2\chi_k}{d\eta^2} + \chi_k\,[k^2+m^2e^{2\eta}] = 0 \quad . \tag{3.2}$$

Changing the "conformal" time η into the "cosmic" time t such that:

$$t = e^{\eta} \quad , \quad t \in \,]0,+\infty[$$

brings (3.2) into the standard form of the Bessel equation (see e.g. Abramowitz & Stegun 1965):

$$\frac{d^2\chi_k}{dz^2} + \frac{1}{z}\frac{d\chi_k}{dz} + \chi_k\,(1 - \frac{\nu^2}{z^2}) = 0 \quad , \quad z \in \,]0,+\infty[\tag{3.3}$$

where $z \equiv mt$ and $\nu \equiv -ik$ $(k > 0)$. (The case $m = 0$ must be treated separatly.)

One solution of (3.3) is the Bessel function $J_\nu(z)$ the asymptotic behaviour of which, when $t \to 0$ (or $\eta \to -\infty$) is (see eq. (A.18) App. A)

$$J_\nu(z) \underset{\eta\to-\infty}{\xrightarrow{z \to 0}} \frac{(z/2)^\nu}{\Gamma(1+\nu)} = \frac{(m/2)^{-ik}}{\Gamma(1-ik)} \exp(-ik\eta) \tag{3.4}$$

which behaves like a positive frequency wave with respect to t (or η) (since $k > 0$). After normalisation (cf. eq. (A.19)) we hence denote:

$$\chi_k^{in}(\eta) = \left[\frac{\pi}{2\sinh\pi k}\right]^{1/2} J_{-ik}(me^{\eta}) \quad . \tag{3.5}$$

Following the second quantisation procedure of section 2 we can then use χ_k^{in} and χ_k^{in*} to define an in-vacuum state in the asymptotic region $t \to 0$ ($\eta \to -\infty$) that is near the Milne "Big-Bang" where $C(\eta) \to 0$.

Another solution of (3.3) is the Hankel function of the second kind $H_\nu^{(2)}(z)$ the asymptotic behaviour of which when $t \to +\infty$ (or $\eta \to +\infty$) is (see Eq. (A.20)):

$$H_\nu^{(2)}(z) \xrightarrow[\eta\to+\infty]{z\to+\infty} \sqrt{\frac{2}{\pi z}} \exp[-i(z - \frac{\pi\nu}{2} - \frac{\pi}{4})] = \sqrt{\frac{2}{\pi m t}} \; e^{\pi k/2} \exp[-i(mt-\pi/4)]$$

which behaves like a positive frequency wave with respect to the cosmic and conformal times t and η. After normalisation (using A.16 and A.19) we hence denote

$$\chi_k^{out}(\eta) = \frac{\sqrt{\pi}}{2} e^{-\pi k/2} H_{-ik}^{(2)}(me^{\eta}) \tag{3.6}$$

and use χ_k^{out} and χ_k^{out*} to define an out-vacuum state in the asymptotic region where $t \to +\infty$ (and $\eta \to +\infty$).

Using the properties of the Bessel and Hankel functions, we have:

$$\chi_k^{in} = \alpha_k \chi_k^{out} + \beta_k \chi_k^{out*}$$

with

$$\alpha_k = \frac{1}{\sqrt{1-e^{-2\pi k}}} \quad , \quad \beta_k = + \frac{1}{\sqrt{e^{2\pi k}-1}}$$

($|\alpha_k|^2 - |\beta_k|^2 = 1$) so that the Bogoliubov coefficient introduced in (2.6) is:

$$\beta_{\vec{p},\vec{k}} = + \frac{1}{\sqrt{e^{2\pi k}-1}} \, \delta(\vec{p},\vec{k})$$

Using the "golden rule" (cf. e.g. Lifchitz & Pitaevskii 1973) that:

$$[\delta(\vec{p},\vec{k})]^2 = \frac{1}{(2\pi)^3}\,\delta(\vec{p},\vec{k}) \int d\Sigma$$

we obtain for the number per unit volume of $\vec{p}$ out-quanta in the in-vacuum state (see (2.8)):

$$dN_{\vec{p}}^{out}/d\Sigma = \frac{1}{(2\pi)^3}\,\frac{1}{e^{2\pi p}-1} \tag{3.7}$$

By analogy with the theory of quantum fields in Minkowski space-time one is led, from the asymptotic behaviour of χ_k^{in} [see (3.5) and (3.4)], to interpret k (or p) as the energy of the ingoing quanta (which hence does not depend on their mass). With this interpretation the out-quanta (3.7) have therefore a Planck spectrum for a Bose-Einstein gas in 4 dimensions at temperature

$$T_0 = 1/2\pi k_B \quad ,$$

where k_B is Boltzmann constant, in the coordinate system using the conformal time η or at a temperature

$$T = \frac{1}{\sqrt{g_{00}}}\,T_0 = \frac{1}{2\pi k_B t} \tag{3.8}$$

in the coordinate system using the cosmic time t.

We note that (3.7) is independent of the mass m of the scalar field. When m = 0 however Eq. (3.2) is trivially solved: $\chi_k^{in} \propto e^{-ik\eta}$, $\chi_p^{out} = \delta_{pk}\chi_k^{in}$ so that $N_{\vec{p}}^{out}$ (m = 0) = 0. There are no massless $\vec{p}$ out-quanta in the in-vacuum state.

Before now concluding that the particle detector of a comoving observer would actually detect the thermal spectrum (3.8) we must enquire whether the in- and out-vacuum states used in the derivation represent actual physical vacua. As one know indeed the Milne universe is flat since the change of coordinates

$$T = e^{\eta}\sqrt{1+r^2} \quad , \quad R = e^{\eta} r \tag{3.9}$$

brings the line element (2.1) with $C(\eta)$ given by (3.1) into a Minkowskian form:

$$ds^2 = dT^2 - dR^2 - R^2(d\theta^2+\sin^2\theta d\phi^2) \tag{3.10}$$

with

$$T^2-R^2 > 0 \quad , \quad T > 0 \tag{3.11}$$

Clearly this metric (3.10-11) which by construction describes only the future cone of the origin of the (T,R,θ,ϕ) coordinate system can be trivially continued to the whole Minkowski manifold by simply dropping the condition (3.11). The Milne comoving observers at constant r (or χ), θ and ϕ are then nothing but inertial observers whose trajectories R = cte T, $T > 0$, radiate from the origin of the (T,R,θ,ϕ) coordinate system. Moreover their proper time (the cosmic time $t = e^{\eta}$) is proportional to the Minkowskian time $T = e^{\eta}\sqrt{1+r^2}$ (but the constant of proportionality $\sqrt{1+r^2}$ differs from one trajectory to another). The Minkowski vacuum, defined by solving the Klein-Gordon equation in the metric (3.8), being invariant under Lorentz transformations, it may thus seem a priori surprising that a Milne observer should detect the particle spectrum (3.8) at the end of its evolution.

