Light Deflection on de-Sitter Space

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We study the light deflection on de-Sitter Space.

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The most general covariant second-order equation for the gravitation field generated by a given (covariant) enegy-matter distribution on the space-time is given by the famous Einstein field equation with a cosmological constant Λ [with dimension $(\text{length})^{-2}$ ([1]), namely.

$$\left(R_{\mu\nu}(g) - \frac{1}{2}\, g_{\mu\nu} R - \Lambda g_{\mu\nu}\right)(x) = 8\pi G T_{\mu\nu}(x) \quad (1)$$

where x belongs to a space-time local chart.

It is well-known that studies on the light deflection by a gravitational field generated by a massive point-particle with a pure time-like geodesic trajectory (a rest particle-"sun" for a three-dimensional spatial space-time section observer!) is always carried out by considering $\Lambda \equiv 0$ ([2],[3]).

Our purpose in this short note is to understand the light deflection phenomena in the presence of a non-vanishing cosmological term in Einstein equation (1), at least on a formal mathematical level of solving trajectory motion equations.

Let us, thus, look for a static spherically symmetric solution of eq. (1) in the standard isotropic term ([2], [3])

$$(ds)^2 = B(r)(dt)^2 - A(r)(dr)^2 - r^2\left[(d\theta)^2 - sen^2\theta(d\phi)^2\right] \quad (2)$$

In the space-time region $r = +|\vec{x}|^2 > 0$, where the matter-energy tensor vanishes identically, we have that the Einstein equation takes the following form

$$R_{\mu\nu}(g)(x) = -\Lambda\left(g_{\mu\nu}(x)\right) \quad (3)$$

In the above cited region, the Ricci tensor is given by

$$-\Lambda g_{\mu\nu} = \begin{pmatrix} -\Lambda B(r) & 0 & 0 & 0 \\ 0 & \Lambda A(r) & 0 & 0 \\ 0 & 0 & \Lambda r^2 & 0 \\ 0 & 0 & 0 & \Lambda r^2 sen^2\theta \end{pmatrix} = \begin{pmatrix} R_{tt} & 0 & 0 & 0 \\ 0 & R_{rr} & 0 & 0 \\ 0 & 0 & R_{\theta\theta} & 0 \\ 0 & 0 & 0 & R_{\phi\phi} \end{pmatrix} \quad (4)$$

we have, thus, the following set of ordinary differential equations in place of Einstein Partial Differential eq. (1)

$$R_{tt} = -\frac{B''}{2A}+\frac{1}{4}\left(\frac{B'}{A}\right)\left(\frac{A'}{A}+\frac{B'}{B}\right)-\frac{1}{r}\left(\frac{B}{A}\right)=-\Lambda B \quad (5)$$

$$R_{rr} \equiv \frac{B''}{2B}+\frac{1}{4}\left(\frac{B'}{B}\right)\left(\frac{A'}{A}+\frac{B'}{B}\right)-\frac{1}{2}\left(\frac{A'}{A}\right)=\Lambda A \quad (6)$$

$$R_{\theta\theta} = -1+\frac{r}{2A}\left(-\frac{A'}{A}+\frac{B'}{B}\right)+\frac{1}{A}=\Lambda r^2 \quad (7)$$

$$R_{\phi\phi} = sen^2\theta R_{\theta\theta}=\Lambda(sen^2\theta)r^2 \quad (8)$$

At this point we note that

$$\frac{R_{tt}}{B(r)}+\frac{R_{rr}}{A(r)}=0 \quad (9)$$

or equivalently

$$A(r)=\frac{\alpha}{B(r)} \quad (10)$$

where α is an integration constant.

Since $R_{\theta\theta}=-1+\frac{r}{\alpha}\ B'+\frac{B}{\alpha}=\lambda r^2$, we get the following expression for the $B(r)$ function

$$B(r)=\frac{\alpha\lambda r^2}{3}+\alpha+\frac{\beta}{r} \quad (11)$$

with β denoting another integration constant.

