Feynman path integral representations for the harmonic oscillator with stochastic frequency

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We propose a Feynman path-integral solution for harmonic oscilator motions with stochastic frequency.

The problem of the (random) motion of a harmonic oscilator in the presence of a stochastic time-dependent perturbation on its frequency is of great theoretical and pratical importance ([1], [2]). In this Brief report we propose a formal path integral solution for the above mentioned problem by following closely our previous studies ([3], [4]). In first section we write a Feynman path integral representation for the external forcing problem. In section two we consider similar problem for the initial condition case.

1. The Green function for external forcing

Let us start our analysis by considering the motion equation of a harmonic oscilator subject to damping and an external forcing

$$\left\{\frac{d^2}{dt^2}+\nu\ \frac{d}{dt}+w_0^2(1+g(t))\right\}x(t)=F(t) \tag{1}$$

Here $w_0^2(1+g(t))$ is the time dependent frequency with stochastic part given by the random function $g(t)$ obeying the gaussian statistics

$$\langle g(t)g(t')\rangle = K(t,t') \tag{2}$$

The solution of eq. (1) is, thus, given by

$$x(t,[g])=\int_0^t G(t,t',[g])F(t')dt' \tag{3}$$

where $G(t,t',[g])$ denotes the problem Green functionally depending on the random frenquency $g(t)$ and the notation emphasise that the objects under study are functionals of the random part $g(t)$ of the harmonic oscilator frequency.

In order to write a path-integral representation for the Green function eq. (3) we follow our previous study ([3]) by using a "proper-time" techique by introducing a related Schrödinger wave equations with an initial point-source and $-\infty \le t \le +\infty$.

$$i\ \frac{\partial\bar{G}(s;(t,t'))}{\partial s}=-\left[\frac{d^2}{dt^2}+\nu\ \frac{d}{dt}+w_0^2(1+g(t))\right]\bar{G}(s;(t,t')) \tag{4}$$

$$\lim_{s\to 0}\bar{G}(s;(t,t'))=\delta(t-t') \tag{5}$$

$$\lim_{s\to\infty}\bar{G}(s;(t,t'))=0 \tag{6}$$

At this point we remark the following indetity between the Schrödinger equation (4), (5), (6) and the searched harmonic oscilator Green function

$$G\left[(t,t',[g])=-i\int_0^\infty ds\bar{G}\left(s;(t,t')\right)\right. \tag{7}$$

Let us, thus, write a path integral for the associated Schrödinger equation (4), (5), (5) and (6) by considering $\bar{G}(s;(t,t'))$ in the operator form (the Feynman Dirac propagator)

$$\bar{G}\left(s;(t,t')\right) = \langle t|\exp(isH)|t'\rangle \tag{8}$$

where H is the differential operator below

$$H = -\left\{\frac{d^2}{dt^2} + \nu\ \frac{d}{dt} + w_0^2(1+g(t))\right\} \tag{9}$$

As in quantum mechanics we write eq. (8) as an infinite product of short-time s propagations

$$\langle t|\exp(isH)|t'\rangle = \lim_{N\to\infty}\prod_{i=1}^{N}\int_{-\infty}^{+\infty} dt_i\langle t_i|\exp i\left(\frac{s}{N}\ H\right)|t_{i-1}\rangle \tag{10}$$

The standard shor-time expansion in the s-parameter for eq. (10) is given by

$$\begin{aligned}
&\lim_{s\to 0^+}\langle t_i|e^{isH}|t_{i-1}\rangle \\
&\qquad = \lim_{s\to 0^+}\int dw_i \exp\left\{is\left[w_i^2 - i\nu w_i + w_0^2\left(1+g^2(t_{i-1})\right)\right]\right\} \\
&\qquad \times\ \exp\left[iw_i(t_i - t_{i-1})\right]
\end{aligned} \tag{11}$$

If we substitute eq. (11) into eq. (10) and take the Feynman limit of $N\to\infty$, we will obtain the following path-integral representation after evaluating the w_i-Gaussian integrals of the representation eq. (8)

$$\begin{aligned}
\bar{G}\left(s;(t,t')\right) &= \int\left(\prod_{\substack{0<\sigma<s \\ t(0)=t';t(s)=t}} dt(\sigma)\right)\times\left(e^{-s\cdot\nu^2}\right) \\
&\quad \exp\left\{\frac{i}{2}\int_0^2 d\sigma\left[\left(\frac{dt(\sigma)}{d\sigma}\right)^2 + \nu\left(\frac{dt}{d\sigma}\right)\right]\right\}
\end{aligned}$$

$$\exp\left\{i\int_0^s\left[w_0^2\left(1+g(t(\sigma))\right)\right]d\sigma\right\} \tag{12}$$

Note that the term $\exp\left\{\frac{i\nu}{2}\int_0^2\frac{dt}{d\sigma}(\sigma)\right\}$ is exactly given by the factor $\exp\left\{\frac{i\nu}{2}(t-t')\right\}$.

