Feynman-Wiener path integral representation for scalar advected diffusion

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We write a path-integral expression for the Green function of a advected scalar on a fluid fluxe.

The problem of passive scalar transport by fluid fluxes is a subject of great interest and practical importance ([1]).

Unfortunately, analyticals treatments on this problem remain very cumbersome from a mathematical point of view. In this Brief Report, we aim to present a analytical closed expression for the above mentioned problem by means of a (Wiener) path integral expression for the Green function of the associated initial value problem of the advected scalar field ([1]).

Let us, thus, consider the motion equation for a scalar field $\phi(x,t)$ advected by a incompressible (*not acoustic waves!*) fluid with velocitty $\vec{V}(x,t)$, namely

$$\frac{\partial\phi(x,t)}{\partial t} = (D(t)D_0)\,\Delta\phi(x,t) - \left([\vec{v}\cdot\vec{\Delta}]\phi\right)(x,t) + j(x,t)\phi(x,t) \tag{1}$$

with the initial value condition

$$\phi(x,t\to 0^+) = f(x) \tag{2}$$

Here $D(t)$ is a time dependente molecular diffusion constant. $j(x,t)$ is an external source field and D_0 a reference value for the scalar diffusion constant.

As a first step to analyze eq. (1), let us consider the following time variable change

$$\tau = \int_0^t D(s)ds \tag{3}$$

$$\phi(x,\tau) \equiv \phi(x,t(\tau)) \tag{4}$$

$$j(x,\tau) \equiv \tau(x,t(\tau))/D((t)(\tau)) \tag{5}$$

$$\vec{\alpha}(x,\tau) \equiv \vec{V}(x,t(\tau))/t(\tau) \tag{6}$$

We obtain, thus, the more amenable form for eq. (1) with a constant molecular diffusion constant in this new time scale τ.

$$\frac{\partial\phi(x,\tau)}{\partial\tau} = D_0\Delta\phi(x,\tau) - \left([\vec{a}\cdot\vec{\nabla}]\phi\right)(x,\tau) + j(x,\tau)\phi(x,\tau) \tag{7}$$

$$\lim_{\tau\to 0^+}\phi(x,\tau) = f(x) \tag{8}$$

At this point let us remark that the pratical case of eq. (1) with $\vec{V}(\vec{x},t)=0$ and $j(\vec{x},t)=0$ ([2]), the problem Green function is easily given by

$$G_{dif}\left((x,t');(x,t')\right) = \left(D_0\int_{t'}^{t} D(s)ds\right)^{-\frac{3}{2}} \times \exp\left\{-\frac{[(\vec{x}-\vec{x}\,')^2]}{D_0\left(\int_{t'}^{t} D(s)ds\right)}\right\} \tag{9}$$

and leading thus to the quadratic mean desviation

$$\langle (\vec{x})^2 \rangle = \frac{3}{2}\, D_0 \int_0^t D(s) ds \qquad (10)$$

which, by its turn, leads to a super-difuse behavior if $D(s) \sim s^\alpha$ for $\alpha > 0$

Usual perturbative calculations may be formally implemented by considering the zeroth-order Green function as given by eq. (9).

Let us write a (non-perturbative) path-integral representation for the Green function $G((x,\tau):(x',\tau'))$ – eq. (7) – eq. (8). In order to implement such analysis we compare it with the analogous problem in Quantum Mechanics of a particle interacting with an eletromagnetic field $\vec{A}$ and a scalar potential V. The Schrödinger equation for this quantum mechanical problem in Landau gauge $\vec{\nabla}\vec{A} = 0$ is given by

$$i\hbar\, \frac{\partial \psi(\vec{x},\tau)}{\partial \tau} = \left\{ -\frac{\hbar^2}{2m}\, \Delta\psi + \frac{ie\hbar}{mc}\, (\vec{A}\cdot\vec{\nabla}\psi) + \frac{ie\hbar}{2mc^2}\, (\vec{\nabla}\cdot\vec{A})\psi + \left(\frac{e^2}{2mc^2}\, (\vec{A})^2 + V \right) \psi \right\} (\vec{x},\tau) \qquad (11)$$

It is well-known that the Green function associated to a initial value problem is given by the following (formal) Feynman path-integral ([3])

$$\begin{aligned} \tilde{G}\left[(x,\tau);(x',\tau')\right] \;&=\; \int_{\vec{r}(\tau')=x';\vec{r}(\tau)=x} D^F[\vec{r}(\sigma)] \times \\ &\exp\left\{ \frac{i}{\hbar} \int_{\tau'}^{\tau} d\sigma \left[\frac{1}{2}\, m \left(\frac{\vec{r}(\sigma)}{d\sigma} \right)^2 + ie\vec{A}\left(\vec{r}(\sigma),\sigma\right) - V\left(\vec{r}(\sigma),\sigma\right) \right] \right\} \end{aligned} \qquad (12)$$

It is straightforward to note that if one makes the following identification on eq. (11)

$$\begin{aligned} \hbar &= -i \quad ; \quad \vec{A} = -\vec{v} \quad ; \quad V = -\frac{1}{4D_0}\, (\vec{a})^2 + j \quad ; \\ m &= \frac{1}{2D_0} \quad ; \quad \frac{e}{c} = \frac{1}{2D_0} \quad ; \quad c = 1 \end{aligned} \qquad (13)$$

one can see that the Schrödinger equation (11) reduces to our scalar advected equation (7).

