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Lucas polynomials and a standard Lax representation for the polytropic gas dynamics

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Abstract

A standard Lax representation for the polytropic gas dynamics is derived by exploiting various properties of the Lucas and Fibonacci polynomials. The two infinite sets of conserved charges are derived from this representation and shown to coincide with the ones derived from the known non-standard representation. The same Lax function is shown to also give the standard Lax description for the elastic medium equations. In addition, some results on possible dispersive extensions of such models are presented.

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1 Introduction:

Systems of hydrodynamic type [1]-[5] have been studied extensively over the last several years. Such systems manifest, among others, in string theories, membrane theories and topological field theories [6]. The polytropic gas equations [7] belong to this general class of systems and are described by

$$u_t + uu_x + v^{\gamma-2}v_x = 0, \qquad v_t + (uv)_x = 0 \tag{1}$$

where u, v denote the two dynamical variables. For different values of the exponent, γ, these equations describe different physical systems, which are dispersionless and integrable [8].

Being dispersionless integrable systems, hydrodynamic systems can be described by a Lax equation on the classical phase space [8]. The Lax description that has been obtained so far for the polytropic gas equations [9]-[10], however, is what is called a non-standard representation, which is not very useful in generalizing this system to other cases, such as the supersymmetric polytropic gas equations [11]. Nevertheless, the non-standard Lax description has been quite useful [9]-[10]. It has led naturally to the two infinite sets of conserved charges of the system (by the standard construction, although the two sets are obtained by expanding the residues at different points). The Lax description also immediately leads to the involution of the charges and clarifies why both the polytropic gas equations and the elastic medium equations share the same set of conserved charges (basically because they are both described by the same Lax function). One can also construct the Hamiltonian structures from this description, although a more convenient construction is through the Moyal-Lax representation of the system [12].

Many integrable systems can have both a standard as well as a non-standard Lax representation and, as we have already mentioned, a standard representation is much more useful. Nonetheless, such a description is lacking so far and, in this paper, we construct such a representation for the polytropic gas equations. Interestingly, such a description involves the use of Fibonacci and Lucas polynomials and their properties [13]. Although a standard Lax representation has unique residues, we show that such a Lax description, nevertheless, leads to the two infinite sets of conserved charges coming from two distinct families of fractional powers of the Lax function. The same Lax function also provides a standard Lax representation for the elastic medium equations [7, 9, 10].

The paper is organized as follows. In section **2**, we review, very briefly, the definitions and some essential properties of the Fibonacci and Lucas polynomials [13]. In section **3**, we derive some identities satisfied by two auxiliary functions dependent on these polynomials. Using these, we construct, in section **4**, the standard Lax representation for the polytropic gas equations. In section **5**, we construct the two infinite sets of conserved charges for this system, which coincide with the earlier known results [9]. The involution of these charges is automatic, since they come from a Lax description. In section **6**, we attempt to construct some dispersive equations, whose dispersionless limit may lead to these equations. This question is extremely difficult and we present only some partial results on this issue. We end with a brief conclusion in section **7**. In the appendix, we show how the same Lax function that leads to a standard Lax description for the polytropic gas equations also describes the elastic medium equations. This also shows that the two systems share the two infinite sets of conserved charges.

2 Definition and properties of the Lucas and Fibonacci polynomials:

The Lucas polynomials are defined recursively as follows:

$$l_{n+1}(x) = x\, l_n(x) + l_{n-1}(x) \tag{2}$$

with $l_0(x) = 2$ and $l_1(x) = x$. Their explicit form for $n \geq 1$ is:

$$l_n(x) = \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} \frac{n}{n-k} \begin{pmatrix} n-k \\ k \end{pmatrix} x^{n-2k} \tag{3}$$

where $\lfloor x \rfloor$ is the largest integer smaller than or equal to x and $\begin{pmatrix} n \\ m \end{pmatrix}$ is the binomial coefficient.

The Fibonacci polynomials are defined by the same recursion relation:

$$f_{n+1}(x) = x\, f_n(x) + f_{n-1}(x) \tag{4}$$

but with $f_0(x) = 0$ and $f_1(x) = 1$. Their explicit form for $n \geq 1$ is:

$$f_n(x) = \sum_{k=0}^{\lfloor \frac{n-1}{2} \rfloor} \begin{pmatrix} n-k-1 \\ k \end{pmatrix} x^{n-2k-1} \tag{5}$$

A good presentation of these two families of polynomials and the many relations involving them can be found in [13]. However, for convenience, we list below some of the relations which we are going to use throughout the paper:

$$l_n(x) = f_{n+1}(x) + f_{n-1}(x) \tag{6}$$

$$(x^2+4)\, f_n(x) = l_{n+1}(x) + l_{n-1}(x) \tag{7}$$

$$l_n^2(x) - (x^2+4)\, f_n^2(x) = 4(-1)^n \tag{8}$$

$$l_n'(x) = n\, f_n(x) \tag{9}$$

3 Various useful identities:

In addition to the Lucas and Fibonacci polynomials, the following two families of polynomials are also useful in deriving some of our results:

$$g_n(x) = \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} \frac{n}{n-k} \begin{pmatrix} n-k \\ k \end{pmatrix} x^k \tag{10}$$

$$h_n(x) = \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} \binom{n-k}{k} x^k \tag{11}$$

If $x \neq 0$, one can easily see from (3) and (5) that:

$$l_n(x) = x^n g_n\left(\frac{1}{x^2}\right) \tag{12}$$

for $n \geq 0$ and:

$$f_n(x) = x^{n-1} h_{n-1}\left(\frac{1}{x^2}\right) \tag{13}$$

for $n \geq 1$. Using this connection with the Lucas and Fibonacci polynomials, we can prove that

Proposition 1 *For any $n \geq 2$, the polynomials g_n and h_n satisfy the following relations:*

