ISSN 0029 - 3865

CBPF-NF-057/88

SU(4) PROPERTIES OF THE DIRAC EQUATION

by

Cesar A. LINHARES* and Juan A. MIGNACO

Centro Brasileiro de Pesquisas Físicas - CBPF/CNPq
Rua Dr. Xavier Sigaud, 150
22290 - Rio de Janeiro, RJ - Brasil

*Instituto de Estudos Avançados
Centro Técnico Aeroespacial
12200 - São José dos Campos, SP - Brasil

CBPF-NF-057/88

Abstract

The Dirac equation in four dimensions has an intimate connection with the representations of the group $SU(4)$. This connection is shown in detail and subsequent properties are displayed in the continuum as well as in the lattice description.

Key-words: Dirac equation; Dirac algebra; Relativistic wave equation; Lattice fermions.

PACS: 11.10.Qr, 11.30.Ly.

1 Introduction

The Dirac equation was introduced almost sixty years ago [1] and immediately shown to nicely describe the properties of spin-1/2 fermions. Its physical content as related to the electromagnetic properties of the electron and the muon has been exploited to a high degree of accuracy in its second quantized version[2].

The symmetry properties of fermions have been exhaustively studied since those early years. The transformation properties under the discrete time reversal, space reflection and charge conjugation operations were of high interest at the time when the structure of weak interactions began to be disclosed[3].

In the study of the strong interactions as described by the non-abelian color gauge theory (QCD) it was first proposed by Wilson[4] to translate field theories onto the lattice, at the expense of Lorentz invariance, to investigate the problem of confinement. The introduction of fermions as described by the Dirac equation was not straightforward. Massless Dirac fields, when taken naively, led to a proliferation of chiral fermions in the continuum limit. Of course, several procedures were soon developed to dispose of this problem: Wilson[4] introduced auxiliary terms, vanishing in the continuum and breaking chiral invariance on the lattice; Banks, Kogut and Susskind[5,6] developed a formalism where fermionic degrees of freedom were spread over lattice points; Drell, Weinstein and Yankielowicz[7] introduced a non-local derivative on the lattice which allowed the elimination of redundant components.

The question of how natural this difficulty is for fermions remains. There is no general agreement on whether it translates a characteristic feature not previously seen or is just an accident of the description in the continuum.

A fundamental leap forward was produced by Nielsen and Ninomiya[8]. Any description of fermions on the lattice satisfying

a) locality of the interaction

b) hermiticity of the Hamiltonian

c) having currents constructed as bilinears of spinors

is bound to produce several species of fermions in the continuum limit. This result is related to the topological properties of the mapping of the continuum onto the lattice.

Currently the practitioners of the game of putting fermions on the lattice play mostly with the alternatives proposed by Wilson and by Banks, Kogut and Susskind. Gliozzi[9] observed that the introduction of a “double-spaced” lattice led to the natural grouping of fermion species and studied geometric properties resembling gravitational features.

An important step towards the understanding of the problem of fermion proliferation as a limit of lattice theories was given by Becher and Joos[10]. They proposed to adapt to the lattice the Kähler formalism[11] in order to deal with fermions in terms of differential forms. It proposes a point of view which is attractive: differential forms provide geometrical structures which are easily adapted to the lattice. The point is that these structures presented a natural $SU(4)$ degeneracy and Kähler's fermions on the lattice transcribed the Kogut-Susskind fermions, the number of species being the same in the continuum. Lately, these features have been applied to the classification of irreducible representations of symmetries to be used in lattice calculations[12].

Recently, we have shown[13] that this $SU(4)$ structure is also contained in the original Dirac equation. It arises as a natural consequence of the fact that the algebra of Dirac matrices is related to the algebra of the generators of the $SU(4)$ group. In general, for even dimensions (n) of spacetime, we conjecture that the relevant groups are $SU(2^{n/2})$. We have been able to show how the proliferation of degrees of freedom on the lattice is related to the mixing of minimal left ideals of the Dirac algebra. It has been claimed before in the mathematical literature[14] that the embedding of the representation of spin-1/2 particles in the Dirac algebra through minimal left ideals is a more appropriate procedure than the current description using spinors.

Several points of our argument deserve, however, further considerations. Among them, the meaning of Lorentz invariance in algebraic terms, and the relationship of symmetry operations to the Cartan subalgebra[15] of the group in an arbitrary representation of the Dirac matrices.

The plan of the article is as follows. In Section 2 we expand in detail the argument relating the Dirac algebra of 4×4 matrices to the Lie algebra of the generators of the unitary group $SU(4)$. The notion of Cartan subalgebra or trunk[16] is made explicit and its connection with the usual concept of representation for the Dirac matrices displayed. The notion of invariance and covariance under a change of representation and Lorentz transformations is presented in some detail.

Section 3 develops the study of ideals and, in particular, minimal left (and right) ideals for the Dirac algebra. The connection between ideals and representations is emphasized. We also try to establish a link between the discrete operations of space inversion (π) and time reversal (τ) of spinors and the transformation of projection operators on ideals. We believe that these properties are quite relevant to the understanding of the physical properties of spin-1/2 fermions and to the alternative choices to their description by four-component spinors.

In Section 4 we examine the translation of the free theory onto the lattice. We deal with the proliferation problem in two dimensions (four dimensions constitute a simple extension) and make an analysis of the importance of minimal left ideals and, indirectly, of the relevance of the representation chosen for the Dirac matrices. The idea of the "reduction group" in the sense introduced by Becher and Joos[10] is recalled.

The final Section 5 elaborates on the results previously obtained, contains suggestions for further research and speculations about the possible physical relevance of the features displayed.

We emphasize in this article several properties of the current presentation of the Dirac equation which are not found in the literature, but can be put into correspondence with the properties appearing in the differential form construction. The reader interested in this correspondence will find a detailed analysis in a forthcoming article[17].

2 The Dirac equation, the algebra of Dirac matrices and $SU(4)$

In this section we shall mainly deal with the four dimensional Minkowski spacetime description. Our conventions are the ones contained in the book by Itzykson and Zuber[18].