Various lines of attach to tackle this paradox have therefore been tried (see e.g. Di Sessa 1974, Gomes et al. 1974, Fulling et al. 1974, Sammerfield 1974, is well as Grove & Ottewill 1983, Letaw & Pfautsch 1980, Hinton 1984, Castagnino & Ferraro 1984, Sanchez 1979).

One of them, a priori sensible but in fact contrary to the spirit of the equivalence principle, consists in endowing with a privileged status the "Minkowski vacuum" constructed from the modes $\phi^{(M)}_{\vec{k}}$ that solve the Klein-Gordon equation in the (T,R,θ,ϕ) coordinate system

$$\phi^{(M)}_{\vec{k}}(T,\vec{R}) = \frac{1}{(2\pi)^{3/2}}\, e^{i\vec{k}.\vec{R}}\ e^{-i\sqrt{\vec{k}^2+m^2}\,T} \qquad (3.12)$$

$\vec{k} \in \mathbb{R}^3$, $\vec{R} = (R\sin\theta\cos\phi, R\sin\theta\sin\phi, R\cos\theta)$. These modes (3.12) together with their complex conjugates form a basis of the solution of the Klein-Gordon equation in our geometry. Therefore the modes $\phi^{in}_{\vec{k}}$ constructed from the Bessel functions (3.5) as well as the modes $\phi^{out}_{\vec{p}}$ constructed from the Hankel functions (3.6) can be expanded on the Minkowski basis $\phi^{(M)}_{\vec{k}}$ and $\phi^{(M)*}_{\vec{k}}$. Now, at least in the particular case of a 2-dimensional space-time (a case however which due to its conformal flatness, is more restrictive than merely considering the modes in a plane θ = cte , ϕ = cte), Fulling et al. 1974, using some integral representation of the Hankel functions (Ryshik & Gradstein 1963), showed that the modes $\phi^{out}_{\vec{k}}$ built with (3.6) were a linear superposition of the positive frequency waves (3.12) only. Hence the Fock spaces constructed with the modes (3.12) or (3.6) are equivalent where as the spaces constructed with the modes (3.12) or (3.5) are not (at least in the 2-dimensional case but, presumably, in the 4-dimensional case here considered as well).

In this perspective then only the modes (3.6) are retained to build the Fock space of a Milne observer: the $|0\rangle^{out}$ vacuum alone, equivalent to the Minkowski vacuum, is considered to be physical. Hence the "number of k-out quanta in the Minkowski vacuum" is zero: the Milne observer should not detect any particle.

4. Creation of matter in an eternal universe: a tractable case

Let us now turn to the "eternal universe" presented in section 1, the line element of which is given by (2.1):

$$ds^2 = C(\eta)\ [d\eta^2 - \frac{dr^2}{1-\varepsilon r^2} - r^2(d\theta^2+\sin^2\theta d\phi^2)] \qquad (4.1)$$

with

$$C(\eta) = a_0{}^2 \cosh^2\eta \qquad (4.2)$$

and $\varepsilon = -1$. (The results however can be straightforwardly extended to the cases $\varepsilon = 0$ or $\varepsilon = 1$.) In terms of the "cosmic" time t:

$$t = a_0 \sinh\eta \qquad (4.3)$$

the function C reads:

$$C(t) = a_0{}^2 + t^2 \quad .$$

We first note that, contrary to the Milne universe (2.1),(3.1), the manifold described by the metric (4.1),(4.2) is geodesically complete (as can easily be checked by studying the geodesics). We also remark that

$$C(\eta) \xrightarrow{\eta\to+\infty} \frac{a_0{}^2}{4}\ e^{2\eta} \quad , \qquad t \to \frac{a_0}{2}\ e^{\eta}$$

$$C(\eta) \xrightarrow{\eta\to-\infty} \frac{a_0{}^2}{4}\ e^{-2\eta} \quad , \qquad t \to -\frac{a_0}{2}\ e^{-\eta}$$

Hence the universe (4.1), (4.2) is asymptotically tangent, in the distante future and distant past respectively, to two distinct Milne universes (3.1) considered in section 3, the first isomorphic to the future cone of the origin of Minkowski space-time, the second to its past cone. The matching of the 2 Milne universes has to be performed through a delicate procedure of analytic continuation through the origin of Minkowski space-time (see e.g. Rumpf 1984) whereas in the eternal universe considered here the matching is perfectly smooth so that a number of pitfalls can hopefully be avoided. Finally the metric (4.1), (4.2), contrary to the Milne metric, is not flat is therefore more akin to the models usually considered in cosmology.

We now again quantise a real scalar field in this geometry following the steps of section 2 and are therefore led to solve eq. (2.4) which reads:

$$\frac{d^2\chi_k}{d\eta^2} + \chi_k[k^2 + m^2 a_0{}^2 \cosh^2\eta] = 0$$

and can be rewritten as

$$\frac{d^2\chi_k}{d\eta^2} - \chi_k\ [\lambda - 2h^2\cosh 2\eta) = 0 \qquad \textbf{(4.4)}$$

with $\lambda \equiv -(k^2+m^2a_0{}^2/2)$ and $h \equiv ma_0/2$.

Equation (4.4) is known as the modified Mathieu equation (see Meixner & Schäfke 1954, Mc Lachlan 1947, Arscott 1964, Abramowitz & Stegun 1965 and Appendix A).

Consider the solution of (4.4) denoted $M_{\tilde{\nu}}^{(3)}(-\eta,h)$, where the index $\tilde{\nu}$ is some complicated function of λ and h (see eq. (A.5) in Appendix A for an explicit expression of

$\tilde{\nu}$ when h is small); we shall just remark here that, because $\lambda = -k^2-2h^2$, $\tilde{\nu}$ is pure imaginary (Floquet therem: see Meixner & Schäfke p. 131 and eq. (A.5)). We shall hence note $\tilde{\nu} = -i\tilde{k}$, $\tilde{k} \in \mathbb{R}^+$. By definition (see eq. (A.10) in Appendix A), the function $M_{\tilde{\nu}}^{(3)}(-\eta,h)$ has the following asymptotic behaviour:

$$M_{\tilde{\nu}}^{(3)}(-\eta,h) \xrightarrow[t\to-\infty]{\eta\to-\infty} H_{\tilde{\nu}}^{(1)}(2h\cosh\eta) \qquad (4.5)$$