(i) $h_n(z) = h_{n-1}(z) + z\, h_{n-2}(z)$

(ii) $g_n(z) = g_{n-1}(z) + z\, g_{n-2}(z)$

(iii) $g_n(z) = h_n(z) + z\, h_{n-2}(z) = h_{n-1}(z) + 2\, z\, h_{n-2}(z)$

(iv) $g_n'(z) = n\, h_{n-2}(z)$

(v) $g_{n+1}(z) - \frac{2}{n+1}\, z\, g_{n+1}'(z) = g_n(z) - \frac{1}{n}\, z\, g_n'(z)$

(vi) $\frac{1}{n+1}\, g_n(z)\, g_{n+1}'(z) - \frac{1}{n}\, g_n'(z)\, g_{n+1}(z) = (-z)^{n-1}$

Proof. To prove (i), we use (13) to rewrite (4) as

$$h_n\left(\frac{1}{x^2}\right) = h_{n-1}\left(\frac{1}{x^2}\right) + \frac{1}{x^2}\, h_{n-2}\left(\frac{1}{x^2}\right)$$

for $x \neq 0$. If we now let $z = x^{-2}$ we find that (i) holds for $z \neq 0$. Since both sides of (i) are polynomial, it follows by continuity that it must also hold at $z = 0$. In a similar way (ii) follows from (2), (iii) follows from (6) and (iv) follows from (9). Let us note from (iii) and (iv) that

$$g_{n+1}(z) - \frac{2}{n+1}\, z\, g_{n+1}'(z) = h_n(z) + 2\; z\, h_{n-1}(z) - 2\; z\, h_{n-1}(z) = h_n(z)$$

as well as

$$g_n(z) - \frac{1}{n}\, z\, g_n'(z) = h_n(z) + z\, h_{n-2}(z) - z\, h_{n-2}(z) = h_n(z)$$

so that (v) holds. Finally, for (vi) we proceed by induction. For $n = 2$ the relation obviously holds. Now suppose that it holds for some n and notice that:

$$\begin{aligned}
&\frac{1}{n+2}\, g_{n+1}(z)\, g_{n+2}'(z) - \frac{1}{n+1}\, g_{n+1}'(z)\, g_{n+2}(z) \\
&= g_{n+1}(z)\, h_n(z) - h_{n-1}(z)\, g_{n+2}(z) \\
&= g_{n+1}(z)\, (h_{n-1}(z) + z\, h_{n-2}(z)) - h_{n-1}(z)\, (g_{n+1}(z) + z\, g_n(z)) \\
&= (-z)\, (h_{n-1}(z)\, g_n(z) - g_{n+1}(z)\, h_{n-2}(z)) \\
&= (-z)\left(\frac{1}{n+1}\, g_{n+1}'(z)\, g_n(z) - \frac{1}{n}\, g_{n+1}(z)\, g_n'(z)\right) \\
&= (-z)^n
\end{aligned}$$

The relation therefore holds for $n+1$, which completes the proof. ■

Let $\phi(z)$ be a function which is analytic in a disc $D = \{z : |z| < r\}$ and let

$$\phi(z) = \sum_{n=0}^{\infty} \gamma_n z^n$$

be its Taylor expansion around $z = 0$. We will denote by $[\phi]_m(z)$ the polynomial consisting of terms in the Taylor expansion of ϕ around $z = 0$ up to z^m. Let $P(z) = a_0 + a_1 z + \ldots + a_n z^n$ be a polynomial. We will denote by $\bar{P}$ the polynomial $\bar{P}(z) = a_n + a_{n-1} z + \ldots + a_0 z^n$ and we define $P_+^{\frac{m}{n}}(z) = z^m \left[\bar{P}^{\frac{m}{n}}\right]_m (z^{-1})$ for any positive integer m. With these notations in place, we can now state the following:

Proposition 2 *The following two relations hold for any $n \geq 2$:*

(i) $\left[g_n^{\frac{n+1}{n}}\right]_{n-1}(z) = \left[g_n^{\frac{n+1}{n}}\right]_{\lfloor\frac{n+1}{2}\rfloor}(z) = g_{n+1}(z)$

(ii) $\left[g_n^{\frac{n-1}{n}}\right]_{n-2}(z) = \left[g_n^{\frac{n-1}{n}}\right]_{\lfloor\frac{n-1}{2}\rfloor}(z) = g_{n-1}(z)$

Proof. For (i) notice that $g_n^{\frac{n+1}{n}}$ solves the differential equation

$$\frac{1}{n+1} g_n(z)\,\phi'(z) - \frac{1}{n} g_n'(z)\,\phi(z) = 0$$

with initial condition $\phi(0) = 1$. From (vi) of Proposition 1 we see that g_{n+1} solves the differential equation

$$\frac{1}{n+1} g_n(z)\,\phi'(z) - \frac{1}{n} g_n'(z)\,\phi(z) = (-z)^{n-1} \tag{14}$$

with initial condition $\phi(0) = 1$. It follows from this that $\psi = g_{n+1} - g_n^{\frac{n+1}{n}}$ solves (14) with initial condition $\phi(0) = \psi(0) = 0$. If we now evaluate both sides in (14) at $z = 0$, with the identification $\phi = \psi$, we get that $\psi'(0) = 0$. Differentiating (14) and evaluating at $z = 0$ (again with the substitution $\phi = \psi$) we get $\psi''(0) = 0$. Repeating the process $n-2$ times we get

$$\psi(0) = \psi^{(1)}(0) = \ldots = \psi^{(n-1)}(0) = 0$$

It follows from this that

$$g_{n+1}(z) = \left[g^{\frac{n+1}{n}}\right]_{n-1}(z)$$

Since g_{n+1} has degree $\lfloor\frac{n+1}{2}\rfloor$ and $\lfloor\frac{n+1}{2}\rfloor \leq n-1$ when $n \geq 2$, it follows that

$$\left[g^{\frac{n+1}{n}}\right]_{n-1}(z) = \left[g^{\frac{n+1}{n}}\right]_{\lfloor\frac{n+1}{2}\rfloor}(z)$$

