Quantization of non-Hamiltonian physical systems

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Abstract

We propose a general method of quantization of non-Hamiltonian physical systems. Applying it, for example, to a dissipative system coupled to a thermal reservoir described by the Fokker-Planck equation, we are able to obtain the Caldeira-Leggett master equation, the non-linear Schrödinger-Langevin equation and the Caldirola-Kanai equation (with an additional term), as particular cases.

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I. INTRODUCTION

Our underlying aim in this paper is to begin to visualize a general answer to the following question: Given a non-relativistic physical system initially described by the laws of classical mechanics, how to describe it in terms of the laws of quantum mechanics?

The most widely accepted answer to this question is given by the Dirac algebraic rules, so-called canonical or standard quantization [1]: First, the physical system is to be described by a Hamiltonian formalism, next the classical dynamical variables p (momentum), q (position), and functions of them $A(p,q)$, should be risen to the category of linear operators $\hat{p}$, $\hat{q}$, and $\hat{A} = A(\hat{p}, \hat{q})$, respectively. Thus, the symplectic structure of classical phase space induced by the Poisson brackets between classical observables $A(p,q)$, $B(p,q)$ corresponds to the algebraic structure given by the commutators of the corresponding quantum observables $\hat{A}, \hat{B}$ divided by $\imath\hbar$:

$$\{A, B\} = \frac{\partial A}{\partial p}\frac{\partial B}{\partial p} - \frac{\partial A}{\partial q}\frac{\partial B}{\partial q} \Longrightarrow \frac{[\hat{A}, \hat{B}]}{\imath\hbar} = \frac{\hat{A}\hat{B} - \hat{B}\hat{A}}{\imath\hbar}. \qquad (1)$$

Along the last seventy years a more critical analysis of the Dirac quantization has been in evidence at least four crucial objections to the suitability of this method to connect the classical and quantum theories, namely

a) In general, the correspondence between classical functions $A(p,q)$ and operators $A(\hat{p}, \hat{q})$ is not unambiguous [1,2,3]. This gives rise to the proliferation of various ordering rules of operators [4] in order to circunvent this difficulty;

b) Even in the cases where there is no ambiguity in the operator ordering, the canonical quantization does privilege the Cartesian frame. For example, the quantization of the Hamiltonian $H(p,q)$ directly in terms of the angle-action variables $H(\alpha, J)$ is not well-defined [5];

c) The existence of the Hamiltonian function generating the classical equations of motion is not sufficient to avoid contradictory physical outcomes by using the canonical

quantization [6]. Therefore, it is necessary the Hamiltonian to be identified with the total energy of the physical system [7,8] in order the quantum system to be physically unique. This condition restricts the applicability of the canonical quantization procedure. It is worth noticing yet that even its application to the quantum mechanical description of classically chaotic Hamiltonian systems (e.g., the kicked harmonic oscillator) leads to very controversial physical and epistemological results [9];

d) Besides its lack of generality, the canonical rules of quantization stress the conceptual abysm between classical and quantum mechanics. The operators $A(\hat{p}, \hat{q})$, obtained unambigously from a classical formalism where $H = E$ (*kinetic energy+potential energy*), act upon wave functions ψ whose physical interpretation still is problematic [10].

Despite the Dirac quantization rules being neither *mathematically* well-defined [2], nor in general *physically* consistent, nor *conceptually* unblurred, we find in the literature its extrapolation to the quantum mechanical study of dissipative systems (non-Hamiltonian systems): Caldirola[11] and Kanai[12], by quantizing an explicitly time dependent Hamiltonian of a damped harmonic oscillator, arrived at a Schrödinger equation depending also on the time explicitly that violates the Heisenberg uncertainty relations[13,14] and does not predict the vacuum fluctuation energy as t(time) $\rightarrow \infty$ [14,15]. Dekker[16], in turn, has proposed a generalization of the Dirac rules for dissipative systems by treating a damped harmonic oscillator in classical mechanics in terms of a complex Hamiltonian other than the total energy of the system in the limit of vanishing damping, and which also is not unique[17], hence the canonical quantization becomes ambiguous. Recently, Tarasov[18] has investigated a generalization of the least action principle for dissipative processes (the Sedov variational principle) obtaining a Hamiltonian formulation that, being canonically quantized, leads to a quantum description of dissipative systems whose operator algebra does not obey the Jacobi identity (nonassociative non-Lie algebra). Although Tarasov had assumed his Hamiltonian to be canonically conjugate to the energy, his approach does not overcome the difficulties arising from the operator ordering when

