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A Non-Perturbative Approach to the Coleman-Weinberg Mechanism in Massless Scalar QED

A. P. C. Malbouisson[1,*], *F. S. Nogueira* and *N. F. Svaiter*

Centro Brasileiro de Pesquisas Físicas - CBPF
Rua Dr. Xavier Sigaud, 150
22290-180 – Rio de Janeiro, RJ – Brazil

[1]Centre de Physique théorique, CNRS UPR14
École Polytechnique
F-91128 Palaiseau Cedex, France

Abstract

We rederive non-perturbatively the Coleman-Weinberg expression for the effective potential for massless scalar QED. Our result is not restricted to small values of the coupling constants. This shows that the Coleman-Weinberg result can be established beyond the range of validity of perturbation theory. Also, we derive it in a manifestly renormalization group invariant way. It is shown that with the derivation given no Landau ghost singularity arises. The finite temperature case is discussed.

Key-words: Effective potential; Finite temperature QED.

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Ideas on situations which a system has a set of degenerate ground states, related by continuous symmetry transformations (spontaneous symmetry breaking) originated in solid state physics in the context of superconductivity theory [1]. The Higgs phenomenon [2] appears as a mechanism to explain spontaneous symmetry breaking. It postulates the existence of a self-interacting scalar field (the Higgs field) which produces massive vector fields in a locally gauge invariant Lagrangean. The Higgs "particle" results from the quantization of a model so defined. For instance, in superconductivity the Higgs particle comes from the breaking of a $U(1)$ symmetry and are the quanta of the elementary excitations due to oscilations of the electron gas, colective modes of vibration, quasi-particles called plasmons. The plasmon mode is associated to the generation of a massive degree of freedom to the photon. This is the origin of the well known Meissner effect.

In particle physics this phenomenon is associated to the so called standard model. The Higgs mechanism was adapted to particle physics in the context of the electroweak phenomenology associated to the spontaneous breaking of a $SU(2)\times U(1)$ symmetry [3]. According to the standard model, the Higgs mechanism is responsible for the masses of the weak interaction vector bosons, the $W^{\pm}$ and Z^0. However, the Higgs particle has never been detected experimentally. The main experimental evidence that the Higgs mechanism in fact works is still associated to superconductivity. The search for the Higgs particle is presently one of the major goals of experimental high energy physics.

In the Higgs model an imaginary mass is attributed to a charged scalar field which is self-interacting and is minimally coupled to a gauge field. Then, symmetry breaking is manifest already at the tree level. However, even if the mass of the scalar field is zero, we still have spontaneous symmetry breaking induced by radiative corrections. This is the Coleman-Weinberg mechanism [4].

The Coleman-Weinberg mechanism and its cosmological consequences was extensively studied in the past decade [5, 6, 7, 8, 9]. It was argued that in theories of grand unification the value of the coupling constant may become large at temperatures typically $\sim 4\times 10^6 Gev$ [6, 8, 9]. In this regime non-perturbative effects may become important in the study of the phase transition. Therefore, it will be of great interest to obtain the Coleman-Weinberg theory in a non-perturbative scheme. We will show that this is possible at least in the Abelian case. In this paper we rederive the Coleman-Weinberg theory non-perturbatively in the case of massless scalar QED.

Let us consider the Euclidean action for scalar QED in $D = 4$:

$$S = \int d^4x \left[\frac{1}{4}F_{\mu\nu}F_{\mu\nu} + (D_\mu\phi)^\dagger(D_\mu\phi) + m^2\phi^\dagger\phi + \frac{\lambda}{3!}(\phi^\dagger\phi)^2\right], \tag{1}$$

where $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ and $D_\mu = \partial_\mu - ieA_\mu$.

The vacuum-vacuum amplitude is given by the functional integral representation:

$$Z = \int \prod_{x\mu}[d\phi^\dagger(x)d\phi(x)dA_\mu(x)]\delta[\partial_\mu A_\mu]\det(-\Box)e^{-S}. \tag{2}$$

Note in the functional measure the delta function $\delta[\partial_\mu A_\mu]$ inforcing the Lorentz gauge condition. The factor $\det(-\Box)$ in the functional measure is the Faddeev-Popov determinant.