The proof for (ii) is similar. ■

Proposition 3 *For any integers $n \geq 3$ and $k \geq 0$ let $r_{n,k}$ be the coefficient of z^{kn+2} in the Taylor expansion of $(g_n(z^2))^{k+\frac{1}{n}}$ around $z=0$ and $\tilde{r}_{n,k}$ the coefficient of z^{kn} in the expansion of $(g_n(z^2))^{k-\frac{1}{n}}$. Then,*

$$r_{n,k} = \begin{cases} \frac{(-1)^{nl}}{n^l l!} \prod_{m=l+1}^{2\,l} (m\,n+1), & k = 2\,l \\ \\ 0, & k = 2\,l+1 \end{cases}$$

and

$$\tilde{r}_{n,k} = \begin{cases} \frac{(-1)^{nl}}{n^l l!} \prod_{m=l+1}^{2\,l} (m\,n-1), & k = 2\,l \\ \\ 0, & k = 2\,l+1 \end{cases}$$

Proof. We are only going to prove the formula for $r_{n,k}$, since the proof for $\tilde{r}_{n,k}$ is almost identical.

For $k=0$, it is easy to see (using (iv) of Proposition 1) that

$$r_{n,0} = \frac{d}{dz}\left(g_n^{\frac{1}{n}}(z)\right)_{z=0} = \frac{1}{n}\,n\,h_{n-2}(0)\,\left(g_n^{\frac{1}{n}-1}(0)\right) = 1$$

For $k=1$, $r_{n,1}$ is the coefficient of z^{n+2} in the Taylor expansion of $g_n^{1+\frac{1}{n}}(z^2)$. If n is odd this is obviously 0, because the Taylor series contains only even powers. If n is even, namely $n=2m$, then $r_{2m,1}$ is equal to the coefficient of z^{m+1} in the Taylor series of $g_{2m}^{1+\frac{1}{2m}}(z)$, which is 0 according to Proposition 2, if $n \geq 3$. Hence, for any $n \geq 3$, $r_{n,1} = 0$.

¿From the residue theorem and the fact that $g_n(z^2) = z^n\,l_n(z^{-1})$ it follows that:

$$\begin{aligned} r_{n,k} &= \frac{1}{2\pi i}\int_\Gamma \frac{1}{z^{kn+3}}\left(z^n\,l_n(z^{-1})\right)^{k+\frac{1}{n}} dz \\ &= \frac{1}{2\pi i}\int_\Gamma \frac{1}{z^{(k-1)n+3}}\,l_n(z^{-1})\,\left(z^n\,l_n(z^{-1})\right)^{k-1+\frac{1}{n}} dz \\ &= \frac{1}{2\pi i}\int_\Gamma \frac{1}{z^{(k-2)n+3}}\,l_n^2(z^{-1})\,\left(z^n\,l_n(z^{-1})\right)^{k-2+\frac{1}{n}} dz \end{aligned} \quad (15)$$

where the contour of integration Γ is contained in a neighborhood of $z=0$ where $(g_n(z^2))^{k+\frac{1}{n}}$ is analytic. From (15) we obtain after integrating by parts:

$$\begin{aligned} r_{n,k} &= -\frac{1}{kn+2}\,\frac{1}{2\pi i}\int_\Gamma \frac{d}{dz}\left(\frac{1}{z^{kn+2}}\right)\left(z^n\,l_n(z^{-1})\right)^{k+\frac{1}{n}} dz \\ &= \frac{kn+1}{kn+2}\,\frac{1}{2\pi i}\int_\Gamma \frac{1}{z^{(k-1)n+3}}\left(l_n(z^{-1}) + z^{-1}\,f_n(z^{-1})\right)\left(z^n\,l_n(z^{-1})\right)^{k-1+\frac{1}{n}} dz \\ &= \frac{2kn+2}{kn+2}\,\frac{1}{2\pi i}\int_\Gamma \frac{1}{z^{(k-1)n+3}}\,f_{n-1}(z^{-1})\,\left(z^n\,l_n(z^{-1})\right)^{k-1+\frac{1}{n}} dz \end{aligned}$$

where we have also used (6) and (4). We can rewrite this last result as:

$$\frac{kn+2}{2kn+2}\,r_{n,k} = \frac{1}{2\pi i}\int_\Gamma \frac{1}{z^{(k-1)n+3}}\,f_{n-1}(z^{-1})\,\left(z^n\,l_n(z^{-1})\right)^{k-1+\frac{1}{n}} dz \quad (16)$$

Moreover, it also follows from (15) that

$$\begin{aligned} r_{n,k} &= \frac{1}{2\pi i} \int_\Gamma \frac{1}{z^{(k-1)n+3}}\, l_n(z^{-1}) \, \left(z^n\, l_n(z^{-1})\right)^{k-1+\frac{1}{n}} \, dz \\ &= \frac{1}{2\pi i} \int_\Gamma \frac{1}{z^{(k-1)n+3}} \, \left(f_{n+1}(z^{-1}) + f_{n-1}(z^{-1})\right) \, \left(z^n\, l_n(z^{-1})\right)^{k-1+\frac{1}{n}} \, dz \\ &= \frac{kn+2}{2kn+2}\, r_{n,k} + \frac{1}{2\pi i} \int_\Gamma \frac{1}{z^{(k-1)n+3}}\, f_{n+1}(z^{-1}) \, \left(z^n\, l_n(z^{-1})\right)^{k-1+\frac{1}{n}} \, dz \end{aligned}$$

and, therefore, we have

$$\frac{kn}{2kn+2}\, r_{n,k} = \frac{1}{2\pi i} \int_\Gamma \frac{1}{z^{(k-1)n+3}}\, f_{n+1}(z^{-1}) \, \left(z^n\, l_n(z^{-1})\right)^{k-1+\frac{1}{n}} \, dz \qquad (17)$$

