Centrally Extended Residual Symmetries in the Presence of a Constant EM Background

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In this talk I discuss the properties of the symmetry algebra (denominated "residual symmetry") of a QFT in the presence of a non-vanishing, constant, electro-magnetic background. It is shown that this problem can be formulated and solved on a purely Lie-algebraic ground (i.e. it is model-independent).

One of its simplest consequences consists in recovering the central extension in the commutator of the translations, associated with the abelian $U(1)$ charge.

For illustrative purposes the case of the $(2+1)$-dimensional Poincaré invariant QFTs coupled with a constant external EM background is discussed in detail. The deformed (and gauge-fixing dependent) surviving Poincaré generators are explicitly computed. In the generic case the residual symmetry algebra is isomorphic to $u(1)\oplus\mathcal{P}_c(2)$, where $\mathcal{P}_c(2)$ is the centrally extended 2-dimensional Poincaré algebra.

The connection with the Noncommutative Field Theories is briefly mentioned. A short discussion concerning the possible physical implications and the outline of the forthcoming research is given.

1. INTRODUCTION

In the present days the issues of Noncommutative Field Theory are vastly explored. Much of the activity on this topic was in consequence of the Seiberg and Witten's observation [1] that a non-commutative gauge theory may be equivalently described by a commuting gauge theory formulated in terms of ordinary (not star) products of a commuting vector potential A_μ, together with an explicit dependence on $\theta^{\alpha\beta}$, which is regarded as a constant background. The equivalence established by Seiberg and Witten motivated much of the following investigations on Field Theories formulated in terms of the Moyal star-product (for more information and references see, e.g., the now available book collecting the Proceedings of the Winter School held here in Karpacz few months ago [2]).

On the other hand, the Seiberg and Witten's observation can inspire a complementary and, so to speak, more "conservative" point of view, namely the investigation of QFT's in a given constant background within the standard commuting space-time framework. This is the viewpoint advocated here.

In this talk I will furnish an answer to the following question: which symmetries survive w.r.t the free case, for a system described by a QFT in a given constant background (which, in practical applications, can be regarded as reached adiabatically). This is why I have employed the notion of "residual symmetry" in the title, even if the term "residual" could be misleading. It appears to suggest that such symmetries form just a subalgebra of the original symmetry algebra. I will prove in the following that this is *not* the case.

My motivation in investigating such a problem came after the work of [3]. Based on some previous results, especially of Cangemi and Jackiw [4,5], the analysis of [3] concerned the very simple, almost trivial, model of a free massive complex boson in $1+1$ dimensions minimally coupled to an external gauge-field.

The main result of [3] consists in the proof that, in the presence of a constant electric field E, the

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Lie algebra of symmetries of the action is the centrally extended Poincaré algebra in $1+1$ dimension. Basically, what happens is that the originally internal $U(1)$ global charge for $E=0$, for $E\neq 0$ appears as a central charge in the commutator of the momenta. Despite the extreme simplicity of the model, the result of [3] is nevertheless noteworthy because it produces another mechanism to produce centrally extended symmetries, besides the known ones based on the arising of anomalies, either quantum [6], or even classically [7]. A systematic analysis of all these features is presented in [8].

On the other hand, the result of [3] concerns a very specific model. In this talk I will prove that the analysis of the residual symmetries and their central extensions can be performed by using solely Lie-algebraic methods, which are model-independent. Due to such a reformulation of the problem it is tempting to say that it is the property of the symmetries themselves which dictates how the residual symmetries are rearranged and the central extensions make their appearance. The generators of a surviving symmetry correspond to a deformation of the original generators.

In consequence of this purely Lie-algebraic investigation, it turns out that the results here obtained have a more general validity (they can be applied, e.g., also to interacting theories). Furthermore, the computations involved are technically simple, which make them easily applicable to more elaborated frameworks, such as standard and higher-dimensional field theories.