The Dirac equation, solution to the problem of time evolution of a particle in the relativistic domain,

$$H\psi(x,t) = i\hbar\frac{\partial\psi}{\partial t}(x,t) \tag{1}$$

is obtained from the first-order Hamiltonian

$$H = -i\hbar c p\cdot\nabla + \beta mc^2, \tag{2}$$

where the matrices β and α_k $(k = 1, 2, 3)$ satisfy anticommutation relations:

$$\{\alpha_i, \alpha_j\} = 2\delta_{ij} \tag{3}$$

$$\{\alpha_i, \beta\} = 0, \qquad i = 1, 2, 3. \tag{4}$$

The Dirac equation may be referred to as the set of four coupled linear first-order equations condensed in

$$(i\gamma^0 \frac{\partial}{\partial x^0} + i\boldsymbol{\gamma} \cdot \nabla - m)\psi(\boldsymbol{x}, t) = 0, \tag{5}$$

where in the last step we have adopted the "natural" system of units $\hbar = c = 1$. As usual,

$$\begin{aligned} \boldsymbol{\gamma} &= \beta\boldsymbol{\alpha} &(6)\\ \gamma^0 &= \beta &(7)\\ \{\gamma^\mu, \gamma^\nu\} &= 2g^{\mu\nu}. &(8)\end{aligned}$$

The matrices $\gamma^0, \gamma^k\,(k = 1,2,3)$ generate the Dirac ring made out of their products (and the identity):

$$\begin{array}{cccccc} & \gamma^0 & \gamma^1 & \gamma^2 & \gamma^3 & \\ \gamma^0\gamma^1 & \gamma^0\gamma^2 & \gamma^0\gamma^3 & \gamma^1\gamma^2 & \gamma^1\gamma^3 & \gamma^2\gamma^3 \\ & \gamma^0\gamma^1\gamma^2 & \gamma^0\gamma^1\gamma^3 & \gamma^0\gamma^2\gamma^3 & \gamma^1\gamma^2\gamma^3 & \\ & & i\gamma^0\gamma^1\gamma^2\gamma^3. & & & \end{array} \tag{9}$$

In Table I we summarize the properties of these matrices. It is well known that all the 15 matrices above have null trace. A glance at Table I readily shows that we can have all matrices hermitian by multiplying by i all matrices of negative square. The set so obtained is, via a well-known theorem[18,19], a basis set in the space of 4×4 matrices. Calling:

$$\begin{aligned} \zeta^0 &= \gamma^0 &&&(10)\\ \boldsymbol{\zeta} &= i\boldsymbol{\gamma} &&&(11)\\ \zeta^{0k} &= \gamma^0\gamma^k, & \zeta^{kl} &= i\gamma^k\gamma^\ell \quad (k,\ell = 1,2,3; k \neq \ell) &(12)\\ \zeta^{0k\ell} &= \gamma^0\gamma^k\gamma^\ell, & \zeta^{123} &= \gamma^1\gamma^2\gamma^3 \quad (k,\ell = 1,2,3; k \neq \ell) &(13)\\ \zeta^5 &= \gamma^5 = i\gamma^0\gamma^1\gamma^2\gamma^3. &&&(14)\end{aligned}$$

Together with the identity, any 4×4 matrix M can be written as

$$M = M_I 1 + M_\alpha \zeta^\alpha + M_{\alpha\beta}\zeta^{\alpha\beta} + M_{\alpha\beta\gamma}\zeta^{\alpha\beta\gamma} + M_5\zeta^5 \tag{15}$$

with the coefficients obtained through:

$$M_H = \frac{1}{4}\mathrm{tr}(\zeta^H M) \quad (H = I, 0, \ldots, 3, 01, \ldots, 23, \ldots, 5). \tag{16}$$

It is simple to show that Dirac matrices form an algebra. Table II contains the result of the commutators:

$$[\gamma^H, \gamma^K] = C^{HK}{}_L \gamma^L \tag{17}$$

and obviously the conditions for ζ^H follow. It is also simple to verify the Jacobi identity:

$$[\gamma^H, [\gamma^K, \gamma^L]] + [\gamma^L, [\gamma^H, \gamma^K]] + [\gamma^K, [\gamma^L, \gamma^H]] = 0. \tag{18}$$

From Table II, the following commutators are seen to vanish:

$$\begin{aligned}
\left[\gamma^0, \gamma^j\gamma^k\right] &= 0 && (19)\\
\left[\gamma^0, \gamma^0\gamma^j\gamma^k\right] &= 0 && (20)\\
\left[\gamma^j, \gamma^0\gamma^k\right] &= 0, \quad j \neq k, && (21)\\
\left[\gamma^j, \gamma^k, \gamma^\ell\right] &= 0, \quad j \neq k \neq \ell, && (22)\\
\left[\gamma^j, \gamma^0\gamma^j\gamma^k\right] &= 0 && (23)\\
\left[\gamma^j, \gamma^1\gamma^2\gamma^3\right] &= 0 && (24)\\
\left[\gamma^0\gamma^j, \gamma^k\gamma^\ell\right] &= 0, \quad j \neq k \neq \ell, && (25)\\
\left[\gamma^0\gamma^j, \gamma^0\gamma^j\gamma^k\right] &= 0 && (26)\\
\left[\gamma^0\gamma^j, \gamma^5\right] &= 0 && (27)\\
\left[\gamma^j\gamma^k, \gamma^0\gamma^j\gamma^k\right] &= 0 && (28)\\
\left[\gamma^j\gamma^k, \gamma^1\gamma^2\gamma^3\right] &= 0 && (29)\\
\left[\gamma^j\gamma^k, \gamma^5\right] &= 0. && (30)
\end{aligned}$$

From the examination of Eqs. (19)-(30), we conclude that it is always possible to have a set of three matrices which are simultaneously diagonal. Several possible sets are:

$\gamma^0\gamma^3, \gamma^1\gamma^2, \gamma^5$ (Kramers-Weyl representation)

$\gamma^0, \gamma^1\gamma^2, \gamma^0\gamma^1\gamma^2$ (Dirac-Pauli representation)

$\gamma^1, \gamma^0\gamma^2, \gamma^0\gamma^1\gamma^2$ (Majorana representation)

$\gamma^1\gamma^2\gamma^3, \gamma^1\gamma^2, \gamma^3$ (Sharatchandra, Thun and Weisz)[20].

The three diagonal matrices in a given representation correspond to the only three available ways to have two -1 and two $+1$ diagonal elements.

We thus have 15 hermitian matrices with null trace, three of which are diagonal, which form an algebra. The classification of all possible matrix algebras was worked out by Lie, Cartan and others[15,16,21]. There is just one possible Lie algebra as the one built starting from the Dirac matrices and their products, having fifteen elements and three of them diagonal: the Lie algebra corresponding to the groups $SU(4)$ or $O(6)$. Our choice favors $SU(4)$ because in two dimensions the corresponding unitary group is $SU(2)$. We conjecture that for higher even dimensions the relevant unitary group for the Dirac equation is $SU(2^{n/2})$.

