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Physical Variables for Chern-Simons-Maxwell Theory

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Abstract

We get the physical Hamiltonian for the CSM theory by working with the symplectic projector method.

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1 Introduction

Some years ago we developed a method to work in gauge field theories[1], [2] in wich we pick out from the original set of variables those which are the "true "or "physical " variables. This would be the first step to treat a gauge theory in a strictly canonical way[3], [4], [5].

We show in this letter what is the physical Hamiltonian in the D=3 Chern-Simons-Maxwell (CSM) model with the Coulomb gauge conditions. Its expression contains a term that has not been found in a work by Devecchi et al[6]

2 The Physical Hamiltonian for CSM theory

We start from the Lagrangian density

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + m\varepsilon^{\alpha\beta\gamma}A_\alpha\partial_\beta A_\gamma \tag{1}$$

with metric (-1,1,1).

The generalized Hamiltonian has the following canonical form:

$$\mathcal{H} = \int d^2x \left[\frac{1}{2}\pi^i\pi^i + \frac{1}{2}\left(\varepsilon^{ij}\partial^i A^j\right)^2 + \frac{1}{2}m^2A^kA^k + m\varepsilon^{ij}A^i\pi^j\right] \tag{2}$$

with the (second class) constraints relations:

$$\Omega^1 = \pi^0 = 0 \tag{3}$$

$$\Omega^2 = \partial^i\pi^i + m\varepsilon^{ij}\partial^jA^i = 0 \tag{4}$$

$$\Omega^3 = A^0 = 0 \tag{5}$$

$$\Omega^4 = \partial^iA^i = 0 \tag{6}$$

To establish a symplectic structure, we use the following correspondence
$\left(A^0, A^1, A^2, \pi^0, \pi^1, \pi^2\right) \Longleftrightarrow \left(\xi^1, \xi^2, \xi^3, \xi^4, \xi^5, \xi^6\right)$

The constraints Ω^i define a local metric g_{ij} , the inverse of $\mathrm{g}^{ij}\left(x,y\right) = \left\{\Omega^i\left(x\right), \Omega^j\left(y\right)\right\}$, which

is in this case:

$$g^{-1} = \begin{pmatrix} 0 & 0 & \delta^2\left(x-y\right) & 0 \\ 0 & 0 & 0 & \nabla^{-2} \\ -\delta^2\left(x-y\right) & 0 & 0 & 0 \\ 0 & -\nabla^{-2} & 0 & 0 \end{pmatrix} \tag{7}$$

The general form of the symplectic projector is given by[1]

$$\Lambda^\mu_\nu\left(x,y\right) = \delta^\mu_\nu\delta^2\left(x-y\right) - \varepsilon^{\mu\alpha}\int d^2\tau d^2\varpi g_{ij}\left(\tau,\varpi\right)\delta_{\alpha(x)}\Omega^i\left(\tau\right)\delta_{\nu(y)}\Omega^j\left(\varpi\right) \tag{8}$$

with $\delta_{\alpha(x)}\Omega^i(\tau) \equiv \frac{\delta\Omega^i(\tau)}{\delta\xi^\alpha(x)}$;
after a straightforward calculation we find:

$$\Lambda = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \delta^2(x-y) - \frac{\partial_1^x\partial_1^y}{\nabla^2} & -\frac{\partial_1^x\partial_2^y}{\nabla^2} & 0 & 0 & 0 \\ 0 & -\frac{\partial_2^x\partial_1^y}{\nabla^2} & \delta^2(x-y) - \frac{\partial_2^x\partial_2^y}{\nabla^2} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -m\frac{\partial_1^x\partial_2^y}{\nabla^2} & m\frac{\partial_1^x\partial_1^y}{\nabla^2} & 0 & \delta^2(x-y) - \frac{\partial_1^x\partial_1^y}{\nabla^2} & -\frac{\partial_1^x\partial_2^y}{\nabla^2} \\ 0 & -m\frac{\partial_2^x\partial_2^y}{\nabla^2} & m\frac{\partial_2^x\partial_1^y}{\nabla^2} & 0 & -\frac{\partial_2^x\partial_1^y}{\nabla^2} & \delta^2(x-y) - \frac{\partial_2^x\partial_2^y}{\nabla^2} \end{pmatrix} \tag{9}$$

Getting the physical variables, $\xi_\mu^*(x)$, is a simple matter of applying the prescription

$$\xi^{\mu *}(x) = \int d^2y \Lambda_\nu^\mu(x,y)\,\xi^\nu(y)\,; \tag{10}$$

from them we get

$$\xi^{1*}(x) = \xi^{4*}(x) = 0 \tag{11}$$

$$\xi^{2*}(x) = A_1^\perp(x) \tag{12}$$

$$\xi^{3*}(x) = A_2^\perp(x) \tag{13}$$

$$\xi^{5*}(x) = \pi_1^\perp(x) - m\int d^2y\left[\left(\partial_1^x\partial_2^y\nabla^{-2}\right)A_1(y) - \left(\partial_1^x\partial_1^y\nabla^{-2}\right)A_2(y)\right] \tag{14}$$

$$\xi^{6*}(x) = \pi_2^\perp(x) - m\int d^2y\left[\left(\partial_2^x\partial_2^y\nabla^{-2}\right)A_1(y) - \left(\partial_2^x\partial_1^y\nabla^{-2}\right)A_2(y)\right] \tag{15}$$

Now, our original constrained Hamiltonian written in symplectic language is:

$$\mathcal{H} = \int d^2x\left[\frac{1}{2}(\xi_5^2+\xi_6^2) + \frac{1}{2}\left(\varepsilon^{ij}\partial^i\xi^j\right)^2 + \frac{1}{2}m^2\left(\xi_2^2+\xi_3^2\right) + m\left(\xi_2\xi_6 - \xi_3\xi_5\right)\right];i,j=2,3 \tag{16}$$

The projected Hamiltonian is thus:

$$\begin{aligned}\mathcal{H}^* &= \int d^2x\left[\frac{1}{2}(\xi_5^{*2}+\xi_6^{*2}) + \frac{1}{2}\left(\varepsilon^{ij}\partial_i\xi_j^*\right)^2 + \frac{1}{2}m^2\left(\xi_2^{*2}+\xi_3^{*2}\right) + \right. \\ &+ m\left(\xi_2^*\xi_6^* - \xi_3^*\xi_5^*\right)];i,j=2,3\end{aligned} \tag{17}$$

Returning to the original phase-space notation, via 11,12,13,14,and 15 we finally have:

$$\mathcal{H}^* = \int d^2x\left[\begin{array}{c}\frac{1}{2}(\pi_i^\perp\pi_i^\perp + m^2A_i^\perp A_i^\perp) + \frac{1}{2}\left(\varepsilon^{ij}\partial_iA_j^\perp\right)^2 + m\left(A_1^\perp\pi_2^\perp - A_2^\perp\pi_1^\perp\right) \\ +\frac{1}{2}m^2\int d^2y\left(\varepsilon^{ij}\partial_iA_j^\perp\right)_x\nabla^{-2}\left(\varepsilon^{kl}\partial_kA_l^\perp\right)_y\end{array}\right] \tag{18}$$

This is the Chern-Simons-Maxwell Hamiltonian written in therms of the so called transverse fields and their associated canonical momenta. We point out that an equivalent expression, derived along another argument line[6], does not contain the last term in 18 .

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