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XYZ Techniques Applied to a Twisted XXX Chain

by

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Abstract

We show that the techniques developed by Baxter to solve the eight-vertex model can be applied to study a spin-1/2 twisted XXX chain.

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The purpose of this letter is to show how a spin-1/2 twisted XXX chain [1] can be treated through the techniques developed by Baxter to solve the eight-vertex model [2], along the lines of the Quantum Inverse Scattering Method (QISM) [3–5]. In the QISM one introduces an auxiliary problem with the help of the so-called Lax operator, $L_\ell(\lambda) \in End(V \otimes \eta_\ell)$ with $V \equiv \mathbb{C}^2$ the auxiliary space and $\eta_\ell \equiv \mathbb{C}^2$ the internal space of the site "ℓ".

Associated to $L_\ell(\lambda)$, one introduces an invertible matrix $R(\lambda) \in End(V \otimes V)$ which satisfies the quantum Yang-Baxter equation (QYBE) [6, 2]

$$R_{12}(\lambda_{12})R_{13}(\lambda_{13})R_{23}(\lambda_{23}) = R_{23}(\lambda_{23})R_{13}(\lambda_{13})R_{12}(\lambda_{12}) \ , \tag{1}$$

with $\lambda_{ij} = \lambda_i - \lambda_j$; $L_\ell(\lambda)$ obeys the Fundamental Commutation Relation (FCR) [3–5]

$$R_{12}(\lambda_{12})L_{1\ell}(\lambda_1)L_{2\ell}(\lambda_2) = L_{2\ell}(\lambda_2)L_{1\ell}(\lambda_1)R_{12}(\lambda_{12}) \ . \tag{2}$$

In (1) and (2)

$$R_{12} \equiv \sum_i a_i \otimes b_i \otimes \mathbb{1} \quad , \quad R_{13} \equiv \sum_i a_i \otimes \mathbb{1} \otimes b_i \quad , \quad R_{23} \equiv \sum_i \mathbb{1} \otimes a_i \otimes b_i \ , \tag{3}$$

with the R-matrix written as

$$R(\lambda) = \sum_1 a_i \otimes b_i \tag{4}$$

and the additional indices, 1 and 2, in the L_ℓ-matrix in (2) follow

$$L_{1\ell} = L_\ell \otimes \mathbb{1} \quad , \quad L_{2\ell} \equiv \mathbb{1} \otimes L_\ell \ . \tag{5}$$

In certain cases, also known as fundamental spin models, the matrices L and R are directly related to each other as

$$L_{1\ell}(\lambda) = R_{1\ell}(\lambda - \eta) \ , \tag{6}$$

where η is a constant. In this case, if the third auxiliary space is identified with the internal space "ℓ" eq. (2) follows from eq. (1).

For the vertex-type models we are considering here, a local Hamiltonian can be obtained as:

$$H = \sum_{\ell=1}^{N} H_{\ell,\ell+1} = C \left. \frac{d \ln T}{d\lambda} \right|_{\lambda=\eta} \tag{7}$$

with C a constant and

$$T = Tr_0 L_{0N} L_{0N-1} \cdots L_{01} \ , \tag{8}$$

where the trace is over the auxiliary space by "0". Thanks to the property (6) and to $R(0)=P$, with P the permutation operator in the tensor product space under consideration, one can write

$$H_{\ell,\ell+1} = C \ \frac{d}{d\lambda} \ (PR)_{\ell,\ell+1}\Big|_{\lambda=\eta} \quad . \tag{9}$$