The abelian subalgebra is known as the Cartan subalgebra or trunk[16]. It turns out that any of the representations of the Dirac matrices corresponds to one of the possible choices within the commuting subsets displayed in Table II and Eqs. (19)-(30).

A change in the representation of Dirac matrices in this point of view means that a given set of three diagonal matrices in an initial Cartan subalgebra transforms into another Cartan subalgebra. The Dirac equation (5) is form-invariant under this operation, which can be expressed in terms of left multiplication by a given matrix of the ring:

$$
\begin{aligned}
i\gamma^\mu \partial_\mu \psi &= m\psi && (31)\\
i\gamma^H \gamma^\mu \partial_\mu &= m\gamma^H \psi && (32)\\
i\gamma'^\mu \partial_\mu \psi' &= m\psi' && (33)\\
\gamma'^\mu &= \gamma^H \gamma^\mu (\gamma^H)^{-1} && (34)\\
\psi' &= \gamma^H \psi. && (35)
\end{aligned}
$$

Given a representation and a corresponding solution of the Dirac equation, a change of representation retains the solution but under a new guise.

In the space of spinors, a representation of a given group of transformations will also admit an expansion in the basis (10)-(14). A symmetry transformation, at the same time, leads to a change of basis. Let us consider the case of a Lorentz transformation, for instance. We mean by this a general homogeneous coordinate transformation:

$$x' = Ax. \tag{36}$$

We then have

$$i\gamma^{\ell} a_{\ell}^{r} \frac{\partial}{\partial x'^{r}} \psi(x = A^{-1}x') = m\psi(x = A^{-1}x') \tag{37}$$

and the equation is covariant under the transformation (36) provided that

$$\gamma'^{m} = a_{\ell}^{m} \gamma^{\ell}. \tag{38}$$

Usually, another requirement is imposed. Since the matrices are irreducible, there exists a similarity transformation

$$\gamma'^{m} = S^{-1}(A)\gamma^{m} S(A), \tag{39}$$

so that we have

$$i\gamma^{r} \frac{\partial}{\partial x'^{r}} S(A)\psi(x = A^{-1}x') = mS(A)\psi(x = A^{-1}x'), \tag{40}$$

which is the requirement of invariance for the Dirac equation.

In the first case, we have the manner in which the old situation is seen in a new coordinate system. In the latter, Eq. (40), we reproduce the old situation in the new coordinate system.

An elementary but instructive example is the following. Assume a $\pi/2$ counterclockwise rotation around the 3 axis:

$$x'^{1} = x^{2} \tag{41}$$

$$x'^{2} = -x^{1}. \tag{42}$$

The original equation, in the Dirac-Pauli representation, is expanded as follows:

$$\psi(x) = \begin{pmatrix} \psi_1 \\ \psi_2 \\ \psi_3 \\ \psi_4 \end{pmatrix} \tag{43}$$

$$i\partial_0\psi_1 + i\partial_1\psi_4 + \partial_2\psi_4 + i\partial_3\psi_3 = m\psi_1 \tag{44}$$

$$i\partial_0\psi_2 + i\partial_1\psi_3 - \partial_2\psi_3 - i\partial_3\psi_4 = m\psi_2 \tag{45}$$

$$-i\partial_0\psi_3 - i\partial_1\psi_2 - \partial_2\psi_2 - i\partial_3\psi_1 = m\psi_3 \tag{46}$$

$$-i\partial_0\psi_4 - i\partial_1\psi_1 + \partial_2\psi_1 + i\partial_3\psi_2 = m\psi_4. \tag{47}$$

Under the change of variables (41), (42) we have:

$$
\begin{aligned}
i\partial_0\psi_1 + \partial_1'\psi_4 - i\partial_2'\psi_4 + i\partial_3\psi_3 &= m\psi_1 && (48)\\
i\partial_0\psi_2 - \partial_1'\psi_3 - i\partial_2'\psi_3 - i\partial_3\psi_4 &= m\psi_2 && (49)\\
-i\partial_0\psi_3 - \partial_1'\psi_2 + i\partial_2'\psi_2 - i\partial_3\psi_1 &= m\psi_3 && (50)\\
-i\partial_0\psi_4 + \partial_1'\psi_1 + i\partial_2'\psi_1 + i\partial_3\psi_2 &= m\psi_4, && (51)
\end{aligned}
$$

where $\partial_k' = \partial/\partial x'^k (k = 1,2)$. The equations (48)-(51) have the form of the Dirac equation provided:

$$
\begin{aligned}
\gamma'^1 &= \gamma_{DP}^2 && (52)\\
\gamma'^2 &= -\gamma_{DP}^1. && (53)
\end{aligned}
$$

The solutions of (48)-(51) correspond to the system rotated $\pi/2$ clockwise.

If, besides this, we perform the following change in ψ:

$$\psi_{2,4}' = -i\psi_{2,4}',$$

eqs. (48)-(51) are brought to the form (44)-(47). That is, the old situation is translated to the new system. One can easily show, by constructing systems of basis vectors in the Dirac ring and in the Lie algebra of $SU(4)$, that what is meant by a change of representation of Dirac matrices in dimension four is a change of reference frame in the Lie algebra. This can also be checked in two dimensions for $SU(2)$.

The analysis above can be taken as an expression of the $SU(4)$ properties of the Dirac equation. It is related to what happens with the translation onto the lattice, which we shall consider after dealing further with the algebraic properties of the Dirac ring.

3 Minimal ideals

Given a ring (or algebra) with the operation of multiplication on the left (or commutation), a set of elements of the ring $\mathcal{I}$ (algebra, $\mathcal{A}$) form an ideal if, for any member $r(a)$ of the ring (algebra), we have

$$r\mathcal{I} \subset \mathcal{I} \qquad ([a, \mathcal{A}] \subset \mathcal{A}). \tag{54}$$

A minimal ideal is an ideal which has no subset with this property[21].

The ideals of the Dirac ring have been studied by Corson[22]. In a given representation of gamma matrices, a minimal left ideal is formed by matrices with just one column filled. For example, the matrices

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}. \tag{55}$$

are a basis for a minimal left ideal.