Now if we multiply (16) by $n-1$ and (17) by $n+1$ and add them we get:

$$\begin{aligned} &\frac{kn^2+n-1}{kn+1}\, r_{n,k} = \frac{1}{2\pi i} \int_\Gamma \frac{1}{z^{(k-1)n+3}} \, \left((n+1)\, f_{n+1}(z^{-1}) + (n-1)\, f_{n-1}(z^{-1})\right) \times \\ &\times \left(z^n\, l_n(z^{-1})\right)^{k-1+\frac{1}{n}} \, dz \\ &= \frac{1}{2\pi i} \int_\Gamma \frac{1}{z^{(k-1)n+1}} \, \frac{d}{dz} \left(l_{n+1}(z^{-1}) + l_{n-1}(z^{-1})\right) \, \left(z^n\, l_n(z^{-1})\right)^{k-1+\frac{1}{n}} \, dz \\ &= -\frac{1}{2\pi i} \int_\Gamma \left(l_{n+1}(z^{-1}) + l_{n-1}(z^{-1})\right) \, \frac{d}{dz} \left(\frac{1}{z^{(k-1)n+1}} \, \left(z^n\, l_n(z^{-1})\right)^{k-1+\frac{1}{n}}\right) \, dz \\ &= -((k-1)n+1)\, \frac{1}{2\pi i} \int_\Gamma \frac{1}{z^{(k-1)n+2}} \, \left(z^{-2}+4\right) \, f_n(z^{-1}) \, \left(z^n\, l_n(z^{-1})\right)^{k-1+\frac{1}{n}} \, dz \\ &+ ((k-1)n+1)\, \frac{1}{2\pi i} \int_\Gamma \frac{1}{z^{(k-2)n+2}} \, \left(z^{-2}+4\right) \, f_n(z^{-1})\, l_n(z^{-1}) \, \left(z^n\, l_n(z^{-1})\right)^{k-2+\frac{1}{n}} \, dz \\ &- ((k-1)n+1)\, \frac{1}{2\pi i} \int_\Gamma \frac{1}{z^{(k-2)n+3}} \, \left(z^{-2}+4\right) \, f_n^2(z^{-1}) \, \left(z^n\, l_n(z^{-1})\right)^{k-2+\frac{1}{n}} \, dz \end{aligned}$$

so that

$$\begin{aligned} \frac{kn^2+n-1}{(kn+1)((k-1)n+1)}\, r_{n,k} &= -\frac{1}{2\pi i} \int_\Gamma \frac{1}{z^{(k-2)n+3}} \, \left(z^{-2}+4\right) \, f_n^2(z^{-1}) \\ &= \times \left(z^n\, l_n(z^{-1})\right)^{k-2+\frac{1}{n}} \, dz \end{aligned} \qquad (18)$$

Finally, adding (17) and (18) we get:

$$\begin{aligned} 4\,(-1)^n\, r_{n,k-2} &= \frac{1}{2\pi i} \int_\Gamma \frac{1}{z^{(k-2)n+3}} \, \left[l_n(z^{-1}) - \left(z^{-2}+4\right) \, f_n^2(z^{-1})\right] \\ &\quad \times \left(z^n\, l_n(z^{-1})\right)^{k-2+\frac{1}{n}} \, dz \\ &= \frac{kn\,(kn+2)}{(kn+1)\,((k-1)n+1)}\, r_{n,k} \end{aligned}$$

In other words:

$$r_{n,k} = 4\,(-1)^n\, \frac{(kn+1)\,((k-1)n+1)}{kn\,(kn+2)}\, r_{n,k-2} \qquad (19)$$

and the desired result follows by induction. ∎

4 The Lax function and the Lax equation:

In this section we show that, for $n \geq 3$, the Lax function

$$L_n = \frac{(\sqrt{-v})^n}{n}\, l_n\left(\frac{n^{\frac{1}{n}}\, p}{\sqrt{-v}}\right) + u \tag{20}$$

provides a standard Lax description for the polytropic gas with parameter $\gamma = n+1$, namely

$$\frac{\partial L_n}{\partial t} = \frac{n}{n+1}\left\{\left(L_n^{1+\frac{1}{n}}\right)_+ ,\, L_n\right\} \tag{21}$$

where $()_+$ denotes the part of the polynomial that contains only non-negative powers of the variable, leads to the dynamical equations of motion

$$u_t + u\, u_x + v^{\gamma-2}\, v_x = 0 \tag{22}$$

$$v_t + (u\, v)_x = 0 \tag{23}$$

which are the polytropic gas equations.

First, let us introduce the following simplifying notations:

$$w = -v, \quad \pi_n = n^{\frac{1}{n}}\, p, \quad \tilde{L}_n = L_n - u$$
$$\{A, B\}_n = \frac{\partial A}{\partial w}\frac{\partial B}{\partial \pi_n} - \frac{\partial A}{\partial \pi_n}\frac{\partial B}{\partial w} = \left(n^{\frac{1}{n}}\, w_x\right)^{-1}\{A, B\}$$

where $\{A, B\}$ denotes the conventional Poisson bracket of A, B. With these notations, we can now rewrite (20) as

$$L_n = \frac{1}{n}\left(\sqrt{w}\right)^n l_n\left(\left(\sqrt{w}\right)^{-1}\pi_n\right) + u = \frac{1}{n}\,\pi_n^n\, g_n\left(w\,\pi_n^{-2}\right) + u \tag{24}$$