If we now consider a matrix R of the form:

$$R = \begin{pmatrix} b & 0 & 0 & c \\ 0 & 0 & a & 0 \\ 0 & a & 0 & 0 \\ c & 0 & 0 & b \end{pmatrix} \quad , \tag{10}$$

the QYBE (1) reduces to

$$\begin{aligned} b(\lambda_{23})a(\lambda_{13})c(\lambda_{12}) + b(\lambda_{12})b(\lambda_{13})c(\lambda_{23}) &= b(\lambda_{23})a(\lambda_{12})c(\lambda_{23}) \ , \\ b(\lambda_{13})b(\lambda_{23})c(\lambda_{12}) + b(\lambda_{12})a(\lambda_{13})c(\lambda_{23}) &= b(\lambda_{12})a(\lambda_{23})c(\lambda_{23}) \ , \\ b(\lambda_{12})b(\lambda_{23})a(\lambda_{13}) + b(\lambda_{13})c(\lambda_{23})c(\lambda_{12}) &= b(\lambda_{23})a(\lambda_{12})a(\lambda_{23}) \ , \end{aligned} \tag{11}$$

which are the QYBE for a XXZ R-matrix of the form

$$R' = \begin{pmatrix} a & 0 & 0 & 0 \\ 0 & c & b & 0 \\ 0 & b & c & 0 \\ 0 & 0 & 0 & a \end{pmatrix} \quad . \tag{12}$$

Thus, giving a XXZ-type matrix R' satisfying the QYBE (1) we immediately get a XYZ-type matrix R satisfying (1). From now on we shall call the matrix R, the XYZ version of the XXZ-type matrix R'.

For simplicity let us consider the XYZ version of the XXX R'-matrix; in this simple case one has

$$a = \lambda + i \quad , \quad b = i \quad , \quad c = \lambda \tag{13}$$

and using (6) with $\eta = i/2$ the L-matrix becomes

$$L_\ell(\lambda) = \begin{pmatrix} \phi(\mathbb{1}_\ell + \sigma^3_\ell) & \Psi\sigma^+_\ell + \theta\sigma^-_\ell \\ \Psi\sigma^-_\ell + \theta\sigma^+_\ell & \phi(\mathbb{1}_\ell - \sigma^3_\ell) \end{pmatrix} = \begin{pmatrix} \alpha_\ell & \beta_\ell \\ \gamma_\ell & \delta_\ell \end{pmatrix} \quad , \tag{14}$$

with $\vec{\sigma} = (\sigma^1, \sigma^2, \sigma^3)$ the Pauli matrices, $\sigma^\pm = (\sigma^1 \pm i\sigma^2)/2$ and

$$\phi = \frac{i}{2} \quad , \quad \Psi = \lambda - i/2 \quad , \quad \theta = \lambda + i/2 \ . \tag{15}$$

Notice that, in contrast to the case of the six-vertex model, the operator γ_ℓ is in general non-degenerate thus rendering more difficult to obtain a local vacuum, either for L_ℓ or for a finite product of such matrices. Before facing this problem let us discuss the Hamiltonian for (10) and (13).

Using (9), (10) and (13) we get

$$H = \sum_{i=1}^{N}(-\sigma_i^1\sigma_{i+1}^1 + \sigma_i^2\sigma_{i+1}^2 + \sigma_i^3\sigma_{i+1}^3) \quad , \quad \vec{\sigma}_{N+1} \equiv \vec{\sigma}_1 \ , \tag{16}$$

apart from an additive constant. The Bethe ansatz equations for (16) are well-known but we think it is useful to reobtain them using the method developed by Baxter to deal with XYZ-type models. Applying this method to simpler matrices like the one given in (10) and (13) may shed some light on the main difficulties of this kind of problem helping us to treat unsolved problems like, for instance, higher spin XYZ chains.

According to Baxter one defines a gauge equivalent matrix $L'_\ell(\lambda)$ [2] as

$$L'_\ell(\lambda) = M_{\ell+1}^{-1}(\lambda)L_\ell(\lambda)M_\ell(\lambda) = \begin{pmatrix} \alpha'_\ell & \beta'_\ell \\ \gamma'_\ell & \delta'_\ell \end{pmatrix} \tag{17}$$

where the $M_n(\lambda)$ are arbitrary non-singular 2×2 numerical matrices, leaving invariant the transfer matrix, $T' = T$, for $M_{N+1} = M_1$.

It turns out that the gauge transformations $M_\ell(\lambda)$ can be chosen in such a way that each matrix $L'_\ell(\lambda)$ has a local vacuum, independent of λ, that is annihilated by its lower left element for all λ.