A representation is characterized by the Cartan subalgebra, whose matrices are diagonal. The minimal left ideals are labelled by the eigenvalues of those subalgebra matrices. For the first column, the corresponding matrices are

$$\begin{aligned} d_1 &\equiv \begin{pmatrix} 1 & & & \\ & 1 & & \\ & & -1 & \\ & & & -1 \end{pmatrix}, \quad d_2 \equiv \begin{pmatrix} 1 & & & \\ & -1 & & \\ & & 1 & \\ & & & -1 \end{pmatrix}, \\ d_3 &\equiv \begin{pmatrix} 1 & & & \\ & -1 & & \\ & & -1 & \\ & & & 1 \end{pmatrix}, \end{aligned} \tag{56}$$

and the eigenvalues are $(+++)$. For the second column, the eigenvalues are $(+--)$. They are $(-+-)$ and $(--+)$ for the third and fourth columns, respectively.

It is possible to define projection operators[22] $\mathcal{P}_i^{\mathrm{CS}}$ on the i-th column of the representation in which a Cartan subalgebra (CS) is diagonal. They satisfy the usual properties; they are hermitian and the following relations hold:

$$\sum_{k=1}^{4} \mathcal{P}_k^{(\mathrm{CS})} = 1 \tag{57}$$

$$\mathcal{P}_k^{(\mathrm{CS})} \mathcal{P}_\ell^{(\mathrm{CS})} = \mathcal{P}_k^{(\mathrm{CS})} \delta_{k\ell}. \tag{58}$$

A minimal left ideal in a given representation is of course not minimal in another. To be precise, let us exemplify with

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} = \frac{1}{4}(1 + \gamma^0 + i\gamma^1\gamma^2 + i\gamma^0\gamma^1\gamma^2),$$

in the Pauli-Dirac representation. It becomes, respectively, in the Kramers-Weyl and Majorana representations, into the following matrices:

$$\text{(KW):}\quad \frac{1}{2}\begin{pmatrix} 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\text{(M):}\quad \frac{1}{4}\begin{pmatrix} 1 & -i & -1 & -i \\ i & 1 & -i & 1 \\ -1 & i & 1 & i \\ i & 1 & -i & 1 \end{pmatrix}$$

We can always write a matrix of the Dirac ring in terms of its components in each left ideal. A change of representation leaves its expression invariant,

$$\gamma^H = \sum_k c_k^H L_k^{(\mathrm{CS})}, \tag{59}$$

where $L_k^{(\mathrm{CS})}$ is the k-th minimal left ideal in the representation where a given Cartan subalgebra is diagonal. In a sense, this is an invariant definition of the index H, a natural consequence of implementing the change of representations $A \to B$ through similarity transformations:

$$\gamma^H_{(B)} = S^{-1}(A \to B)\gamma^H_{(A)} S(A \to B). \tag{60}$$

Thus, the coefficients c_k^H of (59) remain the same for a given H, regardless of the representation (and, consequently, of the Cartan subalgebra chosen as diagonal).

Minimal left ideals have been used by Rabin[23] to represent spinors. His trick is to simply put the components of a spinor in the different rows of a given column, for example,

$$\begin{pmatrix} \psi_1 \\ \psi_2 \\ \psi_3 \\ \psi_4 \end{pmatrix} \to \begin{pmatrix} 0 & \psi_1 & 0 & 0 \\ 0 & \psi_2 & 0 & 0 \\ 0 & \psi_3 & 0 & 0 \\ 0 & \psi_4 & 0 & 0 \end{pmatrix}. \tag{61}$$

The procedure is safe from the standpoint that we recover again a minimal left ideal in the new representation, since minimal left ideals are precisely invariant subspaces under left multiplication, if we multiply by a matrix corresponding to a change of representation like (60).

From the point of view of minimal left ideals, the $SU(4)$ invariance of the Dirac equation translates into the fact that any one out of four minimal left ideals may represent a spinor. Since we can label minimal left ideals through the eigenvalue set of the matrices of the Cartan subalgebra, this can be taken as a kind of "flavor" for each ideal. The point we shall demonstrate later is that the lattice transcription mixes ideals which, in the continuum, evolve independently.

As important and relevant as left ideals are right ideals. For right ideals, the definition analogous to (54) is

$$\mathcal{I}r \subset \mathcal{I}. \tag{62}$$

For right ideals, an analogue of expression (59) is valid. One can easily see that the following properties hold:

i) The minimal right ideal represented by a given row of a representation is orthogonal, by matrix multiplication, to any but the corresponding minimal left ideal (column):

$$R_k L_\ell = \omega \delta_{k\ell}, \tag{63}$$

with ω being a complex number.

ii) The left product of a left ideal with a right ideal generates a full 4×4 matrix.

iii) The set of projection operators $\mathcal{P}_k^{(\mathrm{CS})}$ (idempotents) also projects by right multiplication onto the minimal right ideals.

The minimal left ideal is a kind of invariant subspace of the algebra. It can be put in correspondence with the spinor space. Precisely the $SU(4)$ symmetry of the Dirac ring is present in the fact that there are four minimal left ideals which can be put in correspondence to a given spinor. It should be remarked, however, that while for spinors a linear transformation is represented by a matrix operating on a vector, for a minimal left ideal, the transformation proceeds rigorously through similarity, as in (60). (For right ideals, the corresponding vector space would be that of adjoint spinors or row matrices).

To throw more light on the physical meaning of the decomposition in minimal left ideals, we make the following observation. The multiplication table for the matrices in the Cartan subalgebra (see eqs. (56)) is precisely that of the abelian "four group" constructed out of parity and time reversal. This group is abelian, but not cyclic. Since the group is abelian, its irreducible representations are one dimensional and are essentially equal to the characters, $\chi(\mathcal{R}), (\mathcal{R} = \pi, \tau, \pi\tau)$. The set of matrices in a Cartan subalgebra may be called a "reduction group", to use the name given by Becher and Joos[10] (and introduced in Sharatchandra, Thun and Weisz[20]). See Table III. There we immediately recognize that the χ_n are precisely the labels introduced for the minimal left (or right) ideals in the Dirac algebra (Eq. (56)).

A glance at the properties of Dirac bilinears under these operations is now needed to clarify this matter further. As a simple way of understanding what happens, we shall use the behavior of bilinear expressions as can be found, for instance, in Källén's book[3]:

$$F_i^{ab} = [\bar{\psi}_a(x), O_i \psi_b(x)]$$

with

$$\begin{array}{llll} i = & S, & O = & 1 \\ i = & V, & O = & \gamma^\mu \\ i = & T, & O = & \sigma^{\mu\nu} \\ i = & A, & O = & i\gamma^\mu\gamma^5 \\ i = & P, & O = & \gamma^5. \end{array}$$

Table IV reproduces, with some notational changes, the effects of space reflection or parity (π), time reversal (τ), charge conjugation (κ) and their product ($\kappa\pi\tau$).