It is convenient to reparametrize the theory in the unitary gauge. This amounts in redefine the fields as

$$\phi = \frac{1}{\sqrt{2}}\rho e^{i\theta}, \tag{3}$$

$$B_\mu = A_\mu + \frac{1}{e}\partial_\mu\theta. \tag{4}$$

The action becomes

$$S = \int d^4x \left[\frac{1}{4}F'_{\mu\nu}F'_{\mu\nu} + \frac{1}{2}\partial_\mu\rho\partial_\mu\rho + \frac{1}{2}e^2\rho^2 B_\mu B_\mu + \frac{1}{2}m^2\rho^2 + \frac{\lambda}{4!}\rho^4\right], \tag{5}$$

where we have defined the new field strenght $F'_{\mu\nu} = \partial_\mu B_\nu - \partial_\nu B_\mu$.

In terms of this new parametrization the functional integral is written as

$$Z = \int \prod_{x\mu}[d\rho(x)\rho(x)dB_\mu(x)d\theta(x)]\delta[\partial_\mu B_\mu - \frac{1}{e}\Box\theta]\det(-\Box)e^{-S}. \tag{6}$$

Since the action is independent of the field $\theta(x)$ it is straightforward to integrate out the gauge freedom:

$$\int[d\theta(x)]\delta[\partial_\mu B_\mu - \frac{1}{e}\Box\theta] = \frac{\det(e)}{\det(-\Box)} = \frac{\prod_x e}{\det(-\Box)}. \tag{7}$$

¿From Eq.(7) is readily seen that the Faddeev-Popov determinant is cancelled out from the functional measure.

Integration over the vector field B_μ yields

$$Z = \int \prod_x [d\rho(x)]e^{-S_{eff}[\rho]} \tag{8}$$

where the effective action $S_{eff}[\rho]$ is given by

$$\begin{aligned} S_{eff}[\rho] &= \frac{1}{2}\ln\det[\delta_{\mu\nu}(-\Box + e^2\rho^2) + \partial_\mu\partial_\nu] - \frac{1}{2}\delta^4(0)\int d^4x \ln e^2\rho^2 \\ &+ \int d^4x\left[\frac{1}{2}\rho(-\Box + m^2)\rho + \frac{\lambda}{4!}\rho^4\right]. \end{aligned} \tag{9}$$

The above expression is exact. The factor $-(1/2)\delta^4(0)\int d^4x \ln e^2\rho^2$ arises from the exponentiation of the factor $\prod_x e\rho(x)$ of the functional measure.

The classical equation of motion is given by

$$\frac{\delta S_{eff}[\rho_c]}{\delta\rho(x)} = 0. \tag{10}$$

By considering a solution $\rho_c(x) =<\rho>= const.$, the functional differentiation above becomes an ordinary differentiation and we obtain the following equation:

$$\int d^4x \left(m^2 < \rho > + \frac{\lambda}{3!} < \rho >^3 - \frac{\delta^4(0)}{< \rho >} \right) + e^2 < \rho > Tr D_{\mu\nu}(x - x') = 0, \qquad (11)$$

where $D_{\mu\nu}(x - x')$ is the propagator of the massive vector field with mass $e^2 < \rho >^2$.

Now, we have

$$Tr D_{\mu\nu}(x - x') = \int d^4x \left[3 \int \frac{d^4p}{(2\pi)^4} \frac{1}{p^2 + e^2 < \rho >^2} + \frac{\delta^4(0)}{e^2 < \rho >^2} \right]. \qquad (12)$$

Substituting (12) into (11) one obtains

$$\int d^4x < \rho > \left(m^2 + \frac{\lambda}{6} < \rho >^2 + 3e^2 \int \frac{d^4p}{(2\pi)^4} \frac{1}{p^2 + e^2 < \rho >^2} \right) = 0 \qquad (13)$$