Denoting $M_\ell(\lambda)$ as

$$M_\ell(\lambda) = \begin{pmatrix} r_\ell & r'_\ell \\ r_\ell p_\ell & r'_\ell p'_\ell \end{pmatrix} \ , \tag{18}$$

where r and p are numbers, γ'_ℓ is given by

$$\gamma'_\ell = a_\ell^+\sigma_\ell^+ + a_\ell^-\sigma_\ell^- + a_\ell^3\sigma_\ell^3 + a_\ell^4\,\mathbb{1}_\ell \tag{19}$$

with

$$a_\ell^+ = \frac{1}{\det M_\ell}\; r_\ell r_{\ell+1}(\theta - \Psi p_\ell p_{\ell+1}) \tag{20}$$

$$\begin{aligned} a_\ell^- &= \frac{1}{\det M_\ell}\; r_\ell r_{\ell+1}(\Psi - \theta p_\ell p_{\ell+1}) \\ a_\ell^3 &= \frac{-1}{\det M_\ell}\; r_\ell r_{\ell+1}\phi(p_\ell + p_{\ell+1}) \\ a_\ell^4 &= \frac{-1}{\det M_\ell}\; r_\ell r_{\ell+1}\phi(p_\ell - p_{\ell+1}) \ . \end{aligned}$$

Let $e_\ell^\pm$ be a vector in η_ℓ

$$e_\ell^+ = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad , \quad e_\ell^- = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \ . \tag{21}$$

From (19) and

$$\omega_\ell = f^+ e_\ell^+ + f^- e_\ell^- \ , \tag{22}$$

the condition

$$\gamma'_\ell \omega_\ell = 0 \tag{23}$$

reduces to

$$\begin{aligned} p_{\ell+1} f^+ + (\theta - \Psi p_\ell p_{\ell+1}) f^- = 0 \\ (\Psi - \theta p_\ell p_{\ell+1}) f^+ + p_\ell f^- = 0 \ . \end{aligned} \tag{24}$$

Eq. (24) is a pair of homogeneous linear equations for f^+ and f^-, thus the determinant of the coefficients must vanish leading to

$$p_\ell p_{\ell+1} = 1 \ . \tag{25}$$

We consider the simplest solution of eq. (25), $p_\ell = 1$; substituting it in (24) we have

$$f^+ = f^- \equiv f \ , \tag{26}$$

giving

$$\omega_\ell = f(e_\ell^+ + e_\ell^-) \ . \tag{27}$$

Choosing, for simplicity, $r_\ell = 1$ $r'_\ell = \frac{-1}{2}$ and $p' = -1$ it is easy to obtain the action of the diagonal elements of (17) on ω_ℓ

$$\begin{aligned} \alpha'_\ell \omega_\ell &= \theta \omega_\ell \\ \delta'_\ell \omega_\ell &= -\Psi \omega_\ell \ . \end{aligned} \tag{28}$$

From the local formulae (23) and (28) there follow analogous formulae for the elements of $\Upsilon'(\lambda)$

$$\Upsilon'(\lambda) = L'_N(\lambda) \cdots L'_1(\lambda) = \begin{pmatrix} A(\lambda) & B(\lambda) \\ C(\lambda) & D(\lambda) \end{pmatrix} \tag{29}$$

with respect to

$$\Omega = \omega_1 \otimes \cdots \otimes \omega_N \ . \tag{30}$$

Namely,

$$\begin{aligned} A(\lambda)\Omega &= \theta^N\Omega \\ D(\lambda)\Omega &= (-1)^N\Psi^N\Omega\ , \\ C(\lambda)\Omega &= 0\ . \end{aligned} \tag{31}$$