If we apply these results to the combination that gives the idempotents corresponding to the first column (or minimal left ideal labelled by $(+++)$) we get the results which are summarized in Table V for several representations of the Dirac algebra. The conclusion is that, under discrete operations, there is a mixing of minimal ideals. In other terms, if a spinor is represented by a given left minimal ideal, under these operations it goes into another. This behavior is spectacularly manifest on the lattice, as we shall show in the next section.

To conclude our with analysis, the $SU(4)$ structure apparent for the Dirac ring of matrices is put into evidence through the discrete operations which are incorporated in the Lorentz group as an invariance group for spin-1/2 particles. The possible representations of a spinor through a minimal left ideal of the Dirac algebra were related, as we have shown, to a Cartan subalgebra of the $SU(4)$ Lie algebra, and cannot be considered as equivalent under these operations.

4 Dirac fermions on the lattice

In this section we shall analyze the problem of the proliferation of degrees of freedom in the continuum limit of the lattice treatment of fermions. We shall concentrate mainly on the two-dimensional case, since what is essential to the understanding of the problem already appears at that level. We shall be rather elementary in our exposition. A short version is contained in our previous article[13] and an analogous application for the Kähler treatment of the Dirac equation will be given elsewhere[17]. The elements of the two-dimensional lattice fermion problem are found in the Appendix.

In what we refer to in the Appendix as the Dirac-Pauli representation, the Dirac equation is:

$$\begin{pmatrix} i\partial_0 - m & i\partial_1 \\ -i\partial_1 & -i\partial_0 - m \end{pmatrix} \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} = 0. \tag{64}$$

We write

$$\psi(n_0, n_1) = \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} e^{ikn_1 a - i\omega n_0 a}, \tag{65}$$

where a is the link length of a square bidimensional lattice and n_0, n_1 are the coordinates of a site. Substituting the derivatives by finite differences, we have:

$$\frac{i}{2a}[\psi_1(n_0+1, n_1) - \psi_1(n_0-1, n_1)] - \frac{i}{2a}[\psi_2(n_0, n_1+1) - \psi_2(n_0, n_1-1)]$$
$$= m\psi_1(n_0, n_1) \tag{66}$$

$$-\frac{i}{2a}[\psi_1(n_0, n_1+1) - \psi_1(n_0, n_1-1)] - \frac{i}{2a}[\psi_2(n_0+1, n_1) - \psi_2(n_0-1, n_1)]$$
$$= m\psi_2(n_0, n_1). \tag{67}$$

In terms of the proposed solution, Eq. (65), we find

$$\left(\frac{\sin\omega a}{a} - m\right)c_1 - \frac{\sin ka}{a}c_2 = 0, \tag{68}$$

$$\frac{\sin ka}{a}c_1 - \left(\frac{\sin\omega a}{a} + m\right)c_2 = 0. \tag{69}$$

For the equation to have a solution, it is necessary that

$$\sin\omega a = \pm a\sqrt{\frac{\sin^2 ka}{a^2} + m^2} \tag{70}$$

The solutions corresponding to positive and negative energy are

$$\psi_+(n_0,n_1) = c_+ \begin{pmatrix} 1 \\ \frac{\sin\omega a - ma}{\sin ka} \end{pmatrix} \exp[ikn_1 a - i\omega_+ n_0 a] \tag{71}$$

$$\psi_-(n_0,n_1) = c_- \begin{pmatrix} 1 \\ -\frac{\sin\omega a + ma}{\sin ka} \end{pmatrix} \exp[ikn_1 a + i\omega_- n_0 a]. \tag{72}$$

Let us concentrate on the positive-energy solution (71). We have four continuum limits for the massless case:

$\omega = +k, \quad \omega \approx k \approx 0:$

$$\psi_{+,00}(n_0,n_1) = c_{+,00} \begin{pmatrix} 1 \\ 1 \end{pmatrix} \exp[ikn_1 a - i\omega_+(k)n_0 a] \tag{73}$$

$\omega = -k', \quad k = \frac{\pi}{a} - k':$

$$\psi_{+,0\pi}(n_0,n_1) = c_{+,0\pi} \begin{pmatrix} 1 \\ 1 \end{pmatrix} (-1)^{n_1} \exp[-ik'n_1 a - i\omega_+(k')n_0 a] \tag{74}$$

$\omega = \frac{\pi}{a} - \omega', \quad \omega' \approx k:$

$$\psi_{+,\pi 0}(n_0,n_1) = c_{+,\pi 0} \begin{pmatrix} 1 \\ 1 \end{pmatrix} (-1)^{n_0} \exp[ikn_1 a + i\omega'_+(k)n_0 a] \tag{75}$$

$\omega = \frac{\pi}{a} - \omega', \quad k = \frac{\pi}{a} - k', \quad \omega' = k':$

$$\psi_{+,\pi\pi}(n_0,n_1) = c_{+,\pi\pi} \begin{pmatrix} 1 \\ 1 \end{pmatrix} (-1)^{n_0+n_1} \exp[-ik'n_1 a + i\omega'_+(k')n_0 a]. \tag{76}$$

Notice that (75) and (76) are obtained (up to a phase) from (73) by space inversion, time reversal and both. In principle, all solutions are of positive energy and, of course, positive chirality, since we are in two dimensions and the Hamiltonian of the Dirac equation is proportional to γ^5.

Let us now apply the covariance of the equation under multiplication by any generating matrix of $SU(2)$. To begin with, apply γ^5. We have

$$\begin{pmatrix} i\Delta_0 - m & i\Delta_1 \\ -i\Delta_1 & -i\Delta_0 - m \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} = 0, \tag{77}$$

where Δ_0, Δ_1 represent the finite differences in (66),(67). The compatibility condition is again satisfied but the system of equations (68),(69) now becomes

$$-\frac{\sin ka}{a} c_1 + \left(\frac{\sin \omega a}{a} - m\right) c_2 = 0, \tag{78}$$

$$\left(-\frac{\sin \omega a}{a} - m\right) c_1 + \frac{\sin ka}{a} c_2 = 0. \tag{79}$$

This is identical with (65) if we take $\omega \to -\omega, k \to -k$. In short, we may write

$$\gamma^5 \psi_{+,00}(n_0, n_1) = (-1)^{n_0+n_1} \psi_{+,\pi\pi}(n_0, n_1) \tag{80}$$

Analogously, we obtain:

$$\gamma^1 \psi_{+,00}(n_0, n_1) = (-1)^{n_0} \psi_{+,\pi 0}(n_0, n_1) \tag{81}$$

$$\gamma^0 \psi_{+,00}(n_0, n_1) = (-1)^{n_1} \psi_{+,0\pi}(n_0, n_1). \tag{82}$$

We can then construct Table VI by repeated application of these results. Notice that, though we have used the Dirac-Pauli representation, the table is the same in any representation. We are now able to make the identification of these continuum limits with the ideals of the Dirac ring in the Dirac-Pauli representation. We need the values of the projectors for the minimal left ideals given in the Appendix.