Using now $\cosh\eta \xrightarrow{\eta\to-\infty} e^{-\eta}/2 \sim -t/a_0$ (see eq. (4.3) and the asymptotic expansion of the Hankel function $H_{\tilde{\nu}}^{(1)}$ (see eq. (A.20)), we find the asymptotic behaviour of $M_{\tilde{\nu}}^{(3)}$ is

$$M_{\tilde{\nu}}^{(3)}(-\eta,h) \xrightarrow[t\to-\infty]{\eta\to-\infty} \sqrt{\frac{2}{-\pi mt}}\; e^{-imt}\; e^{-\pi\tilde{k}/2}\; e^{-i\pi/4}$$

which behaves like a positive frequency wave with respect to the cosmic time t. After normalisation (see eqs. (A.16), (A.19)), we hence denote:

$$\chi_k^{in}(\eta) = \frac{\sqrt{\pi}}{2}\, e^{\pi\tilde{k}/2}\, M_{-i\tilde{k}}^{(3)}(-\eta,h) \qquad (4.6)$$

and use $\chi_{\tilde{k}}^{in}$ and $\chi_{\tilde{k}}^{in*}$ to define an in-vacuum in the asymptotic region where $t \to -\infty$ (and $\eta \to -\infty$). We note that the modes (4.6) have the same asymptotic behaviour as the Milne modes (3.15) introduced to define the in-vacuum in the past cone of the origin of Minkowski space-time. Now the Milne modes (3.15), as shown by Fulling et al. 1974 (at least in the two-dimensional case), define a vacuum which is equivalent to the standard Minkowskian vacuum built from the plane waves (3.12). It is therefore intuitively clear (but we shall not attempt here to give a rigorous proof of this assertion) that the Mathieu modes (4.6) also define a vacuum equivalent to the "asymptotic" Minkowskian vacuum.

More precisely, the change of coordinates

$$T = -\frac{a_0}{2}e^{-\eta}\sqrt{1+r^2}\ , \quad R = \frac{a_0}{2}e^{-\eta}\,r \tag{4.7}$$

transforms the metric (4.1), (4.2) into:

$$ds^2 = 4e^{2\eta}\cosh^2\eta\ [dT^2 - dR^2 - R^2(d\theta^2 + \sin^2\theta d\phi^2)]$$

$$= [1 + \frac{a_0^{\ 2}}{4(T^2-R^2)}]\ [dT^2 - dR^2 - R^2(d\theta^2 + \sin^2\theta d\phi^2)] \tag{4.8}$$

with $T^2-R^2 > 0$, $T < 0$, which tends, as $\eta \to -\infty$ or $(T^2-X^2) \to +\infty$, to the Minkowski metric (3.10). The solutions of the Klein--Gordon equation in the metric (4.8), which behave like positive frequency wave with respect to the asymptotically Minkowskian time T and define the asymptotic Minkowski vacuum, necessarily tend to the plane waves (3.12) as $T \to -\infty$. Since (4.6) asymptotically approaches (3.15) which, as Fulling et al. 1974 showed, is a superposition of positive frequency plane waves (3.12) only, the modes (4.6) therefore define a vacuum asymptotically equivalent to the asymptotic Minkowskian vacuum. "QED".

Consider now the solution of eq. (4.4) denoted $M_{\tilde{\nu}}^{(4)}(\eta,h)$. By definition (see eq. (A.10) Appendix A):

$$M_{\tilde{\nu}}^{(4)}(\eta,h) \xrightarrow[t\to+\infty]{\eta\to+\infty} H_{\tilde{\nu}}^{(2)}(2h\cosh\eta) \tag{4.9}$$

Since $\cosh\eta \xrightarrow{\eta\to+\infty} e^{\eta}/2 \sim t/a_0$ according to (4.3) and using the asymptotic expansion (A.20) of the Hankel function (4.9) also reads:

$$M_{\tilde{\nu}}^{(4)}(\eta,h) \xrightarrow[t\to+\infty]{\eta\to+\infty} \sqrt{\frac{2}{\pi m t}}\; e^{-imt}\; e^{\pi\tilde{k}/2}\; e^{+i\pi/4} \tag{4.10}$$

which behaves like a positive frequency wave with respect to the cosmic time t. After normalization (see eq. (A.16),(A.19)) we hence denote

$$\chi_{\tilde{k}}^{out}(\eta) = \frac{\sqrt{\pi}}{2}\; e^{-\pi\tilde{k}/2}\; M_{-i\tilde{k}}^{(4)}(\eta,h) \tag{4.11}$$

and use $\chi_{\tilde{k}}^{out}$ and $\chi_{\tilde{k}}^{out*}$ to define an out-vacuum in the region where $t \to +\infty$ (or $\eta \to +\infty$). As before we note that the modes (4.11) have the same asymptotic behaviour as the Milne modes (3.6) and the same line of argument yields to the conclusion that the out-vacuum defined by (4.11) is asymptoticaly equivalent to an asymptotic Minkowski vacuum.

In order to compute the number of $\tilde{\nu}$-out quanta in the in-vacuum we note the following relation among the Mathieu functions (see Appendix A, eq. (A.13)):

$$-\frac{4i}{\pi} M_{\tilde{\nu}}^{(3)}(-\eta) = \left[\frac{d}{d\eta} M_{\tilde{\nu}}^{(3)} M_{\tilde{\nu}}^{(4)}\right]_{\eta=0} M_{\tilde{\nu}}^{(3)}(\eta) - \left[\frac{d}{d\eta} M_{\tilde{\nu}}^{(3)} M_{\tilde{\nu}}^{(3)}\right]_{\eta=0} M_{\tilde{\nu}}^{(4)}(\eta)$$

We also note that, when $\tilde{\nu} = -i\tilde{k}$, $\tilde{k} \in \mathbb{R}^+$ and $\eta \in \mathbb{R}^+$, $[M_{\tilde{\nu}}^{(4)}(\eta)]^* = M_{\tilde{\nu}^*}^{(3)}(\eta)$, a property which follows from the asymptotic behaviour of $M^{(4)}$ and $M^{(3)}$ (see Appendix A, eq. (A.21)). Hence we have:

$$\chi_{\tilde{k}}^{in} = \alpha_{\tilde{k}} \chi_{\tilde{k}}^{out} + \beta_{\tilde{k}} \chi_{\tilde{k}}^{out*}$$

with

$$\alpha_{\tilde{k}} = -\frac{i\pi}{4} e^{i\tilde{k}} [\frac{d}{d\eta} M^{(3)}_{-i\tilde{k}}(\eta) M^{(3)}_{-i\tilde{k}}(\eta)]_{\eta=0}$$

$$\beta_{\tilde{k}} = -\frac{i\pi}{4} [\frac{d}{d\eta} M^{(3)}_{-i\tilde{k}}(\eta) M^{(3)}_{-i\tilde{k}}(\eta)]_{\eta=0}$$

(In the derivation we made use of the property (A.11) of the Mathieu function.) Since $\chi^{in}_{\tilde{k}}$ is normalised we must have

$$|\alpha_{\tilde{k}}|^2 - |\beta_{\tilde{k}}|^2 = \frac{\pi^2}{16} \left\{ e^{2\pi\tilde{k}} \left| \left[\frac{d}{d\eta} M^{(3)}_{-i\tilde{k}} M^{(3)}_{-i\tilde{k}}\right]_{\eta=0} \right|^2 - \right.$$

$$\left. - \left| \left[\frac{d}{d\eta} M^{(3)}_{-i\tilde{k}} M^{(4)}_{-i\tilde{k}}\right]_{\eta=0} \right|^2 \right\} = 1 \quad ,$$

a relationship which, to our knowledge, does not appear in the literature about the Mathieu functions.