In this talk I will limit myself to treat and compute the residual symmetry algebra associated to a Poincaré-invariant theory in $(2+1)$ dimensions coupled with an abelian $U(1)$ external gauge-field, assumed in a constant electric and magnetic background. The choice of this example is just for illustrative purposes. The analysis of the residual symmetries originated by more complicated Lie and super-Lie algebra is currently under investigation (further comments will be furnished in the Conclusions). The mathematical techniques employed in such analysis, however, are straightforward generalizations of the techniques furnished here.

2. STATEMENT OF THE PROBLEM

For pedagogical reasons it is convenient to formulate and solve the problem by working out a specific example. It will be clear, however, that the problem and the method for its solution can be straightforwardly generalized to more complicated algebraic structures than the one here discussed. The case here treated concerns the computation of the residual symmetry for generic Poincaré-invariant field theories in (2+1)-dimension, coupled with an external constant EM background. The two-dimensional case of [3] is recovered by performing a dimensional reduction.

In the absence of the external electric and magnetic field, the action $\mathcal{S}$ is assumed to be invariant under a 7-parameter symmetry, namely the six generators of the $(2+1)$-Poincaré symmetry which, when acting on scalar fields (the following discussion however is valid no matter which is the spin of the fields) are represented by

$$\begin{aligned} P_\mu &= -i\partial_\mu, \\ M_{\mu\nu} &= i(x_\mu\partial_\nu - x_\nu\partial_\mu), \end{aligned} \quad (1)$$

(the metric is chosen to be $+--$). The remaining symmetry generator corresponds to the internal global $U(1)$ charge that will be denoted as Z.

It is further assumed that in the action $\mathcal{S}$ the dependence on the classical background field is expressed in terms of the covariant gauge-derivatives

$$D_\mu \;=\; \partial - ieA_\mu,$$

with e the electric charge.

In the presence of constant external electric and magnetic fields, the $F^{\mu\nu}=\partial^\mu A^\nu - \partial^\nu A^\mu$ field is constrained to satisfy

$$F^{0i}=E^i, \qquad F^{ij}=\epsilon^{ij}B, \quad (2)$$

where $\mu,\nu=0,1,2$ and $i,j=1,2$. The fields E^i and B are constant. Without loss of generality the x^1, x^2 spatial axis can be rotated so that $E^1\equiv E$, $E^2=0$. Throughout the text this convention is respected.

In order to recover (2), the gauge field A_μ must depend at most linearly on the coordinates $x^0\equiv t$, $x^1\equiv x$ and $x^2\equiv y$.

The gauge-transformation

$$A_\mu \mapsto {A_\mu}' = A_\mu + \frac{1}{e}\partial_\mu \alpha(x^\nu) \quad (3)$$

allows to conveniently choose for A_μ the gauge-fixing

$$\begin{aligned} A_0 &= 0, \\ A_i &= E_i t - \frac{B}{2}\epsilon_{ij}x^j. \end{aligned} \quad (4)$$

The above choice is a good gauge-fixing in the sense that it completely fixes the gauge (no gauge-freedom is left). It is easily proven that the discussion which follows is independent of the choice of the gauge-fixing. In particular, the residual symmetry is a true physical symmetry and its symmetry algebra is the same no matter which gauge-fixing has been chosen.

Due to (4), the action $\mathcal{S}$ explicitly depends on the x^μ coordinates entering A_μ. The simplest way to compute the symmetry property of an action such as $\mathcal{S}$ which explicitly depends on the coordinates consists in performing the following trick. At first A_μ is regarded on the same foot as the other fields entering $\mathcal{S}$ and assumed to transform as standard vector field under the global Poincaré transformations, namely

$${A_\mu}'({x^\rho}') = {\Lambda_\mu}^\nu A_\nu(x^\rho) \quad (5)$$

for ${x^\mu}' = {\Lambda^\mu}_\nu + a^\mu$.