¿From this we see that

$$\begin{aligned}
\left(L_n^{1+\frac{1}{n}}\right)_+ &= \left(\frac{1}{n}\right)^{1+\frac{1}{n}} \pi_n^{n+1}\left[\left(g_n(w\,z^2) + n\,u\,z^n\right)^{1+\frac{1}{n}}\right]_{n+1}(z = \pi_n^{-1}) \\
&= \left(\frac{1}{n}\right)^{1+\frac{1}{n}} \pi_n^{n+1}\left(\left[\left(g_n(w\,z^2)\right)^{1+\frac{1}{n}}\right]_{n+1}(z = \pi_n^{-1}) + (n+1)\,u\,\pi_n^{-n}\right) \\
&= \left(\frac{1}{n}\right)^{1+\frac{1}{n}} \pi_n^{n+1}\left(\left[(g_n(z))^{1+\frac{1}{n}}\right]_{\lfloor\frac{n+1}{2}\rfloor}(z = w\,\pi_n^{-2}) + (n+1)\,u\,\pi_n^{-n}\right) \\
&= \left(\frac{1}{n}\right)^{1+\frac{1}{n}} \pi_n^{n+1}\, g_{n+1}\left(w\,\pi_n^{-2}\right) + (n+1)\left(\frac{1}{n}\right)^{1+\frac{1}{n}} u\,\pi_n \\
&= \left(\tilde{L}_n^{1+\frac{1}{n}}\right)_+ + (n+1)\left(\frac{1}{n}\right)^{1+\frac{1}{n}} u\,\pi_n \\
&= \left(\tilde{L}_n^{1+\frac{1}{n}}\right)_+ + \frac{n+1}{n}\,u\,p
\end{aligned} \tag{25}$$

where we have used Proposition 2. Using the above relation, the Poisson bracket on the right hand side of the Lax equation (21) becomes:

$$\begin{aligned}\left\{\left(L_n^{1+\frac{1}{n}}\right)_+,\ L_n\right\} &= \left\{\left(\tilde{L}_n^{1+\frac{1}{n}}\right)_+,\ \tilde{L}_n\right\} - u_x\,\frac{\partial}{\partial p}\,\left(\tilde{L}_n^{1+\frac{1}{n}}\right)_+ \\ &\quad + \frac{n+1}{n}\left(u_x\,p\,\frac{\partial\,\tilde{L}_n}{\partial\,p} - u\,\frac{\partial\,\tilde{L}_n}{\partial\,x} - u\,u_x\right)\end{aligned} \tag{26}$$

Now, with (v) of Proposition 1 we obtain:

$$\begin{aligned}\frac{\partial}{\partial\,p}\left(\tilde{L}_n^{1+\frac{1}{n}}\right)_+ &= \frac{\partial\,\pi_n}{\partial\,p}\,\frac{\partial}{\partial\,\pi_n}\left(n^{-\left(1+\frac{1}{n}\right)}\,\pi_n^{n+1}\,g_{n+1}(w\,\pi_n^{-2})\right) \\ &= \frac{n+1}{n}\,\pi_n^n\left(g_{n+1}(w\,\pi_n^{-2}) - \frac{2}{n+1}\,(w\,\pi_n^{-2})\,g'_{n+1}(w\,\pi_n^{-2})\right) \\ &= \frac{n+1}{2\,n}\left(2\,\pi_n^n\,g_n(w\,\pi_n^{-2}) - \frac{2}{n}\,w\,\pi_n^{n-2}\,g'_n(w\,\pi_n^{-2})\right) \\ &= \frac{n+1}{2\,n}\left(\pi_n\,\frac{\partial\,\tilde{L}_n}{\partial\,\pi_n} + n\,\tilde{L}_n\right) \\ &= \frac{n+1}{2\,n}\left(p\,\frac{\partial\,\tilde{L}_n}{\partial\,p} + n\,\tilde{L}_n\right)\end{aligned} \tag{27}$$

Furthermore, using (vi) of the same proposition we notice that:

$$\begin{aligned}\left\{\left(\tilde{L}_n^{1+\frac{1}{n}}\right)_+,\ \tilde{L}_n\right\} &= \left(n^{\frac{1}{n}}\,w_x\right)\left\{\left(\tilde{L}_n^{1+\frac{1}{n}}\right)_+,\ \tilde{L}_n\right\}_n \\ &= n^{-2}\,w_x\left\{\pi_n^{n+1}\,g_{n+1}(w\,\pi_n^{-2}),\ \pi_n^n\,g_n(w\,\pi_n^{-2})\right\}_n \\ &= \frac{n+1}{n}\,w_x\,\pi_n^{2\,n-2}\left(\frac{1}{n+1}\,g'_{n+1}(w\,\pi_n^{-2})\,g_n(w\,\pi_n^{-2}) - \right. \\ &\quad \left. - \frac{1}{n}\,g_{n+1}(w\,\pi_n^{-2})\,g'_n(w\,\pi_n^{-2})\right) \\ &= -\frac{n+1}{n}\,v^{n-1}\,v_x\end{aligned} \tag{28}$$

Putting together (26), (27) and (28) we see that the Lax equation (21) is equivalent to

$$\frac{\partial\,L_n}{\partial\,t} = \frac{1}{2}\left(p\,\frac{\partial\,L_n}{\partial\,p} - n\,(L_n - u)\right)\,u_x - \frac{\partial\,L_n}{\partial\,x}\,u - v^{n-1}\,v_x \tag{29}$$

Equating coefficients of powers of p in (29) we see that it holds if and only if the equations of motion (22) and (23) hold.

This shows that the Lax function in (20) does provide a standard Lax description for the polytropic gas dynamics. The same Lax function also provides a standard Lax description for the elastic medium equations, which we show in the appendix. For $\gamma = 3\,(n = 2)$, the polytropic gas equations, under a redefinition of variables, is known to describe two decoupled Riemann equations. This case is slightly tricky, since in this case, both the dynamical variables, u, v have the same dimension. We describe the Lax description for this separately, which does not fall into the above category. Consider the Lax function

$$L = p^2 + \frac{2}{3}u + \frac{1}{6}v^2 p^{-2} \tag{30}$$

The presence of the p^{-2} gives it a different character from the earlier construction. However, it is straightforward to check that the Lax equation

$$\frac{\partial L}{\partial t} = \left\{ \left(L^{\frac{3}{2}} \right)_+ , L \right\} \tag{31}$$

leads to

$$\begin{aligned} u_t &= -uu_x - vv_x \\ v_t &= -(uv)_x \end{aligned} \tag{32}$$

which are the polytropic gas equations for $\gamma = 3$. The two sets of conserved charges, in this case, are obtained from the two possible ways of defining the residues of $L^{n+\frac{1}{2}}$ around $p = 0, \infty$ respectively. We also note here that, for the case $\gamma = 2\,(n = 1)$, the polytropic gas equations only have a nonstandard Lax description.