The averaged out eq. (7) is thus given straightforwardly by the following Feynman Polaron like path integral

$$\begin{aligned}
\langle G(t,t',[g])\rangle_g &= -i\int_0^\infty ds\left(e^{\frac{i\nu}{2}(t-t')}e^{-s\nu^2}e^{iw_0^2 s}\right) \\
&\times \int_{t(0)=t';t(s)=t} D^F[t(\sigma)]\exp\left\{\frac{i}{2}\int_0^s d\sigma\left[\left(\frac{dt}{d\sigma}\right)^2\right]\right\} \\
&\times\ \exp\left\{-w_0^4\int_0^s d\sigma\int_0^s d\sigma' K(t(\sigma);t(\sigma'))\right\}
\end{aligned} \tag{13}$$

The two-point correlation function is still given by a two-full similar path integral, namely

$$
\begin{aligned}
&\langle G(t_1,t_1',[g])\,G(t_2,t_2',[g])\rangle_g \\
&= \int_0^\infty ds_1 ds_2 e^{\frac{i\nu}{2}(t_1-t_1')} e^{\frac{i\nu}{2}(t_2-t_2')} e^{-(s_1+s_2)\nu^2} e^{iw_0^2(s_1+s_2)} \\
&\times \int_{t_1(0)=t_1';t_2(s_1)=t_1;t_2(0)=t_2';t_2(s_2)=t_2} D^F\left[t_1(\sigma),t_2(\sigma)\right] \exp\left\{\frac{i}{2}\left(\int_0^{s_1} d\sigma\left(\dot{t}_1(\sigma)\right)^2 + \int_0^{s_2} d\sigma\left(\dot{t}_2(\sigma)\right)^2\right)\right\} \\
&\times \exp\left\{-w_0^4\left[\int_0^{s_1} d\sigma \int_0^{s_1} d\sigma' K\left(t_1(\sigma),t_1(\sigma')\right) + \int_0^{s_1} d\sigma \int_0^{s_2} d\sigma'\right.\right. \\
&\left(K\left(t_1(\sigma),t_2(\sigma')\right) + K\left(t_2(\sigma),t_1(\sigma')\right)\right) \\
&\left.\left. + \int_0^{s_1} d\sigma \int_0^{s_2} d\sigma' K\left(t_2(\sigma),t_2(\sigma')\right)\right]\right\}
\end{aligned}
\tag{14}
$$

Similar N-iterated path integrals expressions hold true the N-point correlation function $\langle x(t_1,[g])\cdots x(t_N,[g])\rangle_g$. Explicit and approximate evaluations of the path-integrals eq. (13) – eq. (14) follow similar procedures used in the usual contexts of Physics stastistics quantum mechanics and Random Wave Propagation (last reference of ref. [1]).

Let us show such exactly integral representation for eq. (13) in the case of the pratical case of a slowly varying (even function) kernel of the form

$$
\begin{aligned}
&K(t) \sim K(0) - \frac{\ell_0}{2}\,|t|^2 \quad ; \quad |t| << \left(+\frac{K(0)}{\ell_0}\right)^{1/2} = L \\
&K(t) \sim 0 \quad ; \quad |t| >> L
\end{aligned}
\tag{15}
$$

In this case, we have the following exactly result for the path-integral in eq. (13).

$$
\begin{aligned}
\langle G(t,t',[g])\rangle_g \;=\; & -\, e^{\frac{i\nu}{2}(t-t')} \times \int_0^\infty ds e^{-s(\nu^2-iw_0^2)} e^{-(w_0^4 K(0))s^2} \\
\times \; & \left\{(2\pi i s)^{\frac{1}{2}} \times \left[\frac{w_0^2\left(-\frac{\ell_0}{2}\right)^{\frac{1}{2}} S^{\frac{3}{2}}}{sen\left[s^{\frac{3}{2}} w_0^2\left(-\frac{\ell_0}{2}\right)^{\frac{1}{2}}\right]}\right]\right. \\
\times \; & \left.\exp\left(\left[\frac{iw_0^2}{2}\, s^{\frac{1}{2}}\left(-\frac{\ell_0}{2}\right)^{\frac{1}{2}} cot\left(w_0^2\left(-\frac{\ell_0}{2}\right)^{\frac{1}{2}} s^{\frac{3}{2}}\right)\right](t-t')^2\right)\right\}
\end{aligned}
\tag{16}
$$

Another useful formulae is that related to the "mean-field" averaged path integral when the Kernel $K(t,t')$ has a Fourier transform of the general form

$$
K(t,t') = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} dp \cdot e^{ip(t-t')} \tilde{K}(p) \tag{17}
$$

The envisaged integral representation for eq. (13) is, thus, given by

$$\langle G(t,t',[g])\rangle_g = -i\, e^{\frac{i\nu}{2}(t-t')} \int_0^\infty ds\, e^{-s(\nu^2 - i w_0^2)} \times \exp\left\{ -\frac{w_0^4}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} \tilde{K}(\rho) \times M(p,s,t,t') \right\} \tag{18}$$

where

$$M(p,s,t,k') = \int_0^s d\sigma \int_0^s d\sigma' \left\{ \int_{\substack{t(0)=t' \\ t(s)=t}} D^F[t(\sigma)] \exp\left\{ \frac{i}{2} \int_0^s \left[\left(\frac{dt}{d\sigma} \right) \right]^2 \right\} \exp\left[ip\left(t(\sigma) - t(\sigma')\right)\right] \right\} \tag{19}$$