The Bogoliubov coefficient introduced in (2.6) is therefore:

$$\beta_{\tilde{p},\tilde{h}} = \delta(\tilde{p},\tilde{k})(-\frac{i\pi}{4}) \left[\frac{d}{d\eta} M^{(3)}_{-i\tilde{k}}(\eta)\; M^{(4)}_{-i\tilde{k}}(\eta)\right]_{\eta=0} \qquad (4.13)$$

This formula,as such, is not particularly transparent... In the limiting case however when $h \equiv ma_0/2$ is small, which is the case of physical interest when the Compton wavelength 1/m of the created quanta is large compared to the minimum value a_0 of the scale factor of the universe, the Mathieu functions reduce to the Bessel functions (see eq. (A.15) of the Appendix). In particular

$$M^{(3)\,(4)}_{\tilde{\nu}}(\eta,h) \xrightarrow{h\to 0} H^{(1)\,(2)}_{\tilde{\nu}}(2h\cosh\eta)$$

In this approximation then, the exact Bogoliubov coefficient (4.13) reduces to

$$\beta_{\vec{p},\vec{k}} = \delta(\vec{p},\vec{k})\left(-\frac{i\pi}{4}\right)\left[\frac{d}{dy}\, H^{(1)}_{-i\tilde{k}}\; H^{(2)}_{-i\tilde{k}}\;\right]_{y=2h} \qquad (4.14)$$

This is a much more tractable formula that we shall recover and comment upon at the end of the next section.

Another interesting question one may consider in this context is to define your positive frequency modes and an associated O-vacuum near the moment of maximum curvature ($\eta = 0$) and ask for its particle contents as compared to the out-vacuum. For this we consider the normalized solution of eq. (4.4)

$$\chi^{0}_{\tilde{k}}(\eta) = \sqrt{\frac{\pi}{2\sinh\pi\tilde{k}}}\, M^{(1)}_{-ik}\;(\eta,k) \qquad (4.15)$$

With help of eq. (A.14) and (A.9) of Appendix A we obtain its asymptotic behavior near the origin.

$$\chi^{0}_{\tilde{k}}(\eta) \xrightarrow[t\to 0]{\eta\to 0} \text{const. } e^{-i\tilde{k}\eta} \quad . \qquad (4.16)$$

This confirms that they are correctly said to be positive frequency. Interestingly enough we remark that these modes have the same asymptotic behaviour as the Milne modes (3.5) but at $\eta \to +\infty$

$$\chi^{0}_{\tilde{k}}(\eta) \xrightarrow[t\to\infty]{\eta\to\infty} \sqrt{\frac{\pi}{2\sinh\pi\tilde{k}}}\; J_{-i\tilde{k}}\;(mt) \quad . \qquad (4.17)$$

To compute the number of $\tilde{\nu}$ O-quanta in the out-

-vacuum we need the Bogoliubov coefficient of eq. (2.6). As previous remarked they reduce to

$$\alpha_{\tilde{k}} = \left(\sqrt{\frac{\pi}{2\sinh\pi\tilde{k}}}\, M^{(1)}_{-ik}(\eta)\right) \overleftrightarrow{\frac{d}{d\eta}} \left(\frac{\sqrt{\pi}}{2} e^{-\pi\tilde{k}/2}\, M^{(4)}_{-i\tilde{k}}(\eta)\right)^*$$

$$\beta_{\tilde{k}} = -\left(\sqrt{\frac{\pi}{2\sinh\pi k}}\, M^{(1)}_{-i\tilde{k}}(\eta)\right) \overleftrightarrow{\frac{d}{d\eta}} \left(\frac{\sqrt{\pi}}{2}\, e^{-\pi\tilde{k}/2}\, M^{(4)}_{-i\tilde{k}}(\eta)\right)$$

With the help of eq. (A.12) of the Appendix A this is immediately seen to be:

$$\alpha_{\tilde{k}} = \frac{1}{\sqrt{1-e^{-2\pi\tilde{k}}}}$$

$$\beta_{\tilde{k}} = \frac{1}{\sqrt{e^{2\pi\tilde{k}}-1}}$$

We then easily compute the number of $\tilde{k}$ O-quanta in the out-vacuum:

$$\frac{N^{out}_{\tilde{k}}}{d\Sigma} = \frac{1}{(2\pi)^3}\left(\frac{1}{e^{2\pi\tilde{k}}-1}\right) \quad . \tag{4.18}$$

As before we interpret this by saying that the out-vacuum has a Planck spectrum for a Bose gas at the temperature

$$T_0 = \frac{1}{2\pi k_B}$$

or, for observers using the the cosmic time,

$$T = \frac{1}{2\pi k_B\sqrt{a_0{}^2+t^2}} \tag{4.19}$$

The quanta energy density, we remember, is given by

$$\tilde{k} = \sqrt{k^2 + \frac{m^2 a_0{}^2}{2}} + O(m^4 a_0{}^4) \quad . \tag{4.20}$$

5. Creation of matter in an eternal universe: a simplified model

In the preceeding section we had to solve the Mathieu equation

$$\frac{d^2\chi}{d\eta^2} - (\lambda - 2h^2\cosh 2\eta)\chi = 0 \tag{5.1}$$

with $\lambda \equiv -(k^2+m^2a_0{}^2/2)$, $h \equiv ma_0/2$ and where η is related to a "cosmic time" by:

$$t = \sinh\eta \tag{5.2}$$

In this section we shall approximate $\cosh 2\eta$ by $e^{2\eta/2}$ for $\eta \geq 0$ and by $e^{-2\eta/2}$ for $\eta \leq 0$ and hence consider the system:

$$\begin{cases} \dfrac{d^2\chi}{d\eta^2} - (\lambda - h^2e^{-2\eta})\chi = 0 \qquad \eta \leq 0 & (5.3) \\ \dfrac{d^2\chi}{d\eta^2} - (\lambda - h^2e^{2\eta})\chi = 0 \qquad \eta \geq 0 & (5.4) \end{cases}$$

An approximate solution of (5.1) is a solution of (5.3) together with a solution of (5.4) which have same value and derivative at $\eta = 0$.