5 Conserved charges:

A Lax description of an integrable system has the advantage that the conserved charges can be obtained from residues of fractional powers of the Lax function. The polytropic gas, on the other hand, is known to have two infinite sets of conserved charges [9]. In the non-standard Lax description of the polytropic gas, it is known that the two sets of charges arise naturally from calculating the residues around two distinct points. In a standard Lax description, however, the residues are unique. It is interesting, therefore, to see how the two sets of conserved charges will arise in this description. Let us note that a simple dimensional analysis shows that we can assign the dimensions $[v] = 2, [u] = n$ so that $[L_n] = n$. From the explicit forms of the two sets of known conserved charges [9], it is clear that if they are obtained from our Lax function at all, they should arise from the fractional powers $k + \frac{1}{n}$ and $k - \frac{1}{n}$ respectively, where $k = 0, 1, \cdots$ for the first set while $k = 1, 2, \cdots$ for the second set.

Let $H_{n,k} = \mathrm{Res}\left(L_n^{k+\frac{1}{n}}\right)$ and $F_{n,k}(u, w, z) = (g_n(w\, z^2) + n\, u\, z^n)^{k+\frac{1}{n}}$. Let $\rho_{n,k}(u, w)$ be the coefficient of z^{kn+2} in the Taylor expansion of $F_{n,k}$ around $z = 0$. Then:

$$H_{n,k} = n^{-(k+\frac{2}{n})} \rho_{n,k}(u, w) \tag{33}$$

Since

$$\frac{\partial F_{n,k}}{\partial u}(u, w, z) = (k\, n + 1)\, z^n\, F_{n,k-1}(u, w, z) \tag{34}$$

it follows that

$$\frac{\partial \rho_{n,k}}{\partial u}(u,w) = \begin{cases} 0 & \text{if } k = 0 \\ (k\,n+1)\rho_{n,k-1}(u,w) & \text{if } k > 0 \end{cases} \tag{35}$$

and, after repeated integration:

$$\rho_{n,k}(u,w) = \sum_{m=0}^{k} \frac{1}{m!} \left(\prod_{l=k-m+1}^{k} l\,n+1 \right) \rho_{n,k-m}(0,w)\, u^m \tag{36}$$

Using the fact that $\rho_{n,2m}(0,w) = r_{n,2m}\, w^{m\,n+1}$ in conjunction with Proposition 3 we get:

$$\rho_{n,k}(u,w) = \sum_{m=0}^{\lfloor \frac{k}{2} \rfloor} \frac{(-1)^{m\,n}}{m!\,(k-2\,m)!\,n^m} \left(\prod_{l=m+1}^{k} l\,n+1 \right) w^{mn+1}\, u^{k-2m} \tag{37}$$

and therefore

$$H_{n,k} = -n^{-(k+\frac{2}{n})} \prod_{s=0}^{k} (s\,n+1) \sum_{m=0}^{\lfloor \frac{k}{2} \rfloor} \frac{1}{m!\,(k-2\,m)!\,n^m} \prod_{l=0}^{m} \frac{1}{l\,n+1}\, v^{mn+1} u^{k-2m} \tag{38}$$

Similarly, if $\tilde{H}_{n,k} = \text{Res}\left(L_n^{k-\frac{1}{n}} \right)$ then we can show that

$$\tilde{H}_{n,k} = n^{-k} \prod_{s=0}^{k} (s\,n-1) \sum_{m=0}^{\lfloor \frac{k}{2} \rfloor} \frac{1}{m!\,(k-2\,m)!\,n^m} \prod_{l=0}^{m} \frac{1}{l\,n-1}\, v^{mn}\, u^{k-2m} \tag{39}$$

The two sets of conserved charges simply correspond to $\int H_{n,k}\,dx$ and $\int \tilde{H}_{n,k}\,dx$ and coincide with the known conserved charges constructed earlier from a nonstandard Lax representation upto an overall normalization. This construction shows that the two sets of conserved charges can be obtained from the residues of two distinct families of fractional powers of the Lax function in this standard description.

6 Generalization to dispersive cases:

It is an interesting question to ask which dispersive integrable models reduce in the dispersionless limit to the polytropic gas dynamics. It is, of course, well known that the two boson equation, in the dispersionless limit, goes to the polytropic gas equation for $n = 1\,(\gamma = 2)$ and that, for $n = 2\,(\gamma = 3)$, the polytropic gas equation is equivalent to two decoupled Riemann equations which can be thought of as the dispersionless limit of the KdV equation. It is, of course, clear that there may be several dispersive models whose dispersionless limit will give the same equation. However, our interest is to find even one family of such models. Surprisingly, beyond $n = 4$, we have not found any dispersive generalization of these systems.

It is well known that the polytropic gas equations for $\gamma = 2\,(n = 1)$ can be thought of as the dispersionless limit of the two boson equation, which is described by the Lax operator

$$L = \partial - u + \partial^{-1} v \tag{40}$$

and the nonstandard Lax equation

$$\frac{\partial L}{\partial t} = \frac{1}{2}\left[\left(L^2\right)_{\geq 1}, L\right] \tag{41}$$

We will not discuss this system any further.