quantizing a harmonic oscillator with friction, for instance.

From difficulties and limitations in the use of the canonical quantization rules for dissipative systems, above outlined, one can conclude that, in general, they are ambiguous, and hence it is even impossible to obtain a quantum description of these systems from the Lagrangian and Hamiltonian formalisms[7,14,17,19].

A way of trying to save the canonical quantization procedure is to couple a thermal reservoir (system B) to the dissipative system (system A), so that the system A + B as a whole is considered as conservative [20,21,22,23], nevertheless the resultant Hamiltonian can not be identified with the total energy of the system[24] and therefore this approach is not free of physical inconsistencies[25] when one assumes more general physical models to describe the interaction between the dissipative system and the thermal bath[26].

In order to overcome the several problems arising from the Dirac quantization, differ-ent quantization methods of dissipative systems have been proposed, namely: Razavy[17] using the Schrödinger method of quantization [27] to the generalized Hamilton-Jacobi equation, obtained the non-linear Schrödinger-Langevin equation[21]; Pal [28], follow-ing the same method, has arrived at a linear dissipative Schrödinger equation; Skager-stam[29] and Yasue[30] have also derived the non-linear Schrödinger-Langevin equation, but rather using the Nelson stochastic quantization procedure[31]; and Geicke[32], propos-ing a (quasi-canonical) quantization method by replacing p by $-i\hbar\partial/\partial q$ and E by $i\hbar\partial/\partial t$ into the expression of the classically conserved energy $E = p^2/2m + V(q) + \alpha \int (p/m)^n dq$ which governs the motion of the dissipative system, found a linear dissipative Schrödinger equation. Finally, Enz[33] has, recently, suggested a quantization procedure for a dissipa-tive system by remarking a formal resemblance between the classical equations of motion in a generalized Hamiltonian description and the generalization of the Schrödinger equa-tion in terms of a dynamical matrix; Tzani[34], in turn, has made use of a stochastic quantization method for systems with dissipation, starting from the complex Langevin

equations. This brief sketch of the various existent quantization methods of dissipative systems reveals that the quantization of more complex non-Hamiltonian systems holds still as a *terra incognita* .

However, in order to explore this *terra incognita* , in Section 2 we propose formally a general procedure of quantization of non-Hamiltonian systems by introducing the Wigner representation of classical mechanics in phase space, recently suggested by Olavo[35]. In Section 3 we apply it to the quantization of a dissipative system coupled to a thermal reservoir described by the Langevin equations or equivalently by the Fokker-Planck equation and obtain the master equation of Caldeira-Leggett[23], the non-linear Schrödinger equation obtained firstly by Kostin[21] and the Caldirola-Kanai equation[11] with an additional term, as particular cases. In Section 4 we make our concluding remarks.