Setting $x = -he^{-\eta}$ in (5.3) brings it into the standard form of the Bessel equation (see e.g. Abramowitz & Stegun 1965 or Ryshik & Gradstein 1963):

$$\frac{d^2\chi}{dx^2} + \frac{1}{x}\frac{d\chi}{dx} + \chi(1 - \frac{\overline{\nu}^2}{x^2}) = 0 \quad , \quad x \in]-\infty, 0[\qquad (5.5)$$

with $\overline{\nu} = -i\overline{k}$, $\overline{k} \equiv \sqrt{k^2+m^2 a_0{}^2/2}$, $\lambda = \overline{k}^2$. One solution of (5.5) is the Hankel function of the first kind $H_\nu^{(1)}(-x)$, the asymptotic behaviour of which, when $x \to -\infty$, is:

$$H_\nu^{(1)}(-t) \xrightarrow{x\to-\infty} \sqrt{\frac{-2}{\pi x}}\exp\left[-i(x + \frac{\pi\overline{\nu}}{2} + \frac{\pi}{4})\right]$$

which behaves like a positive frequency wave with respect to the cosmic time t (5.2) as t or η tends to $-\infty$. We shall hence denote

$$\chi_{\overline{k}}^{in}(\eta) = \frac{1}{2}\sqrt{\pi}\quad e^{\frac{\pi\overline{k}}{2}}\, H_{-i\overline{k}}^{(1)}\,(he^{-\eta}) \qquad (5.6)$$

We remark the similarity between (5.6) and (3.5), and (4.6). The in-vacuum constructed from (5.6) is then the analogue to the Milne in-vacuum (equivalent as we have seen to a Minkowskian vacuum) of the Milne universe which is isomorphic to the past cone of the origin of the Minkowskian coordinate system. The modes (5.6) and their complex conjugates form a basis of the solutions of (5.3) for $\eta \le 0$. We note that when $z \in \mathbb{R}$ and ν is pure imaginary, $[H_\nu^{(1)}(z)]^* = H_{-\nu}^{(2)}(z)$ where $H_\nu^{(2)}(z)$ is the Hankel function of the second kind.

Setting now $z = he^\eta$ in (5.4) also brings it into the standard form of the Bessel equation (5.5) where x is replaced by z, $z \in]0, +\infty[$. Another solution of (5.5) is $H_\nu^{(2)}(z)$ the asymptotic behaviour of which is, when $z \to +\infty$:

$$H_{\nu}^{(2)}(z) \xrightarrow{z\to+\infty} \sqrt{\frac{2}{\pi z}}\ \exp[-i(z - \frac{\pi\bar{\nu}}{2} - \frac{\pi}{4})]$$

which behaves like a positive frequency wave with respect to the cosmic time t (5.2) as t or η tend to $+\infty$. We shall hence denote:

$$\chi_{\bar{k}}^{out}(\eta) = \frac{1}{2}\sqrt{\pi}\ e^{-\pi\bar{k}/2}\ H_{-i\bar{k}}^{(2)}(he^{\eta}) \qquad (5.7)$$

and remark the similarity between (5.7) and (3.6). The out-vacuum constructed from (5.7) is then the analogue of the Milne out-vacuum (equivalent to a Minkowskian vacuum) of the Milne universe isomorphic to the future cone of the origin of the Minkowskian coordinate system. The mode (5.7) together with its complex conjugate are two independent and normalised solution of (5.4) for $\eta \geq 0$.

The connection between the modes (3.5) and (3.6) in the two distinct Milne universes, respectively isomorphic to the past and future cone of the origin of the Minkowski coordinate system, has to be done through a delicate procedure of analytic continuation through the origin of the coordinate system. In section 4 the connection between the modes (4.6) and (4.11) defining the in-vacuum in the distant past and the out-vacuum in the distant future required a careful analysis of the properties of the Mathieu functions. In this section on the other hand the connection is more simply done. Indeed the approximate solution of (5.1) which behaves like a positive frequency wave when $\eta \to -\infty$ is $\chi^{in}(\eta)$ for $\eta \leq 0$ (eq. (5.6)) and is of the form $\alpha\chi^{out} + \beta\chi^{out*}$ for $\eta \geq 0$. The coefficient α and β are determined by the continuity conditions of the solution and its derivative at $\eta = 0$ that is at $x = -h$ and $z = +h$. They read:

$$
\begin{cases}
e^{\pi\bar{k}/2}\, H^{(1)}_{-i\bar{k}}(h) = \alpha e^{-\pi\bar{k}/2}\, H^{(2)}_{-i\bar{k}}(h) + \beta e^{-\pi\bar{k}/2}\, H^{(1)}_{i\bar{k}}(h) \\
-e^{\pi\bar{k}/2} \dfrac{dH^{(1)}_{-ik}(y)}{dy}\Big|_{y=h} = \alpha e^{-\pi\bar{k}/2} \dfrac{dH^{(2)}_{-i\bar{k}}(y)}{dy}\Big|_{y=h} + \beta e^{-\pi\bar{k}/2} \dfrac{dH^{(1)}_{i\bar{k}}(y)}{dy}\Big|_{y=h}
\end{cases}
$$

which yields

$$
\begin{cases}
\alpha = \dfrac{e^{\pi\bar{k}}}{\langle H^{(2)}_{-i\bar{k}}, H^{(1)}_{i\bar{k}} \rangle_{y=h}} \dfrac{dH^{(1)}_{-ik}\, H^{(1)}_{ik}}{dy}\Big|_{y=h} \\
\beta = - \dfrac{e^{\bar{k}}}{\langle H^{(2)}_{-ik}, H^{(1)}_{i\bar{k}} \rangle_{y=h}} \dfrac{dH^{(2)}_{-ik}\, H^{(1)}_{-ik}}{dy}\Big|_{y=h}
\end{cases}
\qquad (5.8)
$$

where $y \langle H^{(2)}_{\nu}, H^{(1)}_{-\nu} \rangle \equiv H^{(2)}_{\nu}\ y \frac{d}{dy} H^{(1)}_{-\nu} - H^{(1)}_{-\nu}\ y \frac{d}{dy} H^{(2)}_{\nu} = \frac{4i}{\pi} e^{i\pi\nu}$ denotes the Wronskian of $H^{(2)}_{\nu}$ and $H^{(1)}_{-\nu}$.

In the limit when h is large one finds, using the asymptotic expansions of the Hankel functions for large argument:

$$
\beta \to 0 \quad , \quad \alpha \to e^{2i(h-\pi/4)} \qquad (5.8')
$$

so that $|\beta|^2 \to 0$, $|\alpha|^2 \to 1$. In this limit then, the Bogoliubov coefficient β is zero, which means that there are no quanta created in the in-vacuum during the course of the evolution. This result may perhaps be understood by noting that where h is large the coupling between the quantum field and the geometry is weak since the compton wavelength of the quantum particle, 1/m, is small in comparison with a_0, the characteristic length over which the curvature of space time changes appreciably.