As a direct consequence of the above made remark, we obtain the result anounced on the begining of our study. Namely, the Green function $G[(x,\tau),(x',\tau')]$ is given explicitly with a closed form by the following (now well-defined) Wiener path-integral

$$\begin{aligned} G_{dif}\left[(\vec{x},\tau);(\vec{x}',\tau')\right] &= \int_{\vec{Z}(\tau)=\vec{x};\vec{Z}(\tau')=\vec{x}'} D^F\left[\vec{Z}(\sigma)\right] \\ &\times \exp\left\{ -\frac{1}{4D_0} \left(\int_{\tau'}^{\tau} d\sigma \left[\frac{d\vec{Z}}{d\sigma} - \vec{a}\left(\vec{z}(\sigma),\sigma\right) \right]^2 \right) \right\} \times \exp\left\{ -\int_{\tau'}^{\tau} d\sigma j(\vec{z}(\sigma),\sigma) \right\} \\ &\equiv \int_{\vec{Z}(\tau)=\vec{x},Z(\tau')=\vec{\tau}'} d_\mu^{Wiener}[z(\sigma)] \exp\left\{ -\frac{1}{4D_0} \int_{\tau'}^{\tau} \vec{a}(\vec{Z}(\sigma),\sigma)\, \frac{d\vec{Z}}{d\sigma}\, (\sigma) \right\} \times \\ &\exp\left\{ -\int_{\tau'}^{\tau} \left(\frac{\vec{a}^{\,2}}{4D_0} + \vec{j} \right) \left(\vec{Z}(\sigma),\sigma \right) \right\} \end{aligned} \qquad (14)$$

in the other words

$$\phi(x,\tau) = \int_0^\tau d\tau' \int dx' G\left[(x,\tau);(x,\tau')\right]\phi(x,\tau') \quad (15)$$

At this point let us remark that in the pratical important case of large-scale transport where one can set $D_0 \equiv 0$ on eq. (7) (with $j(x,\tau) \equiv 0$ for simplicity), an exactly expression for the first-order resulting equation

$$\frac{\partial \phi(x,\tau)}{\partial \tau} + \left[\vec{\nabla}\cdot(\vec{a}\phi)\right](x,\tau) = 0 \quad (16)$$

is exactly obtained by considering the limit $D_0 \to 0$ on eq. (15) and producing the result

$$\bar{G}_{dif}\left[(\vec{x},\tau);(\vec{x}',\tau')\right] = \delta^{(3)}\left[\vec{x} - \vec{Z}_{(x',\tau')}(\tau)\right] \quad (17)$$

where $\vec{Z}_{(x',\tau')}(\tau)$ satisfies the Saddle-point (minimum) of the positive path-integral weight, namely: $\vec{Z}_{(x',\tau')}(\tau) \equiv \vec{Z}(\sigma)\big|_{\sigma=\tau}$, here $\vec{Z}(\sigma)$ satisfies the Liouville boundary value problem

$$\frac{d\vec{Z}(\sigma)}{d\sigma} = \vec{a}\left(\vec{Z}(\sigma),\sigma\right) \quad (18)$$

with

$$\vec{Z}(\tau') = \vec{x} \qquad \vec{Z}(\tau) = \vec{x} \quad (19)$$

Next D_0 corrections are implemented on the path-integral eq. (12) by similar procedures used in the Feymann path-integral theory. We, thus, consider the following back-ground decomposition of the path manifold on eq. (14)

$$\vec{Z}(\sigma) = \vec{Z}_{(x',\tau')}(\sigma) + \sqrt{D}\,\vec{Y}(\sigma) \quad (20)$$

with the "fractal" path $\vec{Y}(\sigma)$ such that

$$\vec{Y}(\tau') = \vec{Y}(\tau) = \vec{0} \quad (21)$$

As a consequence eq. (20) – eq. (21), we get the next $\sqrt{D}$-correction for the diffusion Green function eq. (12)

$$\begin{aligned} G_{dif}\left[(\vec{x},\tau);(\vec{x}',\tau'\right] \sim\ & \tilde{G}_{dif}\left[(\vec{x},\tau);(\vec{x}',\tau')\right] \times \\ & \det{}_F^{-\frac{1}{2}}\left\{-\frac{d^2}{d^2\sigma}\delta_{AB} + \left[(\partial_A a_s)(\partial_B a_s)\right]\left(\vec{Z}_{x',\tau'}(\sigma)\right)\right. \\ - & \left. 2\left[\partial_A a_B\right]\left(\vec{Z}_{x',\tau'}(\sigma)\right) \times \frac{d}{d\sigma}\right\} + 0(D_0) \end{aligned} \quad (22)$$

where $A, B = 1,2,3$ denote the vectorial indixes on R^3 ($\vec{a}(\vec{x}) \equiv (a_A)(x_B)$) and the functional determinant associated to the fluctuation operator at the one-loop order should be evaluated with Dirichlet boundary conditions eq. (21). Exactly evaluation of the above cited functional determined needs the closed form of the transport fluid fluxe $\vec{a}(\vec{x})$.