In the literature situation, ([2], [3]), one alwyas consider the case $\Lambda \neq 0$ in a pure classical mathematical vaccum situation context (namely eq. (3) holds true even in the region $r=0$!), the so called de-Sitter vaccum pure gravity. However in our case it becomes physical to consider that our solution depends analitically on the cosmological constant. In other words, if the parameter $\Lambda \to 0$ in our solution, it must converges to the usual Schwarzschild solution with a mass singularity at the origem $r=0$. That is our boundary condition hypothesis imposed on our solution.

As a consequence, one gets our proposed Schwarzschild-de-Sitter solution

$$(ds)^2=\left(\frac{\Lambda r^2}{3}+1-\frac{2MG}{r}\right)(dt)^2-\left(\frac{\Lambda r^2}{3}+1-\frac{2MG}{r}\right)^{-1}(dr)^2-r^2\left[(d\theta)^2+(sen^2\theta)(d\phi)^2\right] \quad (12)$$

At this point let us comment that for the space-time region exterior to the spatial sphere $r>\left(\frac{3mG}{\Lambda}\right)^{1/2}$, the field gravitation approximation leads to the anti-gravity (a respulsion gravity force) or to the Einstein cosmological constant mightmare with the only physical brute force solution $\Lambda=0$ (with $r\to\infty$) and leading to a non compact (spatial) space-time manifold-topological characterization.

In what follows we are going to consider a non-vanishing Λ and study the path a light ray on such ever-expansion Einstein manifold eq. (12).

Note that for $r_0=\left(\frac{6MG}{\Lambda}\right)^{1/3}$ the metric eq. (12) is Minkowiskian.

In the usual case of light propagation, we have the following null-geodesic equation for light propagating in $\theta=\pi/2$ plane (Einstein hypothesis) for light propagation on the presence of the sun (see Appendix for the related formulae) presence

$$0=B(r)-\frac{1}{B(r)}\left(\frac{dr}{dt}\right)^2-r^2\left(\frac{d\phi}{dt}\right)^2 \quad (13)$$

At this point we note that

$$\left(\frac{d\phi}{dt}\right)=\left(\frac{B(r)J}{r^2}\right) \quad (14)$$

where J is a integration constant.

After substituting eq. (14) into eq. (13) we have the following differential equation for the light trajectory as a function of the deflection angle ϕ

$$\left(\frac{dr}{d\phi}\ \frac{d\phi}{dt}\right)^2+\frac{J^3B^3(r)}{r^2}-B^2(r)=0 \quad (15)$$

which is exactly integrable

$$d\phi=\frac{dr}{r^2\sqrt{\frac{1}{J^2}-\frac{B(r)}{r^2}}} \quad (16)$$

By supposing a deflection point r_m where $\frac{dr}{dt}=0$ and, thus, $J=r_m/\sqrt{B(r_m)}$, we get the deflection angle

$$\Delta_1\phi = \int_\infty^{r_m} \frac{dr}{r^2\left[\frac{B(r_m)}{r_m^2} - \frac{B(r)}{r^2}\right]^{1/2}} = \int_0^{\frac{1}{r_m}} \frac{dv}{[(U_m^2 - U^2) - 2MG(U_m^3 - U^3)]^{1/2}} \tag{17}$$

which is exactly that one given in the pure ($\Lambda = 0$) Schwarzschild famous case. However, if one suppose that there is no deflection (a continuous monotone trajectory $r = r(\phi)$!), the total deflection angle now depends on the cosmological constant and is given formally by the expression below.

$$\Delta_2\phi = \int_0^{r_n} dr \left\{ \frac{1}{r^2\sqrt{-\frac{1}{r^2} + \frac{2mG}{r^3}}} \left[\frac{1}{\sqrt{1 + \left[\frac{r^3\left(\frac{3-\Lambda J^2}{3J^2}\right)}{2MG - r}\right]}} \right] \right\} \neq \Delta_1\phi \tag{18}$$

As a general conclusion of our note we claim that a non deflected light trajectory on a cosmological scale indicates that $\Lambda \neq 0$.