In the limit now when h is small (the case of strong coupling), one find, using the properties of the Hankel and Bessel functions (see e.g. the Appendix),

$$\begin{cases} \alpha \to \dfrac{\bar{k}\pi}{2\sinh^2\pi k} \left[e^{\pi\bar{k}} \dfrac{(h/2)^{2i\bar{k}}}{\Gamma^2(1+i\bar{k})} - e^{-\pi\bar{k}} \dfrac{(h/2)^{-2i\bar{k}}}{\Gamma^2(1-i\bar{k})} \right] \\ \\ \beta \to \dfrac{\bar{k}}{2\sinh^2\pi\bar{k}} \left[\dfrac{(h/2)^{2i\bar{k}}}{\Gamma^2(1+i\bar{k})} - \dfrac{(h/2)^{-2i\bar{k}}}{\Gamma^2(1-i\bar{k})} \right] \end{cases}$$

which yields, using the properties of the Γ functions (see Appendix):

$$\begin{cases} |\beta|^2 = \dfrac{1}{2\sinh^2\pi\bar{k}} \; [1-\cos(4k\ln h/2 + \phi) \\ |\alpha|^2 = \dfrac{1}{2\sinh^2\pi\bar{k}} \; [\cosh 2\pi\bar{k} - \cos(4\bar{k}\ln h/2+\phi)] \end{cases} \tag{5.9}$$

where ϕ is an irrelevant phase independent of h. (One checks that $|\alpha|^2 - |\beta|^2 = 1$). Hence one sees that the number of out-quanta in the in-vacuum depends, in this limit, on the value of the parameter h and can vary from $|\beta|^2 = 0$ to $|\beta|^2 = \sinh^2\pi\bar{k} \to 4e^{-2\pi\bar{k}}$ for large wave number k. The dependence of $|\beta|^2$ on h is however fairly weak since a variation $h \to h+\alpha$ on resonance (when $|\beta|^2$ is maximum) induces the change $|\beta|^2 = = \sinh^{-2}\pi\bar{h} \to \sinh^{-2}\pi k(1-\varepsilon)$ with $\varepsilon = 8\bar{k}^2\alpha^2$. If this effect is not an artefact due to some oversimplification of the model it would be to our knowledge, the first example where a link between the maximum curvature of the universe, that is its "minimum radius" or in other words the Planck length or mass, and the mass of the observed, standard particles, is knited at: only particles in "resonance" with the curvature of space time at the moment of maximum contraction would be created.

Finally, when $m = 0$ strictly, the Bogoliubov coefficients cannot be obtained by taking the limit $h \to 0$ in (5.9). In this case however the equations (5.3) and (5.4) are identical and trivially solved in terms of exponentials. In this case then there are no out-quanta in the in-vacuum.

6. Conclusion

We have analysed the main properties of a real scalar quantum field in a homogeneous and isotropic universe whose non-singular metric is given by eq. (1.1).It represents a world in wich cosmic evolution begins at $t = -\infty$ in an asymptotic flat phase; goes through a phase of contraction; reaches a point of maximum curvature at $t = 0$; and finally expands untill another asymptotic flat region at $t = +\infty$. This geometry is not an artificial model designed to test quantum field theory in curved space but instead has the good properties needed to be a candidate to represent the geometry of our Universe. This metric has many interesting properties. Among those, the Klein-Gordon equation is exactaly soluble. One can then use the methods of quantum field theory in order to evaluate the rate of particle production in this Universe. This happens to be given by a very simple and fashionable expression. Also the feature of asymptotic ($t = \pm\infty$) flatness allow us some insights in the quest of disentangling coordinate, observer-dependent and curvature effects.

It seems worth to remark that this geometry reproduces the main properties of our expanding phase, without producing the difficulties of the standard model, by the avoidance of a singular origin. This geometry can be obtained from a quantum scalar field (Melnikov & Orlov), in wich case the minimum allowable radius is, as should be expect, of the order of the Planck length. In this case it should contains almost all properties of a classical Big-Bang. From the classical scheme, the minimum radius is not related to Planck length , it becomes a nonpredictable finite value and depends only on the minimals strength of the classical field (Novello & Salim). Defining a vacuum $|0\rangle^{0}$ in the phase of maximum contraction it follows, due to the cosmic expansion that this vacuum will be observed at $t = +\infty$ as a boson gas in thermal equilibrium, whose temperature is given in terms of the geometry as

$$T = \frac{1}{2\, k_B \sqrt{a_0{}^2 + t^2}} \quad , \tag{4.19}$$

and whose energy density is given by the order $\tilde{k}$ of the Mathieu function, wich to first order in ma_0 is:

$$\tilde{k} = \sqrt{k^2 + m^2 a_0{}^2/2} + O(m^2 a_0{}^2) \tag{4.20}$$

It seems worth to point out that the general form of $\tilde{k}$ contains all details needed to characterize completely the problem.

We may also compare observables defined in the two asymptotic flat phases, when the extension of well known concepts of quantum field theory is straight forward. We have shown that the gravitational interaction creates from the vacuum $|0\rangle^{in}$(chosen in the distant past) a non-zero number of particles recognized as such in the distant future. The problem is mathematically equivalent to scattering in a one dimensional potential barrier. This can help us to understand the behavior of $N^{out}_{\tilde{k}}$: For high frequencies the problem can be treated as in a classical domain, in which the reflection coeficient tends to zero (eq. (5.8')); in the other hand the quantum domain behavior is quite distinct: one finds resonance effects between the two lenghts $1/k$ and a_0 wich is responsible for a sort of mass quantization of the created particles. This fact is, as far as we know, new in the literature. This result was confirmed by two approximative methods (Section 5 and Appendix B) and enables one to circunvent the problems of coordinate dependence and interpretation of Quantum Field Theory in Milne universe.

Finally we should like to comment that some open questions still remain to be treated. To understand completely what is going on, it is not enough to know the value of N. There remains to calculate not only the mean-value of the energy-momentum-tensor but also the propagators of the field. The above quoted analogy with the scattering problem suggest us an inverse problem: given the asymptotic conditions to the solutions of eq. (2.4) and the spectrum $N_{\tilde{p}}^{out}$, whath is the form of the conformal factor $C(\eta)$? One could eventually try to generalize the powerfull technics of periodic potentials of solid state physics to the case of potentials with imaginary periodicity. This seems a good perspective to the near future.

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Appendix A: Mathieu and Bessel functions: a formulary

Our references are Meixner & Shåfke 1954 (M.S.) and Abramowitz & Stegun 1964 (A S)

Consider the Mathieu equation

$$d^2y/dz^2 + (\lambda-2h^2\cos 2z)y = 0 \quad \text{(MS p 98)} \tag{A.1}$$

This equation has two essential singularities at $z = \pm\infty$. Its solution are then analytical functions of λ, h^2, $z \in \mathbb{C}$ up to these points.From a theorem by Floquet the solutions must behave like:

$$y(z+\pi) = e^{i\pi\nu}y(z) \qquad \text{(MS p 101)} \tag{A.2}$$

due to the periodicity of the cosine term. If $\nu \in \mathbb{R}$ the solution are then bounded; otherwise they are unstable. The parameter ν is an involved function of λ and h^2 and the conditions on λ and h^2 leading to unstabilities have been the focus of most of the work on Mathieu functions.