II. GENERAL METHOD OF QUANTIZATION

Let us consider a classical physical system with mass m, momentum p and position q described by the general non- Hamiltonian Liouville equation

$$\frac{\partial F}{\partial t} + \frac{\partial H}{\partial p}\frac{\partial F}{\partial q} - \frac{\partial H}{\partial q}\frac{\partial F}{\partial p} + \Omega(p,q,t;\alpha)F = 0, \qquad (2)$$

being $H \equiv H(p,q,t)$ a general Hamiltonian function, and $F \equiv F(p,q,t)$ a probability distribution function, with the following marginal relations

$$f(q,t) = \int F\,dp;\; f(p,t) = \int F\,dq, \qquad (3)$$

such that the average value of any physical quantity $A(p,q,t)$ be given by

$$\langle A \rangle = \int AF\,dpdq. \qquad (4)$$

$\Omega(p,q,t;\alpha)F(p,q,t)$ is a set of terms depending on the parameter of non-Hamiltonianess α, so that as $\alpha \to 0$, we recover the Hamiltonian classical mechanics. Let us now introduce

the Wigner representation of classical mechanics, suggested recently by Olavo[35], by means of the *classical Wigner transformations*

$$\chi(q + \frac{\ell\eta}{2}, q - \frac{\ell\eta}{2}, t) = \int F \exp(\imath p \eta) dp \tag{5}$$

or

$$\chi(p + \frac{\ell\xi}{2}, p - \frac{\ell\xi}{2}, t) = \int F \exp(\imath q \xi) dq, \tag{6}$$

where ℓ, η and ξ have dimensions of *angular momentum*, *(linear momentum)*$^{-1}$ and *(length)*$^{-1}$, respectively.

Without any loss of generality, let us suppose that the Hamiltonian in (2) is given by $H = p^2/2m + V(q,t)$. Applying, then, the transformation (5) on (2) and changing the variables $q + \frac{\ell\eta}{2} = q_1$ and $q - \frac{\ell\eta}{2} = q_2$, we obtain

$$\imath\ell\frac{\partial\chi}{\partial t} = -\frac{\ell^2}{2m}(\frac{\partial^2\chi}{\partial q_1^2} - \frac{\partial^2\chi}{\partial q_2^2}) + (V(q_1,t) - V(q_2,t) + O(q_1,q_2,t))\chi - \imath\ell I(q_1,q_2,t;\alpha)\chi, \tag{7}$$

being $\chi \equiv \chi(q_1, q_2, t)$,

$$I(q_1, q_2, t; \alpha)\chi = \int \Omega(p, q, t; \alpha) F \exp(\imath p \eta) dp \tag{8}$$

and

$$O(q_1, q_2, t) = -\sum_{n=3,5,7,\ldots}^{\infty} \frac{2}{n!}(\frac{q_1 - q_2}{2})^n(\frac{\partial}{\partial q_1} + \frac{\partial}{\partial q_2})^n V(\frac{q_1 + q_2}{2}, t). \tag{9}$$

Eq.(7) is a differential equation fully equivalent to (2) due to the existence of the inverse of (5). We now define the quantization process of non-Hamiltonian systems (2), via Eq.(7), by taking into account the following condition

$$O(q_1, q_2, t) \to 0, \tag{10}$$

and taking the quantum limit

$$\ell \to \hbar = \frac{h}{2\pi}, \tag{11}$$

where ℓ has dimensions of angular momentum and changes continuously while h has dimensions of action and is a universal constant (*Planck's constant*). Thus, the quantized version of Eq.(7) becomes the generalized von Neumann equation in coordinate representation

$$\imath\hbar\frac{\partial\rho}{\partial t} = \frac{-\hbar^2}{2m}(\frac{\partial^2\rho}{\partial q_1^2} - \frac{\partial^2\rho}{\partial q_2^2}) + [V(q_1,t) - V(q_2,t)]\rho - \imath\hbar I(q_1,q_2,t)\rho \qquad (12)$$

describing a non-Hamiltonian quantum system. By going from (7) to (12) we have exchanged $\chi(q_1,q_2,t)$, solution of a classical equation, by $\rho(q_1,q_2,t)$, solution of a quantal equation.