For a generic infinitesimal Poincaré transformation, however, the transformed A_μ gauge-field no longer respects the gauge-fixing condition (4). In the active transformation viewpoint only fields are entitled to transform, not the space-time coordinates themselves. A_μ plays the role of a fictitious field, inserted to take into account the dependence of the action $\mathcal{S}$ on the space-time coordinates caused by the non-trivial background. Therefore, the overall infinitesimal transformation δA_μ should be vanishing. This result can be reached if an infinitesimal gauge transformation (3) $\delta_g(A_\mu)$ can be found in order to compensate for the infinitesimal Poincaré transformation $\delta_P(A_\mu)$, i.e.

$$\delta(A_\mu) = \delta_P(A_\mu) + \delta_g(A_\mu) = 0. \quad (6)$$

Only those Poincaré generators which admit a compensating gauge-transformation satisfying the (6) condition provide a symmetry of the $\mathcal{S}$ action (and therefore enter the residual symmetry algebra). This is a plain consequence of the original assumption of the Poincaré and manifest gauge invariance for the action $\mathcal{S}$ coupled to the gauge-field A_μ.

Notice however that the original Poincaré generators are deformed by the presence of extra-terms associated to the compensating gauge transformation. Let p denote a generator of (1) which "survives" as a symmetry in the presence of the external background. The effective generator of the residual symmetry is

$$\hat{p} = p + \ldots,$$

where ... denotes the terms arising from the compensating gauge transformation associated to p. Such extra terms ... are gauge-fixing dependent. The "residual symmetry generator" $\hat{p}$ can only be expressed in a gauge-dependent manner. However, two gauge-fixing choices are related by a gauge transformation $\mathbf{g}$. It can be easily seen that the same residual symmetry generator expressed in the new gauge-fixing and denoted as $\tilde{p}$, is related by an Adjoint transformation

$$\tilde{p} = \mathbf{g}\hat{p}\mathbf{g}^{-1} \quad (7)$$

to the previous one. Therefore the residual symmetry algebra does not dependent on the choice of the gauge fixing and is a truly physical characterization of the action $\mathcal{S}$.

The extra-terms ... are necessarily linear in the space-time coordinates when associated with a translation generator, and bilinear when associated to a surviving Lorentz generator. They are the reason for the arising of the central term in the commutator of the deformed translation generators.

The scheme being clear, it is just a matter of straightforward computation to perform the analysis of the residual symmetries in different contexts and starting from different symmetry algebras. In the next section the results for the residual symmetry of the Poincaré invariance in $(2+1)$ dimensions are furnished.

3. The residual symmetry for the $(2+1)$ Poincaré case.

The framework and the conventions have been illustrated in the previous section, here just the results are quoted. The residual symmetry algebra of the (2+1)-Poincaré theory involves, besides the global $U(1)$ generator Z, the three deformed translations and just one deformed Lorentz generator (the remaining two are broken, not providing a symmetry).

With the (4) gauge-fixing choice the deformed translations are explicitly given by

$$\begin{aligned} P_0 &= -i\partial_t - eEx, \\ P_1 &= -i\partial_x - \frac{e}{2}By, \\ P_2 &= -i\partial_y + \frac{e}{2}Bx. \end{aligned} \tag{8}$$

The deformed generator of the residual Lorentz symmetry is explicitly given, in the same gauge-fixing and for $E \neq 0$, by

$$\begin{aligned} M &= i(x\partial_t + t\partial_x) - i\frac{B}{E}(y\partial_x - x\partial_y) + \\ &\quad \frac{e}{2}(Et^2 + Ex^2 - Bty). \end{aligned} \tag{9}$$

The residual symmetry algebra is given by the following commutation relations

$$\begin{aligned} [P_0, P_1] &= iEZ, \\ [P_0, P_2] &= 0, \\ [P_1, P_2] &= iBZ, \end{aligned} \tag{10}$$

together with

$$\begin{aligned} [M, P_0] &= -iP_1, \\ [M, P_1] &= -iP_0 - i\frac{B}{E}P_2, \\ [M, P_2] &= i\frac{B}{E}P_1. \end{aligned} \tag{11}$$

The $U(1)$ charge Z is no longer decoupled from the other symmetry generators, but it appears in (10) as a central charge.