Let us exemplify this last point for the two-dimensional vortex configuration with *constant vorticity* ($\vec{x} = (x,y)$)

$$\vec{a}(\vec{x},t) = \left(-\frac{1}{2}\ wy\ ,\ \frac{1}{2}\ wx\right) \quad (23)$$

In this case the classical trajectory equations eq. (18) – eq. (19) are given exactly by

$$z_1(\sigma) = A_1 sen\left(-\frac{w}{2}\ \sigma + \rho_1\right) \quad (24)$$

$$z_2(\sigma) = A_2 sen\left(-\frac{w}{2}\ \sigma + \rho_2\right) \quad (25)$$

where the integration constants $(A_1, A_2, \rho_1, \rho_2)$ must be choosen in order to satisfy the boundary conditions eq. (19). Namely,

$$A_1 sen\left(\frac{w}{2}\ \tau' + \rho_1\right) = x_1' \quad (26)$$

$$A_1 sen\left(\frac{w}{2}\ \tau + \rho_1\right) = x_1 \quad (27)$$

$$A_2 sen\left(\frac{w}{2}\ \tau' + \rho_2\right) = y_1' \quad (28)$$

$$A_2 sen\left(\frac{w}{2}\ \tau' + \rho_2\right) = y_1 \quad (29)$$

The functional determinant on eq. (22) is easily evaluated by the usual path-integral techniques applied to the problem of a particle can the presence of a harmonic oscillator and a constant magnetic field

$$det_F^{-\frac{1}{2}}\left\{\begin{bmatrix} -\frac{d^2}{dv^2}+\frac{1}{4}\,w^2 & -w\,\frac{d}{dv} \\ w\,\frac{d}{dv} & -\frac{d^2}{dv^2}+\frac{1}{4}\,w^2 \end{bmatrix}\right\} = \frac{w}{4\pi(\tau-\tau')\cdot sen\left(\frac{w(\tau-\tau')}{2}\right)} \tag{30}$$

As a last point worth remarking let us consider the Boltzman-Vlasov advected *damped* equation on R^6 wiht an external stirring $f(\vec{x},t)$ ([4] – pg. 22. eq. (7.12))

$$\frac{\partial N(\vec{x},t)}{\partial t} = -\nu N(\vec{x},t) - \left([\vec{V}\cdot\vec{\nabla}]N\right)(\vec{x},t) + f(\vec{x},t) \tag{31}$$

with the inital condition

$$\lim_{t\to 0^+} N(\vec{x},t) = f(\vec{x}) \tag{32}$$

By following the above exposed study, it is straighforward to write the solution of eq. (32) as the sum of the homogneous case with non zero initial condition added with that of the non-homogenous case but now with zero initial condition, namely

$$\begin{aligned} N(\vec{x},t) &= e^{-\nu t}\int d^6\vec{x}\,'\delta^{(6)}\left[(\vec{x}-\vec{Z}_{x',0}(t))\right]g(x') \\ &+ \int_0^t dt' e^{\nu(t'-t)}\int d^6\vec{x}\,'\delta^{(6)}\left[\vec{x}-\vec{Z}_{x',t'}(t))\right]f(\vec{x}\,',t') \end{aligned} \tag{33}$$

Here $\vec{Z}_{x',t'}(t) \equiv \vec{Z}_{x',t'}(\sigma)\big|_{\sigma=t}$ satisfies the equations (18) – (19)).

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References

[1] – Luiz C.L. Botelho – Modern Physics Letters 13B, 317, (1999);
– V. Gurarie and A. Migdal – Phys. Rev. E54, 4908, (1996);
– A.A. Migdal – Int. J. Mod. Physics A9, 1997 (1994)
– Peter B. Weichman and Roman E. Glazman – Phys. Rev. Letters vol. n. 83, n. 24, 5011, (1999);
– A.S. Monin and A.M. Yaglon – "Statistical Fluid Mechanics" (MIT Press, Cambridge) (1975).

[2] – G.I. Taylor – Proc. London Math. Soc. 20A, 196, (1921).

[3] – S. Schulman – "Techniques and applications of path-integration", (Wiley, New York (1981).

[4] – P.C. Clemmow and J.P. Dougherty – "Electrodynamics of Particles and Plasma", Addison-Wesley – (1990).