Note that the divergent factor $\delta^4(0)$ has been cancelled out. Eq. (13) gives us two possible solutions:

$$< \rho >= 0, \qquad (14)$$

and

$$< \rho >^2 = -\frac{6m^2}{\lambda} - \frac{18e^2}{\lambda} \int \frac{d^4p}{(2\pi)^4} \frac{1}{p^2 + e^2 < \rho >^2}. \qquad (15)$$

It is worth to emphasize the self-consistent character of Eq.(15). By defining the parameters $M^2 = e^2 < \rho >^2$, $M_0^2 = -6m^2/\lambda$, and $g = 36e^4/\lambda$ we can rewrite (15) in the form

$$M^2 = M_0^2 - \frac{g}{2} \int \frac{d^4p}{(2\pi)^4} \frac{1}{p^2 + M^2}. \qquad (16)$$

Eq.(16) corresponds to the superdaisy resummation [10] for a ϕ^4 theory with negative coupling $-g$. The role of the mass is played by the parameter M_0^2.

By considering a x-independent background field $\overline{\rho}$ we obtain the following expression for the effective potential:

$$V(\overline{\rho}) = \frac{3}{2} \int \frac{d^4p}{(2\pi)^4} \ln(p^2 + e^2\overline{\rho}^2) + \frac{1}{2} m^2 \overline{\rho}^2 + \frac{\lambda}{24} \overline{\rho}^4. \qquad (17)$$

Note again the cancellament of the divergent factor $\delta^4(0)$.

Evaluating the integral in (15) by using a cutoff Λ we obtain

$$\begin{aligned} < \rho >^2 \left(\lambda + \frac{9e^4}{8\pi^2} \ln \frac{\mu^2}{\Lambda^2} \right) &= -6m^2 - \frac{9e^4}{8\pi^2} \Lambda^2 \\ &\quad - \frac{9e^4}{8\pi^2} < \rho >^2 \ln \frac{e^2 < \rho >^2}{\mu^2}, \end{aligned} \qquad (18)$$

where μ is an arbitrary renormalization scale. From Eq.(18) we define the renormalized quantities:

$$m_R^2 = m^2 + \frac{3e^2}{16\pi^2}\Lambda^2, \tag{19}$$

$$\lambda_R = \lambda + \frac{9e^4}{8\pi^2}\ln\frac{\mu^2}{\Lambda^2}. \tag{20}$$

Eq.(18) becomes

$$\lambda_R < \rho >^2 = -6m_R^2 - \frac{9e^4}{8\pi^2} < \rho >^2 \ln\frac{e^2 < \rho >^2}{\mu^2}. \tag{21}$$

Also, by evaluating the integral in (17) using a cutoff Λ one obtains

$$V(\overline{\rho}) = \frac{m_R^2}{2}\overline{\rho}^2 + \frac{\lambda_R}{24}\overline{\rho}^4 + \frac{3e^4}{64\pi^2}\overline{\rho}^4\ln\frac{e^2\overline{\rho}^2}{\mu^2} - \frac{3e^4}{128\pi^2}\overline{\rho}^4. \tag{22}$$

Eq.(21) together with Eq.(14) gives us the position of the stationary points of the effective potential, Eq.(22). If $m_R^2 > 0$ the physical solution is given by Eq.(14) and corresponds to an absolute minimum at the origin. On the other hand, if $m_R^2 < 0$ Eq.(14) will correspond to a local maximum at the origin while Eq.(21) gives the position of the two degenerate absolute minima. This solution describes a broken symmetry state. It is also possible to obtain a broken symmetry solution by using just the Eq.(21). In order to eq.(21) to describe such a solution we must demand that $< \rho >_{max} = 0$, corresponding to a local maximum at the origin, must be a solution to Eq.(21). This is possible only if $m_R^2 = 0$. Thus, assuming $m_R^2 = 0$ and considering the solution to Eq.(21) $< \rho >_{min} = \sigma$, σ corresponding to the minimum, we can solve Eq.(21) for λ_R:

$$\lambda_R = -\frac{9e^4}{8\pi^2}\ln\frac{e^2\sigma^2}{\mu^2}. \tag{23}$$

Substituting Eq.(23) into Eq.(22) with $m_R^2 = 0$ we have

$$V(\overline{\rho}) = \frac{3e^4}{64\pi^2}\overline{\rho}^4\left(\ln\frac{\overline{\rho}^2}{\sigma^2} - \frac{1}{2}\right). \tag{24}$$

which is just the Coleman-Weinberg potential [4]. It is important to note the non-perturbative character of our approach. This result was not obtained perturbatively by computing Feynman graphs. In the usual loop expansion it is assumed that λ_R is of the same order as e^4 and then terms proportional to λ_R^2 are neglected. No such assumption is necessary here since our result does not depends upon perturbation theory. In perturbation theory this assumption is justified through renormalization group arguments [4]. In our approach this is not necesssary because from Eq.(22) with $m_R^2 = 0$ and Eq.(23) the indenpendence of physical results on the renormalization point is manifest. Thus, our results are obtained in a renormalization group invariant way.

We can give a mean field interpretation of our result. Mean field theories are generally characterized by a gap equation. In the Landau approximation to ϕ^4 theory the gap equation gives the mean field critical indices of the Ising model. In our case the gap equation is given by Eq.(15). It is this gap equation that, in the massless case, allows the elimination of the arbitrary renormalization scale. Thus, we have obtained the Coleman-Weinberg theory in a mean field like fashion. Note that we achieve this result in a *physical* gauge, the unitary gauge. In the Higgs mechanism this gauge exhibits the physical degrees of freedom already at the tree level. We have just shown that the same is true in the case of the Coleman-Weinberg mechanism in an Abelian theory. The Coleman-Weinberg potential emerges as a "tree level" potential for the effective action Eq.(9).

Our derivation, being non-perturbative, is not restricted to a small value of e^2. A large value of coupling constants in the perturbative scheme is generally not allowed since this may cause a breakdown of the perturbation series. We have shown that the Coleman-Weinberg potential, originally a perturbative result, can be established beyond the range of validity of perturbation theory. We can do a renormalization group analysis of the coupling constant in this framework. By taking the fourth derivative at σ of the potential, Eq.(22), we obtain

$$V^{(4)}(\sigma) = \lambda_R + \frac{9e^4}{8\pi^2}\ln\frac{e^2\sigma^2}{\mu^2} + \frac{33e^4}{8\pi^2}. \tag{25}$$

We obtain that $V^{(4)}(\sigma) = \lambda_R$ if and only if we have

$$\frac{33e^4}{8\pi^2} = -\frac{9e^4}{8\pi^2}\ln\frac{e^2\sigma^2}{\mu^2}. \tag{26}$$

¿From Eqs.(23) and (26) we obtain

$$\lambda_R = \frac{33e^4}{8\pi^2}, \tag{27}$$

which agrees with ref.[4]. Solving Eq.(26) for e^2 one obtains

$$e^2(\mu) = \frac{\mu^2}{\sigma^2}\exp\left(-\frac{11}{3}\right). \tag{28}$$

Thus, it does not matter at which energy scale we are working. We can always choose to measure e^2 in units of μ^2/σ^2. Let us define $\alpha = e^2$ and see what happens if we change the renormalization scale in a such a way that $\mu \longrightarrow e^t\mu$. ¿From Eq.(28) we obtain the following equation:

$$\frac{d\alpha}{dt} = 2\alpha. \tag{29}$$

Solving the above equation we get

$$\overline{\alpha}(t) = \alpha\exp(2t), \tag{30}$$

which defines the running coupling constant $\overline{\alpha}(t)$. It is easily seen that the theory is infrared free as $t \longrightarrow -\infty$. Indeed, this is already manifest in Eq.(28). Also, in the ultraviolet end, $t \longrightarrow \infty$, the running coupling constant diverges. This is not problematic for us since we are not performing a perturbative expansion. Also, we have already seen that no matter how large is μ, we can always choose to measure the physical quantities in units of μ^2/σ^2. Note that our analysis does not coincide to the usual one obtained from the conventional loop expansion [4]. In the usual perturbative scheme it is found that the running coupling diverges at a *finite* t, which implies that a Landau ghost develops. However, it is argued that this singularity arises in a scale which is beyond the range of validity of the one loop result. Here we established the one loop result non-perturbatively. Our result implies that the running coupling diverges only for infinite t and there is no Landau ghost. This result has implications concerning triviality questions. It has been established rigorously that the ϕ^4 theory is trivial for $d \geq 5$ [12]. It is almost certain that this is also true at $d = 4$. It has been speculated the impact of such a situation (if it is true) in the Higgs models which involves a scalar ϕ^4 sector [13]. In the non-Abelian Higgs model asymptotic freedom saves the theory and the model is non-trivial although the pure scalar sector appears to be trivial. However, in the Abelian case there is no asymptotic freedom and the ghost trouble occurs. In the ϕ^4_4 theory a Landau ghost seems to develop even at high orders in perturbation theory. In our derivation of the Coleman-Weinberg potential the self-coupling of the scalar field is related to the coupling e^2 just as in the perturbative treatment. However, the one loop renormalization group analysis shows that, at least at this level, a Landau singularity arises. In the present approach this does not happen. Thus, we conjecture that in the case of the Coleman-Weinberg mechanism there is no problem associated to the possible triviality of the pure scalar theory. The coupling to the gauge field induces the phenomenon of dimensional transmutation which protects the theory against the triviality trouble. This seems to happen also in the ϕ^4_4 theory. Recently, a non-trivial phase to ϕ^4_4 was found [11]. This phase is also associated to the phenomenon of dimensional transmutation. This phenomenon does not occur in the perturbative phase of ϕ^4_4 [4]. However, dimensional transmutation does occur in a non-perturbative regime in the massless case. It is found that the resulting expression for the effective potential contains only one free parameter given by $M_0 = \mu^2 \exp(96\pi^2/\lambda_R)$ which has dimension of mass squared [11]. Therefore, dimensional transmutation seems to be a possible path towards a non-trivial theory in situatutions which there is no asymptotic freedom nor non-trivial ultraviolet fixed points.

Let us consider now the finite temperature case. We can treat this case by standard methods [10]. At high temperature we have

$$\lambda_R < \rho >_T^2 = -6m_R^2 - \frac{3e^2T^2}{2} + \frac{9e^3T| < \rho >_T |}{2\pi} - \frac{9e^4 < \rho >_T^2}{8\pi^2} \ln \frac{e^2 < \rho >_T^2}{\mu^2}, \tag{31}$$

where we have assumed as before that $m_R^2 = 0$. Eq.(31) is established by generalizing Eq.(15) to finite temperature using the prescription $\int \frac{d^4p}{(2\pi)^4} \longrightarrow T \sum_n \int \frac{d^3p}{(2\pi)^3}$, where the discrete sum is over the Matsubara boson frequency $\nu_n = 2n\pi T$ [10]. From Eqs.(23) and (31) we obtain the following equation:

$$\frac{3e^2}{4\pi^2} < \rho >_T^2 \ln \frac{< \rho >_T^2}{\sigma^2} = \frac{3eT}{\pi} | < \rho >_T | - T^2 \tag{32}$$

At high temperature we obtain the following expression for the effective potential:

$$V(\overline{\rho}) = \frac{3e^4}{64\pi^2} \overline{\rho}^4 \left(\ln \frac{\overline{\rho}^2}{\sigma^2} - \frac{1}{2} \right) + \frac{e^2}{8} T^2 \overline{\rho}^2 - \frac{e^3}{4\pi} T |\overline{\rho}|^3 \tag{33}$$

Note that from the above equation we can also obtain Eq.(32) by imposing that $V'(<\rho>_T) = 0$.

In summary, we have obtained non-perturbatively the effective potential for massless scalar QED. We showed that with the treatment given here the theory is free of Landau ghost singularity. It is a question if we can treat the non-Abelian case similarly to the Abelian case. The generalization of our approach to the non-Abelian case is not straightforward because in a non-Abelian gauge theory the gauge fields interact and it is not possible to integrate out the fields directly. Maybe this problem can be solved partially by introducing auxiliary fields to obtain a set of gap equations. This situation is under study.

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