2. The homogeneous problem

Let us start this section by considering now the problem of determining two linearly independent solutions of the homogeneous harmonic oscilator problem

$$\left\{ \frac{d^2}{dt^2} + \nu\, \frac{d}{dt} + w_0^2(1+g(t)) \right\} x(t) = 0 \tag{20}$$

with the initial conditions

$$x(0) = x_0 \quad ; \quad x'(0) = v_0 \tag{21}$$

It is straightforwardly to see that two L.I – solutions are given by the following expressions

$$x_1(t,[g]) = \exp\left\{ \int_0^t e^{-\nu\sigma} y^2(\sigma,[g]) \right\} \tag{22}$$

$$x_2(t,[g]) = x_1(t,[g]) \times \int_0^t e^{-\nu\sigma} \left(x_1(\sigma,[g])\right)^{-2} d\sigma \tag{23}$$

where $y(t,[g])$ satisfy the first-order non-linear ordinary differential equation

$$\frac{dy}{dt}(t) + e^{-\nu t}\left(y(t)\right)^2 = -w_0^2 e^{+\nu t}\left(1+g(t)\right) \tag{24}$$

In order to obtain a path-integral representation for eq. (24) we remark that the whole averaging (stochastic) information is contained in the characteristic functional

$$Z[j(t)] = \langle \exp\left\{ i \int_0^\infty dt\; y(t,[g]) j(t) \right\} \rangle_g \tag{25}$$

In order to write a path-integral representation for the characteristic functional eq. (25) we rewrite eq. (25) as a Gaussian functional integral in $g(t)$:

$$Z[j(t)] = \int D^F[g(t)] \exp\left(-\frac{1}{2} \int_0^\infty dt\; dt' g(t) K^{-1}(t,t') g(t') \right) \exp\left\{ i \int_0^\infty dt\; y(t,[g]) \right\} \tag{26}$$

At this point we observe the validity of the following functional integral representation for the characteristic functional eq. (26) after considering the functional change $g(t) \to y(t)$ defined by eq. (24) namely:

$$\begin{aligned} Z[j(t)] = \int D^F[y(t)] \quad &\times \quad \exp\left(-\frac{1}{2(w_0)^4}\int_0^\infty dt\, dt' \left[\left(\frac{dy}{dt}+e^{-\nu t}y^2\right)(t)e^{-\nu t}+w_0^2\right]\right. \\ &\times \quad K(t,t')\times \\ &\quad\left.\left[\left(\frac{dy}{dt'}+e^{-\nu t'}y^2\right)e^{-\nu t'}+w_0^2\right]\right) \\ &\times \quad \exp\left\{i\int_0^\infty dt\; j(t)y(t)\right\} \end{aligned} \tag{27}$$

where we have used that the Jacobian associated to the functional change $g(t)\to y(t)$ is unity

$$det_F\left[\frac{d}{dt}+2e^{-\nu t}y\right]=\frac{\delta g(t)}{\delta y(t)}=1 \tag{28}$$

At this point it is instructive to remark that at the important case of a white noise frequency process with strenght γ

$$K(t,t')=\gamma\delta(t-t) \tag{29}$$

the path integral representation for the characteristic functional eq. (27) takes the more amenable form

$$\begin{aligned} Z[j(t)] \quad = \quad & \int D^F[y(t)]\exp\left\{-\frac{\gamma}{2(w_0)^4}\int_0^\infty dt\times\right. \\ & \left.\left[e^{-\nu t}\frac{dy}{dt}+e^{-2\nu t}y^2(t)+w_0^2\right]^2\right\} \\ & \exp\left\{i\int_0^\infty dt y(t)j(t)\right\} \end{aligned} \tag{30}$$

After the variable path integral change

$$y(t)=e^{\nu t}\bar{y}(t) \tag{31}$$

$$D^F[y(t)]=e^{\nu t}D^F[\bar{y}(t)] \tag{32}$$

we get, thus, the standard $\lambda\varphi^4$-zero dimensional path integral as a functional integral representation for the characteristic function eq. (25) in the white noise case

$$\begin{aligned} Z[j(t)] \quad = \quad & e^{\nu t}\int D^F[\bar{y}(t)] \\ \times \quad & \exp\left\{-\frac{\gamma}{2(w_0)^4}\int_0^\infty dt\left[\left(\frac{d\bar{y}}{dt}\right)^2+\nu^2\bar{y}^2\right.\right. \\ + \quad & \left.\left. 2w_0^2\bar{y}^2+\bar{y}^4-2\nu\bar{y}^3-2w_0^2\nu\bar{y}+2\nu\,\frac{d\bar{y}}{dt}\,\bar{y}\right](t)\right\} \\ & \exp\left\{i\int_0^\infty dt e^{\nu t}\bar{y}(t)j(t)\right\} \end{aligned} \tag{33}$$

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