By quantizing a classical system described by the Liouville equation (2), with $\Omega(p,q,t;\alpha = 0)F = 0$, we obtain obviously Eq.(12) without the non-Hamiltonian term $I(q_1,q_2,t;\alpha)$

$$\imath\hbar\frac{\partial\rho}{\partial t} = \frac{-\hbar^2}{2m}(\frac{\partial^2\rho}{\partial q_1^2} - \frac{\partial^2\rho}{\partial q_2^2}) + [V(q_1,t) - V(q_2,t)]\rho \qquad (13)$$

which in turn can be derived from the Schrödinger equation at point q_1

$$\imath\hbar\frac{\partial\psi(q_1,t)}{\partial t} = \frac{-\hbar^2}{2m}\frac{\partial^2\psi(q_1,t)}{\partial q_1^2} + V(q_1,t)\psi(q_1,t) \qquad (14)$$

and its complex conjugate at point q_2. Note that the Hamiltonian that generates the Newtonian equations of motion in phase space does not need necessarily be identified with the total energy of the system [6].

For the particular case $I(q_1,q_2,t;\alpha) = I(q_1,\alpha) + I(q_2,\alpha)$, Eq.(12) is reducible to any generalized Schrödinger equation, i.e., $\rho(q_1,q_2,t) = \psi^*(q_2,t)\psi(q_1,t)$:

$$\imath\hbar\frac{\partial\psi(q_k,t)}{\partial t} = \frac{-\hbar^2}{2m}\frac{\partial^2\psi(q_k,t)}{\partial q_k^2} + V(q_k,t)\psi(q_k,t) - \imath\hbar I(q_k,t;\alpha)\psi(q_k,t), (k = 1,2). \qquad (15)$$

However, in general, the quantization of non-Hamiltonian systems leads to master equations where the density matrix ρ can not be factorized $\rho \neq \psi^*\psi$.

After these general considerations about the formal peculiarities of the master equation (12), in next Section we shall take into account a specific example of non-Hamiltonian system: Dissipation aroused by the interaction between a particle and a thermal reservoir described by the Langevin equations or equivalently by the Fokker-Planck equation.

III. APPLICATION: THE FOKKER-PLANCK EQUATION

Let us consider a particle of mass m, momentum p and position q immersed in a reservoir whose temperature at thermal equilibrium is T. The one-dimensional movement of this particle is described by the Newton equation

$$\frac{dp}{dt} = F_1 + F_2 + F_3 + F_4, \tag{16}$$

with $p = m\frac{dq}{dt}$ and being

$$F_1 = -\frac{dV}{dq} \tag{17}$$

the force derived of a external potential $V = V(q,t)$ applied to the particle, and

$$F_2 = -2\gamma m\frac{dq}{dt} - 2\gamma\frac{\ell}{m}\frac{\partial Z}{\partial q} \tag{18}$$

the friction force dependent of the speeds $\frac{dq}{dt}$ and $v = \frac{\ell}{m}\frac{\partial Z}{\partial q}$, being v derived of a function $Z = Z(q,t)$. In (18) 2γ is the friction coefficient and ℓ a physical variable with dimensions of angular momentum. In (16) $F_3 = F(t)$ and $F_4 = -\frac{\partial V_R}{\partial q}$ are forces due to the reservoir, $V_R(q,t)$ is a general random potential. These stochastic forces give rise fluctuations in the particle motion. Supposing that the particle follows a Markovian evolution, i.e., the friction term (18) does not contain memory effects, and having the fluctuating force $F(t)$ the following statistical properties

$$\langle F(t)\rangle = 0, \tag{19}$$

$$\langle F(t)F(t')\rangle = 4m\gamma k_B T\delta(t - t'), \tag{20}$$

where $\langle...\rangle$ denotes the average value, k_B the Boltzmann constant and δ the Dirac delta function, (16) gives us the Langevin equations

$$dp = -[\frac{dV}{dq} + 2\gamma p + 2\gamma\frac{\ell}{m}\frac{\partial Z}{\partial q} + \frac{\partial V_R}{\partial q}]dt + (4m\gamma k_B T)^{1/2}dW(t) \tag{21}$$