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Appendix – The Trajectory Motion Equation

The body trajectory $(t(p), r(p), \theta(p), \varphi(p))$ on the presence of the gravitational field generated by the metric eq. (12) is described by the following geodesic equations

$$\frac{d^2t}{d^2p} + \frac{B'}{B}\left(\frac{dr}{dp}\right)\left(\frac{dt}{dp}\right) = 0 \tag{1}$$

$$\frac{d^2r}{d^2p} + \frac{A'}{2A}\left(\frac{dr}{dp}\right)^2 - \frac{r}{A}\left(\frac{d\theta}{dp}\right)^2 - \frac{r\ sen^2\theta}{A}\left(\frac{d\phi}{dp}\right)^2 + \frac{B'}{2A}\left(\frac{dt}{dp}\right)^2 = 0 \tag{2}$$

$$\frac{d^2\phi}{d^2p} + \frac{2}{r}\frac{d\theta}{dr}\frac{dr}{dp} - sen\theta \cdot cos\theta\left(\frac{d\phi}{dp}\right)^2 = 0 \tag{3}$$

$$\frac{d^2\phi}{d^2p} + \frac{2}{r}\frac{d\phi}{dp}\frac{dr}{dp} + 2cotg(\theta)\frac{d\phi}{dp}\frac{d\theta}{dp} = 0 \tag{4}$$

At this point we remark that by multiplying eq. (A.1) by $B(r(p))$, it reduces to the exactly integral form

$$\frac{dt}{dp} = \frac{1}{B(r)} \tag{5}$$

We remark either that eq. (A.4) can be rewritten in the form

$$\frac{d}{dp}\left(\ell n\frac{d\phi}{dp} + \ell n r^2 + 2\ell n\ sen\theta\right) = 0 \tag{6}$$

which reduces to the following form

$$\left(\frac{d\phi}{dp}\ r^2_{(p)}\ sen(\theta(p))\right) = J \tag{7}$$

where J is a integration constant.

By substituting eq. (A.5) and eq. (A.7) into equations (A.2) and (A.3) we obtain the full set of equations describing the body trajectory in relation to the (r,θ) variables

$$\frac{d^2r}{d^2p} - \frac{B'}{2B}\left(\frac{dr}{dp}\right)^2 - rB\left(\frac{d\theta}{dp}\right)^2 - \frac{J^2B}{r^3} + \frac{B'}{2B} = 0 \tag{8}$$

$$\frac{d^2\theta}{d^2p} + \frac{2}{r}\frac{d\theta}{dp}\frac{dr}{dp} - \frac{cos\theta}{sen\theta}\frac{J^2}{r^4} = 0 \tag{9}$$

For Einstein hypothesis of light propagation on the plane $\theta = \pi/2$, eq. (A.9) vanishes and eq. (A.8) takes the form

$$\frac{d^2r}{d^2p} - \frac{B'}{2B}\left(\frac{dr}{dp}\right)^2 - \frac{J^2B}{r^3} + \frac{B'}{2B} = 0 \tag{10}$$

or in a more manageable alternative form after multiplying eq. (A.10) by $\frac{2}{B}\left(\frac{dr}{dp}\right)$ and by using eq. (A.5) for exchange the geometrical parameter p by the time manifold coordinate

$$\left(\frac{dr}{dt}\right)^2\frac{1}{B^3} + \frac{J^2}{r^2} - \frac{1}{B} + E = 0 \tag{11}$$

where E denotes another integration constant.

By writing r is a function of ϕ and using eq. (A.7) $\left(\frac{d\phi}{dt}\frac{r^2}{B} = J!\right)$, we get our final trajectory equation

$$\frac{dr}{d\phi} = \pm\ r^2\left[\frac{1}{J^2} - \frac{B}{r^2} - \frac{BE}{J^2}\right]^{1/2} \tag{12}$$

which leads to the body trajectory geometric form

$$\phi = \pm\int\frac{dr}{r^2B^{1/2}\left[\frac{1}{J^2B} - \frac{E}{J^2} - \frac{1}{r^2}\right]^{1/2}} \tag{13}$$

Note that for light propagation the integration constant E always vanishes, a result used on the text by means of eq. (16).