The case $n = 2$ is tricky even at the dispersionless limit, as we have already pointed out. We note that the Riemann equation is well understood to be the dispersionless limit of the KdV equation, whose Lax description is one of the first to have been studied [14]. However, for $n = 2$, we have two decoupled Riemann equations in the dispersionless limit. The dispersive generalization of this system is not at all clear. However, with some work, we have found that the Lax operator

$$L = \partial^2 + \frac{1}{2}u + \frac{1}{2}\partial^{-1} u \partial + \frac{1}{4}\partial^{-1} v \partial^{-1} v \tag{42}$$

through the nonstandard Lax equation

$$\frac{\partial L}{\partial t} = -\frac{2}{3}\left[\left(L^{\frac{3}{2}}\right)_{\geq 1}, L\right] \tag{43}$$

leads to

$$\begin{aligned} u_t &= -\frac{2}{3}u_{xxx} - uu_x - vv_x \\ v_t &= -\frac{2}{3}v_{xxx} - (uv)_x \end{aligned} \tag{44}$$

With a simple redefinition of variables to $u \pm v$, this becomes two decoupled KdV equations, which, therefore, provides a trivial dispersive generalization of the polytropic gas equations for $\gamma = 3$ (or $n = 2$). It is worth pointing out here that this Lax operator also gives through

$$\frac{\partial L}{\partial t} = -\left[\left(L^{\frac{1}{2}}\right)_{\geq 1}, L\right] \tag{45}$$

leads to

$$u_t = -u_x, \qquad v_t = -v_x \tag{46}$$

which are the elastic medium equations for $\gamma = 3$. Under this description, they do not seem to pick up any dispersive terms.

For $\gamma = 4\,(n = 3)$, we expect the dispersive generalization to be related to the Boussinesq hierarchy, simply from the counting of dimensions. In fact, if we choose

$$L = \partial^3 + v\partial + u \tag{47}$$

the standard Lax equation

$$\frac{\partial L}{\partial t} = \left[\left(L^{\frac{4}{3}}\right)_+, L\right] \tag{48}$$

leads to

$$\begin{aligned} u_t &= \frac{1}{9}\left(3u_{xxxx} - 2v_{xxxxx} - 6v_{xxx}v - 12v_{xx}v_x + 6(vu_x)_x + 12uu_x - -4v^2v_x\right) \\ v_t &= \frac{1}{3}\left(2u_{xxx} - v_{xxxx} - (v^2)_{xx} + 4(uv)_x\right) \end{aligned} \tag{49}$$

These clearly provide a dispersive generalization of the polytropic gas equations for $\gamma = 4$. These equations are known to admit the W_3 algebra as one of the Hamiltonian structures.

Let us further note that the Lax equation

$$\frac{\partial L}{\partial t} = \left[\left(L^{\frac{2}{3}}\right)_+, L\right] \tag{50}$$

leads to the set of equations

$$\begin{aligned} u_t &= \frac{1}{3}\left(3u_{xx} - 2v_{xxx} - 2vv_x\right) \\ v_t &= -v_{xx} + 2u_x \end{aligned} \tag{51}$$

which provides a dispersive generalization of the elastic medium equations for $\gamma = 4$. In general, we note that the equations

$$\frac{\partial L}{\partial t} = \left[\left(L^{\frac{k}{3}}\right)_+, L\right] \tag{52}$$

with $k = 3m \pm 1$ defines consistent equations corresponding to the two different hierarchies.

For $\gamma = 5\,(n = 4)$, consider the Lax operator [15]

$$L = \partial^4 + v\partial^2 + v_x\partial + u \tag{53}$$

This leads to consistent equations through

$$\frac{\partial L}{\partial t} = \left[\left(L^{\frac{k}{4}}\right)_+, L\right] \tag{54}$$

for $k = 4m \pm 1$. Thus, for $k = 5$, the dynamical equations turn out to be

$$\begin{aligned} u_t &= \frac{1}{32}\Big(12u_{xxxxx} - 5v_{xxxxxxx} - 5(2vv_{xxxx} + 4v_xv_{xxx} + 3v_{xx}^2)_x + 20(u_xv)_{xx} \\ &\quad + 40uu_x - 5(v^2v_{xx} + vv_x^2)_x + 5v^2u_x\Big) \\ v_t &= \frac{1}{32}\left(40u_{xxx} - 18v_{xxxxx} - 15(2vv_{xx} + v_x^2)_x + 40(uv)_x - 15v^2v_x\right) \end{aligned} \tag{55}$$

With a simple change of variables, $\tilde{u} = u - \frac{1}{8}v^2, \tilde{v} = v$, it is easy to check that these equations reduce, in the dispersionless limit, to the polytropic gas equations with $\gamma = 5$.

Therefore, this model provides a dispersive generalization of these equations. It is worth noting here that this system of equations admits as a Hamiltonian structure (after suitable redefinition of fields) the nonlinear W algebra, $W(2,4)$, which is uniquely characterized by the presence of a spin 2 Virasoro field as well as a spin 4 primary field.

Similarly, it can be easily checked that, for $k=3$, the Lax equation leads to

$$\begin{aligned} u_t &= u_{xxx} - \frac{3}{8} v_{xxxxx} - \frac{3}{8}(v v_{xx})_x + \frac{3}{4} u_x v \\ v_t &= -\frac{5}{4} v_{xxx} + 3u_x - \frac{3}{4} v v_x \end{aligned} \tag{56}$$

which, under the same redefinition of variables, goes over to the elastic medium equations for $\gamma = 5$ in the dispersionless limit. This system of equations, therefore, gives a dispersive generalization of both these systems.

For $\gamma = 6\,(n=5)$ as well as $\gamma = 7\,(n=6)$, we have explicitly verified that there is no standard Lax equation that leads to consistent equations with two dynamical variables. For higher values of n (and, therefore, γ), it is unlikely that a standard Lax description would lead to consistent equations with two dynamical fields, since the number of consistency conditions increases rapidly. However, we have not checked this explicitly beyond $n=6$. The dispersive generalization of the polytropic gas equations for higher values of γ, therefore, remains an open question. It is possible that they arise only as nonstandard equations or that one may have to introduce additional dynamical variables, which, somehow, disappear in the dispersionless limit.