One solution of (A.1), $me_\nu(z;h^2)$, is obtained as a Fourier series:

$$me_\nu(z,h^2) = \sum_{j=-\infty}^{+\infty} c_{2j}^{\nu}(h^2)e^{i(\nu+2j)z} \qquad \text{(M S p 111)} \tag{A.3}$$

Substituting this series in the Mathieu equation (A.1) yields recurrence relations for $c_{2j}^{\nu}(h^2)$ which read, when expanded in series of h^2

$$\frac{c_{2j}^{\nu}}{c_0^{\nu}} = (-1)^j \frac{\Gamma(1+\nu)}{2^{2j} j! \Gamma(1+\nu+j)} h^{2j} + O(h^{2j+4}) \qquad \text{(M S p 121)} \quad \text{(A.4)}$$

Similarly we have

$$\lambda_{\nu}(h^2) = \nu^2 + \frac{1}{2(\nu^2-1)} h^4 + O(h^8) \qquad \text{(M S p. 119)} \qquad \text{(A.5)}$$

so that:

$$me_{\nu}(z;h^2) = e^{i\nu z} - h^2 \left(\frac{1}{4(\nu+1)} e^{i(\nu+z)z} - \frac{1}{4(\nu-1)} e^{i(\nu-z)z}\right) + O(h^4) \qquad \text{(M S p. 122)} \quad \text{(A.6)}$$

The functions $me_{\nu}(z)$ and $me_{-\nu}(z)$ $(\nu \notin Z)$ form a set of independent solutions of (A.1) (M S p 102).

The functions $me_{\nu}(z,h^2)$ have the following properties $(\nu \notin Z)$:

$$me_{-\nu}(z) = me_{\nu}(-z) \qquad \text{(M S p. 102)} \qquad \text{(A.7)}$$

Consider now the modified Mathieu equation:

$$d^2Y/dz^2 - (\lambda - 2h^2 \cosh 2z)Y = 0 \quad \text{(M S p. 130)} \quad \text{(A.8)}$$

which has the following solution:

$$\begin{aligned} Me_{\nu}(z,h^2) &= me_{\nu}(-iz,h^2) \\ &= \sum_{r=-\infty}^{+\infty} c_{2r}^{\nu}(h^2)\, e^{(\nu+2r)z} \quad \text{(M S p 130)} \end{aligned} \qquad \text{(A.9)}$$

The properties of $Me_{\nu}(z,h^2)$ hence follow from (A.3-7).

We will also be interested in the solutions $M_{\nu}^{(j)}$ of

(A.8) the asymptotic behaviour of which is imposed to be:

$$\lim_{Rez\to+\infty} M_\nu^{(j)}(z;h) = Z_\nu^{(j)}(2h\cosh z) \quad \text{(M S p. 166)} \quad \text{(A.10)}$$

where $Z_\nu^{(j)}$ stands for the Bessel (or cylinder) functions: $Z_\nu^{(1)}$ is the Bessel function (of the first kind) $J_\nu(z)$; $Z_\nu^{(2)}$ is the Bessel function of the second kind $N_\nu(z)$ (or $Y_\nu(z)$ also known as the Neumann or Weber function); $Z_\nu^{(3)}$ and $Z_\nu^{(4)}$ are the Hankel functions of the first and second kind $H_\nu^{(1)}$ and $H_\nu^{(2)}$ respectively.

This functions have the additional interesting properties, $\nu \notin Z$:

$$\begin{cases} M_\nu^{(3)}(z,h) = M_\nu^{(1)}(z,h) + i\, M_\nu^{(2)}(z,h) \\ M_\nu^{(4)}(z,h) = M_\nu^{(1)}(z,h) - i\, M_\nu^{(2)}(z,h) \end{cases}$$

$$\begin{cases} M_{-\nu}^{(3)}(z,h) = e^{i\pi\nu}\, M_\nu^{(3)}(z,h) \\ M_{-\nu}^{(4)}(z,h) = e^{-i\pi\nu}\, M_\nu^{(4)}(z,h) \end{cases} \quad \text{(A.11)}$$

$$\begin{cases} i\sin\nu\pi M_\nu^{(3)}(z,h) = M_{-\nu}^{(1)}(z,h) - e^{-i\pi\nu}\, M_\nu^{(1)}(z,h) \\ -i\sin\nu\pi M_\nu^{(4)}(z,h) = M_{-\nu}^{(1)}(z,h) - \cos\nu\pi\, M_\nu^{(1)}(z,h) \end{cases}$$

(M S p. 169).

The Wronskians of two $M^{(j)}$ are the same as the Wronskians of the associated $Z^{(j)}$. Hence, denoting:

$$[j,k] \equiv M_\nu^{(j)} \overleftrightarrow{\frac{d}{dz}} M_\nu^{(k)} \equiv M_\nu^{(j)} \frac{d}{dz} M_\nu^{(k)} - M_\nu^{(k)} \frac{d}{dz} M_\nu^{(j)} ,$$

we have:

$$[3,4] = -4i/\pi \ ; \ [1,3] = -[1,4] = 2i/\pi \ ; \qquad (A.12)$$
$$[1,2] = -[2,3] = -[2,4] = 2/\pi$$

[(M S p 171) and (A·S p 360)], and:

$$[jk]M_\nu^{(j)}(-z) = \left[\frac{d}{dz} M_\nu^{(k)} M_\nu^{(j)}\right]_{z=0} M_\nu^{(j)}(z) - \left[\frac{d}{dz} M_\nu^{(j)} M_\nu^{(j)}\right]_{z=0} M_\nu^{(k)}(z) \qquad (A.13)$$

(M S p 171).

Finally we note that the functions $M_\nu^{(j)}$ and Me_ν are related by:

$$M_\nu^{(1)}(z;h) = \left[\frac{M_\nu^{(1)}(0,h)}{me_\nu(0,h^2)}\right] Me_\nu(z;h^2) \qquad (A.14)$$

(M S p 181) and that

$$M_\nu^{(j)}(z;h) \xrightarrow{h\to 0} Z_\nu^{(j)}(2h\cosh z) \qquad (A.15)$$

for all Rez > 0 (M S p 171).

To conclude let us file a number of basic properties of the Bessel functions (A S p 358 et seq.):

$$\begin{cases} H_\nu^{(1)}(z) = \dfrac{i}{\sin\nu\pi}\,[e^{-\nu\pi i}\, J_\nu(z) - J_{-\nu}(z)] \\ H_\nu^{(2)}(z) = \dfrac{i}{\sin\nu\pi}\,[J_{-\nu}(z) - e^{\nu\pi i} J_\nu(z)] \end{cases} \qquad (A.16)$$

$$H_{-\nu}^{(1)}(z) = e^{\nu\pi i}\, H_\nu^{(1)}(z) \quad ; \quad H_{-\nu}^{(2)}(z) = e^{-\nu\pi i}\, H_\nu^{(2)}(z) \qquad (A.17)$$

$$J_\nu(z) \xrightarrow{z\to 0} (\tfrac{1}{2}z)^\nu / \Gamma_{(\nu+1)} \qquad (A.18)$$

$$J_\nu(z) \overleftrightarrow{\frac{zd}{dz}} J_{-\nu}(z) = -\frac{2\sin\nu\pi}{\pi} \qquad (A.19)$$

$$\begin{cases} J_\nu(z) \xrightarrow{|z|\to\infty} \sqrt{2/\pi z}\ \cos(z - \frac{1}{2}\nu\pi - \frac{1}{4}\pi) \\ H_\nu^{(1)} \xrightarrow{|z|\to\infty} \sqrt{2/\pi z}\ e^{i(z - \frac{1}{2}\nu\pi - \frac{1}{4}\pi)} \\ H_\nu^{(2)}(z) \xrightarrow{|z|\to\infty} \sqrt{2/\pi z}\ e^{-i(z - \frac{1}{2}\nu\pi - \frac{1}{4}\pi)} \end{cases} \qquad (A.20)$$

if $z \in \mathbb{R}$,

$$[J_\nu(z)]^* = J_{\nu^*}(z) \qquad \text{(from A S p 360)}$$
$$[H_\nu^{(1)}(z)]^* = H_{\nu^*}^{(2)}(z) \qquad (A.21)$$