Let us now derive the FCR satisfied by $L'_\ell(\lambda)$. From (5) and (17) we have

$$\begin{aligned} L_{1\ell}(\lambda) &= (M_{\ell+1}(\lambda)\otimes \mathbb{1})L'_{1\ell}(\lambda)(M_\ell^{-1}(\lambda)\otimes \mathbb{1}) \\ L_{2\ell}(\lambda) &= (\mathbb{1}\ \otimes M_{\ell+1}(\lambda))L'_{2\ell}(\lambda)(\mathbb{1}\ \otimes M_\ell^{-1}(\lambda))\ . \end{aligned} \tag{32}$$

Substituting (32) in (2) we have

$$\begin{aligned} &R_{12}(\lambda_{12})(M_{\ell+1}(\lambda_1)\otimes M_{\ell+1}(\lambda_2))L'_{1\ell}(\lambda_1)L'_{2\ell}(\lambda_2)(M_\ell^{-1}(\lambda_1)\otimes M_\ell^{-1}(\lambda_2)) = \\ &(M_{\ell+1}(\lambda_1)\otimes M_{\ell+1}(\lambda_2))L'_{2\ell}(\lambda_2)L'_{1\ell}(\lambda_1)(M_\ell^{-1}(\lambda_1)\otimes M_\ell^{-1}(\lambda_2))R_{12}(\lambda_{12}) \qquad , \end{aligned} \tag{33}$$

which leads to

$$R_{12}^{(\ell+1)}(\lambda_{12})L'_{1\ell}(\lambda_1)L'_{2\ell}(\lambda_2) = L'_{2\ell}(\lambda_2)L'_{1\ell}(\lambda_1)R_{12}^{(\ell)}(\lambda_{12})\ , \tag{34}$$

where

$$R_{12}^{(\ell)}(\lambda_{12}) = (M_\ell^{-1}(\lambda_1)\otimes M_\ell^{-1}(\lambda_2))R_{12}(\lambda_{12})(M_\ell(\lambda_1)\otimes M_\ell(\lambda_2))\ . \tag{35}$$

In the special case we are considering,

$$M_\ell = M = \begin{pmatrix} 1 & -1/2 \\ 1 & 1/2 \end{pmatrix} \tag{36}$$

does not depend on ℓ and λ and $R^{(\ell)}$ reduces to

$$R^\ell(\lambda) = \tilde{R}(\lambda) = \begin{pmatrix} \lambda+i & 0 & 0 & 0 \\ 0 & -\lambda & i & 0 \\ 0 & i & -\lambda & 0 \\ 0 & 0 & 0 & \lambda+i \end{pmatrix}\ . \tag{37}$$

The above $\tilde{R}(\lambda)$ matrix is also obtained taking the non-deformed limit of the twisted ($\sigma^\pm_{N+1} = (-1)^N\sigma^\pm_1$) XXZ model (see for instance [7]), which is the twisted ($\sigma^\pm_{N+1} = (-1)^N\sigma^\pm_1$) XXX model. In fact from

$$\tilde{R}(\lambda_{12})\Upsilon'(\lambda_1)\Upsilon'(\lambda_2) = \Upsilon'(\lambda_2)\Upsilon'(\lambda_1)\tilde{R}(\lambda_{12}) \tag{38}$$

one derives easily the well-known Bethe ansatz equations

$$\left(\frac{\lambda_\beta - i/2}{\lambda_\beta + i/2}\right)^N = (-1)^N \prod_{\substack{\alpha=1 \\ \alpha\neq\beta}}^{M} \frac{\lambda_\beta - \lambda_\alpha - i}{\lambda_\beta - \lambda_\alpha + i} \quad ; \quad \beta = 1, \cdots, M \leq N \ . \tag{39}$$

The Hamiltonian (16) becomes one of the twisted ($\sigma^{\pm}_{N+1} = (-1)^N \sigma^{\pm}_1$) XXX chain after a suitable redefinition of axis in the spin-space and a similarly transformation [7].

In summary, we have developed in this letter a different treatment of a spin-1/2 twisted XXX chain which is not a trivial specialization of refs. [2, 3]. This method has the advantage of presenting some of the important techniques used by Baxter, when solving the eight-vertex model, in a very simple way and may shed some light on unsolved problems like, for instance, higher spin XYZ chains.

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