$$dq = \frac{p}{m}dt, \tag{22}$$

being $dW(t) = (4m\gamma k_B T)^{-1/2}F(t)dt$ the Wiener process, with $\langle dW(t)\rangle = 0$ and $\langle dW(t)dW(t')\rangle = \delta(t - t')dt$. Thus, the particle described by the Langevin equations (21) and (22) realizes a stochastic process of the Brownian type that may also be described equivalently by the Fokker-Planck equation in phase space [36]

$$\frac{\partial F}{\partial t} = [-\frac{\partial D_1^{(1)}}{\partial q} - \frac{\partial D_2^{(1)}}{\partial p} + \frac{\partial^2 D^{(2)}}{\partial p^2}]F. \tag{23}$$

$F = F(p, q, t)$ is the probability density evaluated from the initial conditions $F(p, q, t = 0) = \delta(p-p')\delta(q-q')$, ($p' = p(t = 0)$, $q' = q(t = 0)$), $D_1^{(1)}$ and $D_2^{(1)}$ are the drift coefficients

$$D_1^{(1)} = \frac{p}{m} \tag{24}$$

$$D_2^{(1)} = -[\frac{dV}{dq} + 2\gamma p + 2\gamma\frac{\ell}{m}\frac{\partial Z}{\partial q} + \frac{\partial V_R}{\partial q}], \tag{25}$$

and

$$D^{(2)} = 4m\gamma k_B T \tag{26}$$

the diffusion coefficient.

Performing now the classical Wigner transformation (5) on (23), changing the variables $q + \frac{\ell\eta}{2} = q_1$, $q - \frac{\ell\eta}{2} = q_2$, next quantizing via Eqs.(10) and (11), we obtain the generalized quantum von Neumann equation

$$i\hbar\frac{\partial\rho}{\partial t} = -\frac{\hbar^2}{2m}\left\{\frac{\partial^2\rho}{\partial q_1^2} - \frac{\partial^2\rho}{\partial q_2^2}\right\} + \{V(q_1, t) - V(q_2, t)\}\rho$$
$$+ \{V_R(q_1, t) - V_R(q_2, t)\}\rho + \frac{2\gamma\hbar}{m}\{Z(q_1, t) - Z(q_2, t)\}\rho$$
$$-i\hbar\gamma(q_1 - q_2)\left\{\frac{\partial\rho}{\partial q_1} - \frac{\partial\rho}{\partial q_2}\right\} - \frac{2im\gamma k_B T}{\hbar}(q_2 - q_1)^2\rho \tag{27}$$

with the initial conditions $\rho(q_1, q_2, t = 0) = \delta(\frac{q_1+q_2}{2} - \frac{q_1'+q_2'}{2})[1 + \frac{\imath p'}{\hbar}(q_1 - q_2) - \frac{p'^2}{\hbar^2}(q_1 - q_2)^2]$.

Eq.(27) reduces, in form, to the master equation found by Caldeira and Leggett[23], making V_R, $Z = 0$. In contrast to the Caldeira-Leggett approach, our Eq.(27) is valid for any temperature and no coupling-induced renormalization of the external potential was introduced here. Furthermore, assuming the reservoir to be at thermal equilibrium at the initial time $t = 0$, requirements of translational and time reversal invariance imply that the initial coordinates of the reservoir are given by $Q'_{eq} = \frac{q_1'+q_2'}{2}$ [37]. So in our approach, the process of quantization of the system particle + reservoir is performed on the particle variables alone, whereas the reservoir variables enter only through their initial values[38]:

$$\rho(q_1, q_2, t = 0) = \delta\left(\frac{q_1 + q_2}{2} - Q'_{eq}\right)[1 + \frac{\imath p'}{\hbar}(q_1 - q_2) - \frac{p'^2}{\hbar^2}(q_1 - q_2)^2]. \tag{28}$$

These initial conditions (28) correlate the particle with the thermal reservoir. This relevant physical detail is neglected in the Caldeira-Leggett theory based on the factorization assumption.