Analytic continuation: (A S p 361):

$$H_\nu^{(1)}(Ze^{\pi i}) = -e^{-\nu\pi i}\ H_\nu^{(2)}(z)$$
$$H_\nu^{(2)}(Ze^{-\pi i}) = -e^{\nu\pi i}\ H_\nu^{(1)}(z) \qquad (A.22)$$

We also used in the calculations the following properties of the Γ functions:

$$\Gamma(1-ik)\Gamma(1+ik) = \frac{k\pi}{\sinh k\pi} \qquad \text{(A S p 256)} \qquad (A.23)$$

$$\Gamma(1+iy) = ig\ \Gamma(ig) \qquad \text{(A S p 256)} \qquad (A.24)$$

$$R \ln\Gamma(iy) = \frac{1}{2}\ln\frac{\pi}{y\sinh\pi y} \qquad \text{(A S p 257)} \qquad (A.25)$$

Appendix B: A WKB approximate solution

Consider the equation

$$d^2\chi/d\eta^2 - W(\eta)\chi = 0 \quad , \quad \eta \in \mathbb{R} \quad , \tag{B.1}$$

where $W(\eta)$ is a real, C^∞ and negative function of η so that the solutions of (B.1) have everywhere an oscillatory behaviour. Consider the solution which represents a sum of an incident and reflected wave at $\eta \to -\infty$ and a purely transmitted wave at $\eta \to +\infty$. The problem is to obtain an approximate expression of the transmission coefficient T, that is the square of the ratio of the moduli of the transmitted and incident waves.

When $W(\eta)$ is everywhere negative the standard WKBJ iteration method (see e.g. Morse & Feschbach 1953, Bender & Orszag 1978 or Landau & Lifchitz 1967) consists in setting $\chi = Ae^{is}$, A and S real, so that (B.1) is equivalent (up to an integration constant) to ,

$$A^4 = (-W + A''/A)^{-1} \quad , \quad S' = A^{-2}$$

which, as long as $A''/A < -W$ can be rewritten as:

$$A^2 = \pm (-W+A''/A)^{-1/2} \quad ; \; S = \pm \int^{\eta} (-W+A''/A)^{1/2} \, d\eta' \tag{B.2}$$

The iteration method consists in neglecting at lowest order A''/A in (B.2), obtaining $A \sim A_0 = (-W)^{-1/4}$, computing $(A''/A)_0$, using this expression to obtain A_1, and iterating. As long as $(A''/A)_n < -W$ the normalised nth order solution of (B.1) corresponding to a purely transmitted wave then is

$$\chi^{(n)}(\eta) = [-W+(A''/A)_{n-1}]^{-1/4} \exp\left[+i\int^{\eta} [-W+(A''/A)_{n-1}]^{1/2} d\eta'\right]$$

which is a good approximation to χ in the short wavelength limit. Hence in this approximation scheme, the transmission coefficient is always unity. This result is at the root of many of the features of the "adiabatic vacuum" introduced when studying quantum fields in curved space times (see e.g. Birrell & Davies 1984).

In order to obtain a better approximation for T another method must therefore be designed.

To this end we note that if $W(\eta)$ were positive in some region (for, say, $\alpha \leq \eta \leq \beta$), the WKBJ method would give for the transmission coefficient through this potential barrier:

$$T \cong e^{-2\zeta} \quad , \quad \zeta = \int_{\alpha}^{\beta} \sqrt{W}\, d\eta \tag{B.3}$$

(see e.g. Morse & Feschach 1953, Landau & Lifschitz 1967 and Bender & Orzag 1978) that the WKBJ method can be extended to the complex η-plane. Consider first, as an example, the potential:

$$W(\eta) = -(a^2 + b^2\eta^2) \tag{B.4}$$

which is encountered when dealing with a quantum field interacting with a Robertson-Walker geometry (see e.g. Birrell & Davis 1984 p 70) or with a constant electric field in Minkowski spacetime (see e.g. Damour 1977). If we allow η to be complex, $W(\eta)$ has two zeros on the imaginary axis: $\eta = \pm i(a/b)$. A meaning can therefore be given to (B.3) by changing the sign of W and rotating the integration axis by $\pi/2$ that is by setting $\eta = iz$, $z \in \mathbb{R}$. Then:

$$\zeta = \int_{-a/b}^{a/b} dz\sqrt{a^2-b^2z^2} = \pi a^2/2b \tag{B.5}$$

and hence $T = \mathrm{enp}[-\pi a^2/b]$. Now the exact solution of (B.1), with W given by (B.4), in terms of parabolic cylinder functions (see e.g. Birrell & Davies 1984 or Damour 1977) yields for the transmission coefficient of an incident wave exactly the same results. (The fact that (B.5) happens to be the exact rather than some approximate result is commented upon in Damour 1977.)

Let us now apply the same procedure to the case when

$$W(\eta) = \lambda - 2h^2 \cosh 2\eta$$

with $\lambda = -(k^2+m^2a_0{}^2)$ and $h = ma_0/2$. The zeros of $W(\eta)$ are: $\eta\pm = i\frac{r}{2} \pm \alpha$ $\cosh \alpha = \frac{1}{2}\,\mathrm{argCh}(1+2k^2/m^2a_0{}^2)$. To compute (B.3) we therefore change the sign of W, set $\eta = ir/2+z$ and integrate of z, $z \in \mathbb{R}$. Then:

$$\zeta = \int_{-\alpha}^{+\alpha} \sqrt{-\lambda-2h^2\cosh 2z}\; dZ \tag{B.6}$$

In the limit when $k^2 << m^2a_0{}^2/2$, $\cosh 2z \sim 1+2z^2$ in the integrand of (B.6) which then reduces to:

$$\zeta \sim \pi k^2/4h$$

so that $$T \sim e^{-rk^2/2h} \quad .$$

In the other limit when $k^2 >> m^2a_0{}^2/2$ we can set $2\cosh 2z \sim e^{2z}$ for $z > 0$ in the integrand of (B.6) which is then approximated by:

$$\zeta \sim 2\int_0^{\ln(-\lambda/h^2)} \sqrt{-\lambda-h^2e^{2z}}\; dz \quad .$$

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