The first component of $\psi_{+,00}$ in the first minimal left ideal is

$$\psi^{(1,1)} = \frac{1}{2}(1 + \gamma^0)\psi_{+,00} = \frac{1}{2}(\psi_{+,00} + (-1)^{n_1} \psi_{+,0\pi}), \tag{83}$$

whereas the second one is

$$\psi^{(1,2)} = -\frac{1}{2}(\gamma^1 - \gamma^5)\psi_{+,00} = -\frac{1}{2}(-1)^{n_0}(\psi_{+,\pi 0} - (-1)^{n_1}\psi_{+,\pi\pi}). \qquad (84)$$

For the second minimal left ideal, we have:

$$\psi^{(2,1)} = \frac{1}{2}(\gamma^1 + \gamma^5)\psi_{+,00} = \frac{1}{2}(-1)^{n_0}(\psi_{+,\pi 0} + (-1)^{n_1}\psi_{+,\pi\pi}), \qquad (85)$$

$$\psi^{(2,2)} = \frac{1}{2}(1 - \gamma^0)\psi_{+,00} = \frac{1}{2}(\psi_{+,00} - (-1)^{n_1}\psi_{+,0\pi}). \qquad (86)$$

We see that the function with values at the origin of the Brillouin zone, eigenfunction of the discretized Hamiltonian at the continuum limit, is a combination of two ideals in the Dirac algebra.

The traditional treatment found, for instance, in the review article by Kogut[5], uses the Kramers-Weyl representation. From that review, we take Figure 1 where, after the spin diagonalization by Kawamoto and Smit[25], the fermions are represented on a lattice by two superimposed lattices on which the degrees of freedom are distributed as seen.

Kogut works in the Euclidean domain, with the following representation for the Dirac matrices

$$\gamma^0 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \sigma^1$$

$$\gamma^1 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} = \sigma^2$$

$$\gamma^5 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = \sigma^3 = -i\gamma^0\gamma^1$$

and shows that the action on the lattice can be written in diagonal form with two fermion "flavors" by defining

$$u_1 = \frac{1}{\sqrt{2}}(\chi(1) - i\chi(1')\gamma^5)\begin{pmatrix} 1 \\ 0 \end{pmatrix} \qquad (87)$$

$$
\begin{aligned}
u_2 &= \frac{1}{\sqrt{2}}(\chi(2)\gamma^0 - \chi(2')\gamma^1)\begin{pmatrix}1\\0\end{pmatrix} && (88)\\
\tilde{d}_1 &= \frac{1}{\sqrt{2}}(\chi(1) + i\chi(1')\gamma^5)\begin{pmatrix}1\\0\end{pmatrix} && (89)\\
\tilde{d}_2 &= \frac{1}{\sqrt{2}}(\chi(2)\gamma^0 + \chi(2')\gamma^1)\begin{pmatrix}1\\0\end{pmatrix}. && (90)
\end{aligned}
$$

The $\chi(k)$ are independent components of a spinor in two dimensions. Recalling the properties of the idempotents (see Appendix), we have, in the continuum limit,

$$
u_1 = \frac{1}{\sqrt{2}}[\psi_{+,00} - i(-1)^{n_0+n_1}\psi_{+,\pi\pi}] \qquad (91)
$$

and equivalent expressions. In the limit, we can identify the continuum limit of the naive fermion treatment with the Kogut-Susskind reduction of fermions on the lattice.

The relationship also translates the content of the "reduction group" of Sharatchandra, Thun and Weisz[20] which is the implementation of Table VI.

In four dimensions, everything goes through in the same manner. To be specific, the basic relations are

$$
\begin{aligned}
\gamma^0\psi_{+,0000} &= (-1)^{n_1+n_2+n_3}\psi_{+,0\pi\pi\pi} && (92)\\
\gamma^1\psi_{+,0000} &= (-1)^{n_0+n_2+n_3}\psi_{+,\pi 0\pi\pi} && (93)\\
\gamma^2\psi_{+,0000} &= (-1)^{n_0+n_1+n_3}\psi_{+,\pi\pi 0\pi} && (94)\\
\gamma^3\psi_{+,0000} &= (-1)^{n_0+n_1+n_2}\psi_{+,\pi\pi\pi 0}. && (95)
\end{aligned}
$$

It requires only industrious work to produce the equivalent of Table VI. Again, everything is linked to [20].

We now have four "flavors", instead of the sixteen corresponding to the naive fermion approach. As Sharatchandra *et al.*[20] have shown, the Kogut-Susskind fermions realize the minimal spreading of spin-1/2 degrees of freedom through the reduction group.

5 Discussion and tentative conclusions

We have tried in this article to support the view that the relationship between a unitary group ($SU(4)$) and the description of spin-1/2 fermions in four dimensions may be of relevance for the understanding of the physical properties of these particles.

For this, we have exhibited in some detail the features of the algebra of Dirac matrices and their products and its connection with the Lie algebra of $SU(4)$. The rôle of the Cartan subalgebra has been shown in the definition of the representation in which the Dirac equation is written and the spinor solutions are obtained.

We have also looked at the purely algebraic properties that allow one to define ideals, their relation to the given representation and the construction of sets of (idempotent) projection operators on them. We have analyzed the link between the Cartan subalgebra and the discrete operations of space reflection, time reversal and their product, which transform idempotents among themselves. We have shown that the meaning of the reduction group of Becher and Joos, and of Sharatchandra, Thun and Weisz is related to the Cartan subalgebras we have considered.

All this seems to be of importance for the understanding of the proliferation of spin-1/2 particles in the continuum limit of massless particles described on the lattice. We have shown this by emphasizing the relation between the Kogut-Susskind treatment of fermions on the lattice and the characterization in terms of ideals of the solutions of the Dirac equation near different edges of the first Brillouin zone. Needless to stress, the "chiral" projection operators on ideals in the "chiral" Kramers-Weyl representation are relevant.