Making the approximation $q_1 - q_2 \ll 1$ into (27), we find

$$\imath\hbar\frac{\partial \rho}{\partial t} = -\frac{\hbar^2}{2m}\left\{\frac{\partial^2 \rho}{\partial q_1^2} - \frac{\partial^2 \rho}{\partial q_2^2}\right\} + \{V(q_1, t) - V(q_2, t)\}\rho$$
$$+ \{V_R(q_1, t) - V_R(q_2, t)\}\rho + \frac{2\gamma\hbar}{m}\{Z(q_1, t) - Z(q_2, t)\}\rho. \tag{29}$$

which can be obtained, by introducing $\psi = (\rho)^{1/2}exp(\imath Z)$, from the generalized Schrödinger equation

$$\imath\hbar\frac{\partial \psi(q_1, t)}{\partial t} = -\frac{\hbar^2}{2m}\frac{\partial^2 \psi(q_1, t)}{\partial q_1^2} + \{V(q_1, t) + V_R(q_1, t)\}\psi(q_1, t) - \frac{\imath\hbar\gamma}{m}log[\frac{\psi(q_1, t)}{\psi^*(q_1, t)}]\psi(q_1, t)$$
$$\tag{30}$$

at point q_1 and its complex conjugate at point q_2. Eq.(30) is called the Schrödinger-Langevin equation originally derived by Kostin[21] from the Heisenberg-Langevin equation for a Brownian particle interacting with a thermal enviroment. In our derivation the condition $q_1 - q_2 \ll 1$ means that (30) describes a deterministic process, i.e. , a *without*

diffusion process, from the initial conditions $\psi(q_1, t) = \delta(q_1 - Q'_{eq})$, and also without dissipation due to friction forces depending on the velocity dq/dt, in contrast to the Nelson stochastic quantization procedure followed by Skagerstam[29] and Yasue[30] on deriving Eq.(30).

Now going back to Eq.(23) given explicitly by

$$\frac{\partial F}{\partial t} = -\frac{p}{m}\frac{\partial F}{\partial q} + \frac{\partial V}{\partial q}\frac{\partial F}{\partial p} + 2\gamma F + 2\gamma p\frac{\partial F}{\partial p} + 2\gamma m k_B T \frac{\partial^2 F}{\partial p^2}, \qquad (31)$$

with V_R, $Z = 0$, and performing the following non-canonical transformation $(p, q) \mapsto (P, Q)$

$$P = e^{2\gamma t} p$$

$$Q = \quad q \qquad\qquad\qquad (32)$$

we arrive at

$$\frac{\partial f}{\partial t} = -\frac{P}{m}e^{-2\gamma t}\frac{\partial f}{\partial q} + e^{2\gamma t}\frac{\partial f}{\partial q} + e^{2\gamma t}\frac{\partial V}{\partial q}\frac{\partial f}{\partial P} + 4\gamma f + 2\gamma P\frac{\partial f}{\partial P} + 2\gamma m k_B T e^{4\gamma t}\frac{\partial^2 f}{\partial P^2}, \quad (33)$$

where $f(P, q, t) = e^{2\gamma t} F(p, q, t)$. Introducing the classical Wigner function in the form

$$\chi(q + \frac{\ell\zeta}{2}, q - \frac{\ell\zeta}{2}, t) = \int f(P, q, t) exp(\imath P\zeta) dP, \qquad (34)$$

defining new variables $q_1 = q + \frac{\ell\zeta}{2}$, $q_2 = q - \frac{\ell\zeta}{2}$, using (10) and quantizing $\ell \to \hbar$, and next taking into account the approximation $q_1 - q_2 \ll 1$, we obtain the evolution equation for the matrix density ρ