The residual symmetry algebra of the $(1+1)$ dimensions of ref. [3] is recovered from the P_0, P_1, M, Z subalgebra and corresponds to the centrally extended $2D$ Poincaré algebra thoroughly studied in [5]. This is, however, the residual symmetry algebra for any $(1+1)$-dimensional theory coupled with an external constant electric background, not only the symmetry algebra of the specific model studied in [3].

The 5-generator solvable, non-simple Lie algebra of residual symmetries admits a canonical presentation, obtained by a careful choice of the generators in its presentation.

At first it should be noticed that

$$\tilde{Z} =_{def} BP_0 + EP_2 \tag{12}$$

commutes with all the other $*$ generators

$$[\tilde{Z}, *] = 0, \tag{13}$$

so that the residual symmetry algebra is given by a direct sum of $u(1)$ and a 4-generator algebra. The latter algebra is isomorphic to the centrally extended two-dimensional Poincaré algebra. Such an algebra is of Minkowskian or Euclidean type according to whether $E > B$ or respectively $E < B$ (the case $E = B$ is degenerate). This point can be intuitively understood due to the predominance of the electric or magnetic effect (in the absence of the electric field the theory is manifestly rotational invariant, so that the Lorentz generator is associated with the Euclidean symmetry). We have explicitly, for $B > E$, that the algebra

$$\begin{aligned} [\overline{M}, S_1] &= iS_2, \\ [\overline{M}, S_2] &= -iS_1 \end{aligned} \tag{14}$$

is reproduced by

$$\begin{aligned} \overline{M} &= EM\frac{1}{\sqrt{B^2 - E^2}}, \\ S_1 &= P_0 + \frac{B}{E}P_2, \\ S_2 &= \frac{\sqrt{B^2 - E^2}}{E}P_1, \end{aligned} \tag{15}$$

while for $E > B$ the algebra

$$\begin{aligned} [\overline{M}, T_1] &= iT_2, \\ [\overline{M}, T_2] &= iT_1 \end{aligned} \tag{16}$$

is reproduced by

$$\begin{aligned} \overline{M} &= \frac{1}{\sqrt{E^2 - B^2}}M, \\ T_1 &= P_0 + \frac{B}{E}P_2, \\ T_2 &= -\sqrt{E^2 - B^2}P_1. \end{aligned} \tag{17}$$

In both cases the commutator between the translation generators S_1, S_2, and respectively T_1, T_2, develops the central term proportional to Z which can be conveniently normalized.

The residual symmetry algebra of the $(2+1)$ case for generic values of E and B (the $E=B$ case is degenerate) is therefore given by the direct sum

$$u(1) \oplus \mathcal{P}_c(2). \tag{18}$$

No new residual symmetry algebra is produced starting from the three-dimensional case (and in contrast with the 2-dimensional and ordinary Poincaré cases).

Three central charges are found, namely Z, $\tilde{Z}$ and the order two Casimir of the centrally extended Poincaré algebra (see [5]).

The introduction of a constant EM background implies, in some sort, that for the peculiar three-dimensional case the effective theory corresponds to a field theory in one dimension less (albeit a noncommutative one), with respect to the physical space-time dimension of the formulated problem. Notice that this is true for $(2+1)$, while it is not so in the case of $(1+1)$ dimensions since the unique Lorentz generator "survives" as a symmetry generator. In $D=4$ the residual symmetry algebra admits seven generators (instead of five). Its explicit form will be reported elsewhere.