There is undoubtedly further work to be performed.

Our results show a close connection with the description of fermions with differential forms, to be presented in a forthcoming article[17]. The analogy however is not yet complete.

The generalization to higher even dimensions and presumably to any (even or odd) number of dimensions may be considerably helped by the analogy. For instance, the fact that in three dimensions a pair of $SU(2)$ groups is needed for the treatment of fermions may be simply connected to the fact that the eight basic differential forms split naturally in two sets of four related by taking the adjoint. It is easy to enlarge these considerations to other dimensions and work along this line is in progress. As stated before, our conjecture is that for even dimensions the unitary group whose Lie algebra is related to the corresponding Dirac matrices is $SU(2^{n/2})$ whereas for odd dimensions it is $SU(2^{(n-1)/2}) \times SU(2^{(n-1)/2})$.

A main point raised by our study is how crucial it is to understand the relation of spin-1/2 fermions to a unitary group. Is an electron, in the real world, properly speaking, an $SU(4)$ object? This may be an interesting question and we think a hint on it may be provided by the following observation. The square of the operator of time reversal, which is antiunitary, produces a minus sign in the usual treatment of spinors. It is the fourth power which brings one back to the original description. This is reminiscent of the situation of spin-1/2 particles in non-relativistic quantum mechanics. A rotation of 2π generates a minus sign, which was only relatively recently checked experimentally[26], through careful and nice experiments involving slow neutrons.

In the Euclidean regime, a 2π rotation along *any* axis produces a change of sign for spin-1/2 particles. In the same sense that a physical rotation means an incomplete rotation on $SU(2)$, it may be that the minus sign in time reversal in the real world implies that what we need is a complete $SU(4)$ rotation, which, of course, cannot be approximated by (or inside of) $SU(2)$. The matter is complicated since to devise an experiment to eventually check this property must involve a kind of spacetime rotation (a Lorentz transformation, in fact) in Minkowski space. It is possible that the geometric properties uncovered by Gliozzi[9] on the lattice provide a hint for the setting of an appropriate experiment.

We think that several of the results we present under separate guises might be globally phrased in a more sophisticated language. This is perhaps a rewarding task for the future.

Acknowledgments

The authors are indebted to Prof. A. P. Balachandran for discussions and his interest in this work and to Prof. B. V. Carlson for a revision of the manuscript.

Appendix. The Dirac equation in two dimensions

To stick to the conventions in four dimensions, we shall call the Dirac-Pauli representation the one in which γ^0 is diagonal, and the Kramers-Weyl representation the one with γ^5 hermitian and diagonal.

Dirac-Pauli:

$$\gamma^0 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = \sigma_3; \quad \gamma^1 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = i\sigma_2;$$

$$\gamma^5 = \gamma^0\gamma^1 = \sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

Kramers-Weyl:

$$\begin{aligned} \gamma^0 &= \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix} = -\sigma_1; \\ \gamma^1 &= \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = i\sigma^2; \\ \gamma^5 &= \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \end{aligned}$$

The commutator algebra of the hermitian matrices is the one of the generators of $SU(2)$, as is well known. The solution to the Dirac equations are:
Dirac-Pauli:

$$\begin{aligned} \psi^{\pm}(k,x) &= c^{\pm} \begin{pmatrix} 1 \\ -\frac{m-\omega}{k} \end{pmatrix} e^{ikx-i\omega t} \\ \omega &= \pm(k^2+m^2)^{1/2} \end{aligned}$$

Kramers-Weyl:

$$\begin{aligned} \psi^{\pm}(k,x) &= c^{\pm} \begin{pmatrix} 1 \\ -\frac{m}{k+\omega} \end{pmatrix} e^{ikx-i\omega t} \\ \omega &= \pm(k^2+m^2)^{1/2} \end{aligned}$$

Minimal left ideals:

	Dirac-Pauli	*Kramers-Weyl*
$\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$	$\frac{1}{2}(1+\gamma^0)$	$\frac{1}{2}(1+\gamma^5)$
$\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$	$\frac{1}{2}(-\gamma^1+\gamma^5)$	$-\frac{1}{2}(\gamma^0+\gamma^1)$
$\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$	$\frac{1}{2}(\gamma^1+\gamma^5)$	$\frac{1}{2}(-\gamma^0+\gamma^1)$
$\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$	$\frac{1}{2}(1-\gamma^0)$	$\frac{1}{2}(1-\gamma^5)$

A complementary table is the following:

	Dirac-Pauli	*Kramers-Weyl*
$\frac{1}{2}(1+\gamma^0)$	$\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$	$\frac{1}{2}\begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}$
$\frac{1}{2}(-\gamma^1+\gamma^5)$	$\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$	$\frac{1}{2}\begin{pmatrix} 1 & -1 \\ 1 & -1 \end{pmatrix}$
$\frac{1}{2}(\gamma^1+\gamma^5)$	$\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$	$\frac{1}{2}\begin{pmatrix} 1 & 1 \\ -1 & -1 \end{pmatrix}$
$\frac{1}{2}(1-\gamma^0)$	$\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$	$\frac{1}{2}\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$
$\frac{1}{2}(1+\gamma^5)$	$\frac{1}{2}\begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}$	$\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$
$\frac{1}{2}(-\gamma^0+\gamma^1)$	$\frac{1}{2}\begin{pmatrix} -1 & -1 \\ 1 & 1 \end{pmatrix}$	$\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$
$\frac{1}{2}(-\gamma^0+\gamma^1)$	$\frac{1}{2}\begin{pmatrix} -1 & 1 \\ -1 & 1 \end{pmatrix}$	$\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$
$\frac{1}{2}(1-\gamma^5)$	$\frac{1}{2}\begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}$	$\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$