7 Conclusion:

We have derived a standard Lax description for the polytropic gas dynamics. The Lax function, in this case, is intimately connected with Lucas polynomials, which are also related to the Fibonacci polynomials. The two infinite sets of conserved charges have been obtained from the residues of two distinct sequences of fractional powers of the Lax function. We have shown that the same Lax function also provides a standard Lax description for the elastic medium equations. In addition, we have presented some results on possible dispersive generalizations of such systems.

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A Standard Lax description for elastic medium equations:

In this appendix, we show how the same Lax function of (20) leads to a standard Lax description for the elastic medium equations [7],9.

$$\left(L_n^{1-\frac{1}{n}}\right)_+ \;=\; n^{\frac{1}{n}-1}\,\pi_n^{n-1}\,\left[\left(g_n(w\,z^2) + n\,u\,z^n\right)^{1-\frac{1}{n}}\right]_{n-1} (z = \pi_n^{-1})$$

$$
\begin{aligned}
&= n^{\frac{1}{n}-1}\,\pi_n^{n-1}\,\left[\left(g_n(w\,z^2)\right)^{1-\frac{1}{n}}\right]_{n-1}\,(z=\pi_n^{-1})\\
&= n^{\frac{1}{n}-1}\,\pi_n^{n-1}\,\left[\left(g_n(z)\right)^{1-\frac{1}{n}}\right]_{\lfloor\frac{n-1}{2}\rfloor}\,(z=w\,\pi_n^{-2})\\
&= n^{\frac{1}{n}-1}\,\pi_n^{n-1}\,g_{n-1}(w\,\pi_n^{-2}) = \left(\tilde{L}_n^{1-\frac{1}{n}}\right)_+
\end{aligned}
$$

$$
\left\{\left(L_n^{1-\frac{1}{n}}\right)_+,\ L_n\right\} = \left\{\left(\tilde{L}_n^{1-\frac{1}{n}}\right)_+,\ \tilde{L}_n\right\} - u_x\,\frac{\partial}{\partial p}\,\left(\tilde{L}_n^{1-\frac{1}{n}}\right)_+
$$

$$
\begin{aligned}
\frac{\partial}{\partial p}\left(\tilde{L}_n^{1-\frac{1}{n}}\right)_+ &= \frac{\partial\,\pi_n}{\partial p}\,\frac{\partial}{\partial\,\pi_n}\,\left(n^{\frac{1}{n}-1}\,\pi_n^{n-1}\,g_{n-1}(w\,\pi_n^{-2})\right)\\
&= n^{\frac{2}{n}-1}\,(n-1)\,\pi_n^{n-2}\,\left(g_{n-1}(w\,\pi_n^{-2}) - \frac{2}{n-1}\,(w\,\pi_n^{-2})g'_{n-1}(w\,\pi_n^{-2})\right)\\
&= n^{\frac{2}{n}-1}\,(n-1)\,\pi_n^{n-2}\,\left(g_{n-2}(w\,\pi_n^{-2}) - \frac{1}{n-2}\,(w\,\pi_n^{-2})g'_{n-2}(w\,\pi_n^{-2})\right)\\
&= n^{\frac{2}{n}-1}\,(n-1)\,\left(\sqrt{w}\right)^{n-2}\,\left(l_{n-2}\left(\frac{\pi_n}{\sqrt{w}}\right) - f_{n-3}\left(\frac{\pi_n}{\sqrt{w}}\right)\right)\\
&= n^{\frac{2}{n}-1}\,(n-1)\,\left(\sqrt{w}\right)^{n-2}\,f_{n-1}\left(\frac{\pi_n}{\sqrt{w}}\right)\\
&= n^{\frac{2}{n}-1}\,(n-1)\,\pi_n^{n-2}\,h_{n-2}(w\,\pi_n^{-2})\\
&= n^{\frac{2}{n}-1}\,(n-1)\,\frac{\partial\,\tilde{L}_n}{\partial\,w}
\end{aligned}
$$

$$
\begin{aligned}
\left\{\left(\tilde{L}_n^{1-\frac{1}{n}}\right)_+,\ \tilde{L}_n\right\} &= n^{\frac{1}{n}}\,w_x\,\left\{\left(\tilde{L}_n^{1-\frac{1}{n}}\right)_+,\ \tilde{L}_n\right\}_n\\
&= n^{2\left(\frac{1}{n}-1\right)}\,w_x\,\left\{\pi_n^{n-1}\,g_{n-1}(w\,\pi_n^{-2}),\ \pi_n^n(w\,\pi_n^{-2})\right\}_n\\
&= n^{\frac{2}{n}-1}\,(n-1)\,w_x\,\pi_n^{2n-4}\\
&\quad\times\left(\frac{1}{n-1}\,g_n(w\,\pi_n^{-2})\,g'_{n-1}(w\,\pi_n^{-2}) - \frac{1}{n}\,g'_n(w\,\pi_n^{-2})\,g_{n-1}(w\,\pi_n^{-2})\right)\\
&= (-1)^{n-1}\,n^{\frac{2}{n}-1}\,(n-1)\,w^{n-2}\,w_x
\end{aligned}
$$

Using these results, the Lax equation

$$
\frac{\partial\,L_n}{\partial\,t} = -n^{1-\frac{2}{n}}\,\frac{1}{n-1}\,\left\{\left(L_n^{1-\frac{1}{n}}\right)_+,\ L_n\right\}
$$

can be written

$$
w_t\,\frac{\partial\,\tilde{L}_n}{\partial\,w} + u_t = u_x\frac{\partial\,\tilde{L}_n}{\partial\,w} + (-1)^n\,w^{n-2}\,w_x
$$

which holds if and only if

$$
\left\{\begin{array}{l} v_t = -u_x\\ u_t = -v^{n-2}\,v_x\end{array}\right.
$$

These are none other than the elastic medium equations. Thus, we have shown that the same Lax function also provides a standard Lax description for the elastic medium equations.

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