Some considerations are in order. The above residual symmetry algebra should be regarded on the same foot as the ordinary Poincaré invariance for theories formulated in a constant background. Issues such as the spin-statistics connection in the presence of a non-trivial background could be formulated in terms of the residual symmetry algebra. Furthermore it should be possible, e.g., to discuss the consequences of such symmetries at the level of Feynman diagrams. The notion of residual symmetry could be helpful in analyzing questions like the renormalization property for perturbative field theories in the presence of a given background. A vast range of concrete applications can be found for the realm of residual symmetries.

I conclude this section with two further remarks. The first one concerns the noncommutativity. The noncommutativity is implied by the presence, in the commutator of the deformed translation generators, of a central extension. Such deformed symmetry generators are associated with the corresponding Noether charges, which should develop the same central extensions. In reality we should further check that no extra anomaly is produced by the quantum Noether charges. This analysis, however, requires a detailed investigation of the concrete given model and is less likely being conducted in a model-independent way. To avoid such kind of problems, we can just assume working with *classical* field theories.

The non-commutativity of the momenta is a converse picture with respect to the noncommutativity of the spacetime coordinates which is usually discussed in the current literature. At least in some particular cases, the connection between the two pictures is well-established. In [9] such a connection is explained in detail for the case of a point-particle moving on a plane, in the presence of a strong, perpendicular to the plane, magnetic field and at the lowest Landau level. It should be noticed that this particular case can be recovered by specializing the residual symmetry algebra obtained in this paper and given by the formulas (10) and (11). Due to the rather complete discussion of [9] there is no need here to spend more words on that.

Finally, let me point out another reference, given by [10], in which the noncommutativity of a $(2+1)$ theory (with just Galilean, instead of Poincaré invariance) was discussed.

4. CONCLUSIONS

In this talk I have discussed the issue of the residual symmetry in the presence of a constant electro-magnetic background. I worked out in detail the case of the originally $(2+1)$-invariant Poincaré theory and shown that its residual symmetry is isomorphic to the direct sum

$$u(1) \oplus \mathcal{P}_c(2), \tag{19}$$

the latter being the centrally extended two-dimensional Poincaré algebra. It should be noticed that the $u(1)$ generator *does not* correspond to the $U(1)$ global charge generator Z which, in the presence of a constant background, appears as a central charge in the commutator algebra of the deformed translation generators. Despite the fact that the explicit computations presented regarded a particular algebra, the method employed is quite general and can be applied straightforwardly to any initial symmetry algebra. Such symmetries include Poincaré invariance in any dimension and the conformal transformations, as well as their supersymmetric extensions. Issues related with the partial breaking of extended supersymmetries can be studied in the light of this method. All such cases are currently under investigation and the final results will be reported in a forthcoming paper [11].

The algebra of the residual symmetries, *being not* a subalgebra of the symmetry algebra in the absence of electro-magnetic background, is interesting in itself. It can define for instance some dynamical models. It is tempting for instance to use, for more general residual symmetries, the approach of [4] in which a gravitational theory based on the centrally extended two-dimensional Poincaré algebra is constructed. For what concerns the residual symmetry of the $(2+1)$-dimensional Poincaré invariance, the results of [4] can be immediately borrowed, due to the (19) decomposition of such an algebra. In more complicated cases such as the Minkowskian $(3+1)$-dimensional case, decompositions like (19) no longer apply.

The present talk concerns only the residual symmetries arising from a constant background of an abelian $U(1)$ gauge-invariance. Generalizations can be done in two ways, i.e. considering the residual symmetries for constant background in the presence of higher rank antisymmetric tensor fields. Such backgrounds are relevant in the string/brane context [12].

Conversely, residual symmetries can be studied also for non-constant background. In the presence of a linear homogeneous electric background it is known that a phenomenon of particle production is produced [13]. It is quite tempting to investigate such a problem in the light of the residual symmetries of the model. Investigations in this direction are currently under consideration.

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