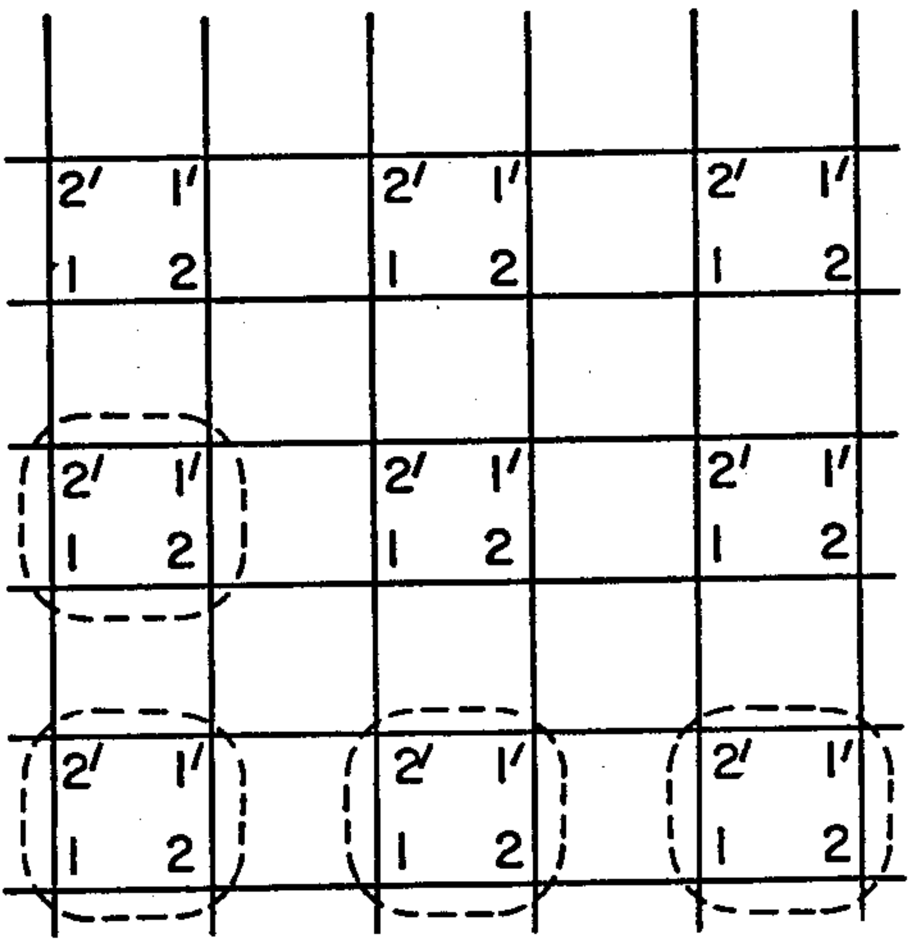

FIGURE 1

Figure 1. Kogut-Susskind fermions on a two-dimensional lattice.

References

[1] P.A.M. Dirac, Proc. Roy. Soc. (London) A117, 610 (1928).

[2] *Present Status and Aims of Quantum Electrodynamics*, Mainz Symposium, edited by G. Graff, E. Klempt and G. Werth, Lecture Notes in Physics 143 (Springer-Verlag, New York, 1981).

[3] See, for instance, P. Roman, *Theory of Elementary Particles* (North Holland, Amsterdam, 1960); J. Leite Lopes, *Inversion Operators in Quantum Field Theory* (Fac. de Ciencias Exactas y Naturales, Buenos Aires, 1960); G. Källén, *Elementary Particle Physics* (Addison-Wesley, Reading, 1965); T.D. Lee and C.S. Wu, Annual Rev. of Nucl. Sci. 15, 381 (1965); 16, 471(1966).

[4] K.G. Wilson, in *New Phenomena in Subnuclear Physics A, Erice, 1975,* edited by A. Zichichi (Plenum Press, New York, 1979).

[5] J.B. Kogut and L. Susskind, Phys. Rev. D 11, 395 (1975); 11, 3594 (1975); L. Susskind, *ibid.* 16, 3031 (1977); J.B. Kogut, Rev. Mod. Phys. 55, 775 (1983).

[6] T. Banks, J.B. Kogut and L. Susskind, Phys. Rev. D 13, 1043 (1976); 15, 1111 (1977).

[7] S.D. Drell, M. Weinstein and S. Yankielowicz, Phys. Rev. D 14, 487 (1976); 14, 1627 (1976).

[8] H.B. Nielsen and N. Ninomiya, Nucl. Phys. B 185, 20 (19781); 193, 173 (1981); 195, 541 (1982)(E); L.H. Karsten, Phys. Lett. 104B, 315 (1981).

[9] F. Gliozzi, Nucl. Phys. B 204, 419 (1982).

[10] P. Becher and H. Joos, Zeits. fur Physik C 15, 343 (1982).

[11] E. Kähler, Abh. Dt. Akad. Wiss. Berlin, Kl. für Math., Phys. u. Tech., Jahrg. 1960, 4(1960); 1(1961); Rendiconti di Matematica (Roma), Ser. V, 21, 425 (1962).

[12] H. Joos and M. Schaefer, Z. Phys. C 34, 465 (1987).

[13] C.A. Linhares and J.A. Mignaco, Phys. Lett. 153B, 82 (1985).

[14] M. Seweryński, preprint 85-66, Inst. Theor. Phys., University of Göteborg (1985). See references for previous mathematical literature in this article.

[15] M. Gourdin, *Basics of Lie Groups* (Frontières, Gif-sur-Yvette, France, 1982).

[16] H. Freudenthal and H. de Vries, *Linear Lie Groups* (Academic, New York, 1969).

[17] C.A. Linhares and J.A. Mignaco, *$SU(4)$ properties of the Dirac-Kähler equation*, in preparation.

[18] C. Itzykson and J.-B. Zuber, *Quantum Field Theory* (McGraw-Hill, New York, 1981).

[19] B. de Witt, *Dynamical Theory of Groups and Fields* (Gordon and Breach, New York, 1965).

[20] H.S. Sharatchandra, H.J. Thun and P. Weisz, Nucl. Phys. B 192, 205 (1981).

[21] M. Naïmark and A. Stern, *Théorie des Representations des Groupes* (MIR, Moscow, 1979); L.S. Pontriaguin, *Grupos Continuos* (MIR, Moscow, 1978).

[22] E.M. Corson, *Introduction to Tensors, Spinors and Relativistic Wave Equations* (Blackie, Glasgow, 1953).

[23] J.M. Rabin, Nucl. Phys. B 201, 315 (1982).

[24] See, for instance, appendix A3 of J. M. Jauch and F. Rohrlich, *The Theory of Photons and Electrons* (Addison-Wesley, Reading, 1955).

[25] N. Kawamoto and J. Smit, Nucl. Phys. B 192, 100 (1981).

[26] H. Rauch, A. Zeilinger, G. Badurek, A. Wilfing, W. Bauspiess and U. Bonse, Phys. Lett. 54A, 425 (1975); S.A. Werner, R. Colella, A.W. Overhauser and C.F. Eagen, Phys. Rev. Lett. 35, 1053 (1975); A. G. Klein and G.I. Opat, *ibid.*, 37, 238 (1976).