$$\imath\hbar\frac{\partial\rho}{\partial t} = \frac{-\hbar^2}{2m}e^{-2\gamma t}\left\{\frac{\partial^2\rho}{\partial q_1^2} - \frac{\partial^2\rho}{\partial q_2^2}\right\} + e^{2\gamma t}\left\{V(q_1, t) - V(q_2, t)\right\}\rho - 2\imath\hbar\gamma\rho \qquad (35)$$

which is reducible to the following deterministic Schrödinger equation

$$\imath\hbar\frac{\partial\psi}{\partial t} = [-\frac{\hbar^2}{2m}e^{-2\gamma t}\frac{\partial^2}{\partial q^2} + e^{2\gamma t}V(q, t) - \imath\hbar\gamma]\psi \qquad (36)$$

at a generic point q of Euclidean space. Apart the term $-\imath\hbar\gamma$ Eq.(36) is formally equal to the Caldirola-Kanai equation [11,12], obtained making use of the Dirac canonical quantization procedure from the classical Hamiltonian $H(p, q, t) = e^{-2\gamma t}p^2/2m + e^{2\gamma t}V(q, t)$,

and that describes *exactly* an isolated particle dissipating *all* its energy [14,15] and staying localizable as $t \to \infty$ [13,14].

It is straightforward to verify that Eq.(36) does predict the zero point energy. For instance, for the case of the harmonic oscillator $V = \frac{1}{2}m\omega^2 q^2$ at the fundamental state

$$\psi_0 = \left(\frac{m\omega}{\pi\hbar}\right)^{1/4} e^{-\frac{1}{2}(\gamma+i\Omega)t} e^{-\frac{m}{2\hbar}(i\gamma+\Omega)q^2 e^{2\gamma t}}, \tag{37}$$

and $\omega > \gamma$, the mean energy $\langle p^2/2m + m\omega^2 q^2/2 \rangle$ is

$$\frac{\hbar\omega^2}{4\Omega}(1 + e^{-4\gamma t}) \tag{38}$$

where $\Omega = (\omega^2 - \gamma^2)^{1/2}$ is the damped frequency. Consequently, as $t \to \infty$ Eq.(38) yields correctly the zero point energy. However, by evaluating the relation $\triangle q \triangle q$ we find

$$\triangle q \triangle q = \frac{\hbar\omega}{2\Omega} e^{-2\gamma t}. \tag{39}$$

This result shows the incompatibility of Eq.(38) with the Heisenberg uncertainty principle in case $t \to \infty$.

Therefore, our Eq.(36) describes *approximately* an isolated dissipative system becoming well-localizable with a determined quantum energy in the limit $t \to \infty$.

IV. CONCLUDING REMARKS

Based on the Wigner representation of classical mechanics we have proposed a general method of quantizing non-Hamiltonian systems. We have then obtained a generalized von Neumann equation irreducible to any Schrödinger equation. That is, *subjacent to the von Neumann matrix, in general, does not exist a wavefunction.* Analysing the specific case of a particle interacting with a thermal bath, we successfully derived three equations of motion describing quantum dissipative systems, as *approximations*, namely

a) The Caldeira-Leggett stochastic equation describing a quantum Brownian motion caused by the thermal fluctuations coming from the reservoir, and the dissipation arising from the friction force depending on the velocity dq/dt;

b) The deterministic Schrödinger-Langevin equation whose non-linearity is solely associated with the dissipation due to the friction force depending on the velocity $v = \frac{\ell}{m}\partial Z/\partial q$; and

c) The deterministic Caldirola-Kanai equation with an additional term (not introduced *ad hoc*) which predicts correctly the zero-point fluctuation energy for a particle well-localizable as $t \to \infty$. Here, the dissipation arises kinematically from a non-canonical transformation.

To conclude, we would like to remark that the *operator structure* induced by the Wigner representation of classical mechanics (Eqs.(5), (6)) and the classical limit $\hbar \to 0$ of the dissipative quantum equations of motion shall be examined elsewhere.

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