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Nonextensive Thermostatistics and Deformed Structures

A. B. Pinheiro[a] and *I. Roditi*[b]

Centro Brasileiro de Pesquisas Físicas - CBPF

Rua Dr. Xavier Sigaud, 150

22290-180, Rio de Janeiro, RJ, Brasil

Abstract

We obtain a generalized Planck law within the framework of nonextensive statistics making use of a deformed oscillator system. Our results are used to fit the data from the COBE (Cosmic Background Explorer satellite). Best fit values for the entropy parameter q, the deformation parameter r and for the temperature are found.

e-mail:

(a) baptista@cat.cbpf.br

(b) roditi@cat.cbpf.br

The Boltzmann-Gibbs Statistical Mechanics (BGSM) is one of the most successful theories of Physics. Its particular cases, the Fermi-Dirac and Bose-Einstein quantum statistics (and their common high temperature asymptotic limit, the Maxwell-Boltzmann statistics), give an excellent description of physical systems characterized by extensive statistical mechanics. Nevertheless, extensivity is not an universal property, and BGSM cannot give an accurate description of the problem. Well-known examples of nonextensive statistical mechanics appears in cosmology, gravitation and astrophysics [1]. The source for the nonextensive behavior of those systems is the presence of long range interactions or, conversely, if the elements of the system interact across distances comparable to or larger than the linear size of the system.

A succesful proposal for the investigation of nonextensive statistical mechanics has been formulated by Tsallis [2], within the framework of Information Theory and multi-fractals. He coined a generalized form for the entropy

$$S_q = k\frac{1-\sum_{i=1}^{W} p_i^q}{q-1}. \tag{1}$$

as a starting point for generalizing BGSM. In the limit $q \to 1$ the Shannon entropy, $-k_B \sum_s p_s ln p_s$ is recovered. Eq.(1) is also concave (convex) for $q > 0 (q < 0)$ and all p_i, but it is nonextensive, i.e.,

$$S_q(\Sigma_1 U \Sigma_2) = S_q(\Sigma_1) + S_q(\Sigma_2) + (1-q)S_q(\Sigma_1)S_q(\Sigma_2), \tag{2}$$

where Σ_1 and Σ_2 are two non-interacting systems. From this entropy, the ensembles (Microcanonical, Canonical and Grand Canonical)and the corresponding partition functions can be derived [3]. This formulation has been successfully applied to many concepts of statistical mechanics, such as mean-field Ising model [4], Langevin and Fokker-Planck equations [5], Boltzmann H-theorem [6], Ehrenfest theorem [7], Bogolyubov inequality [8] and others.

Recently, it has been shown that his formalism has connections [9] with quantum groups[10–14] through deformed oscillator algebras, where a parameter r [15] is introduced such that, in the limit $r \to 1$, the corresponding oscilator algebra is recovered. These deformed systems turn out to be nonextensive, and provide a natural ground to apply the Tsallis generalized statistical mechanics.

Both formalisms have been separately used to generalize the Planck radiation law [16, 17, 18]. Particularly in [16] it was assumed that due to long range gravitational interaction one could assume that nonextensivity causes slight deviations from the usual Planck's blackbody radiation law. We shall deal with that question by describing the radiation law through deformed harmonic oscillators and studying this nonextensive system within the framework of the generalized statistical mechanics. Then, we refer the corresponding radiation law to the cosmic background radiation data from Mather *et al* [19], obtained from the COBE - Cosmic Background Explorer satellite, assuming that its internal reference has a Planck spectrum.

The r-deformed oscillators algebra is generated by the elements a, $a^{\dagger}$ that act on a Fock space with base $|n\rangle$, $n = 0, 1, 2, ...$ and

$$a|0\rangle = 0 \tag{3}$$

and the excited states

$$|n\rangle = \frac{(a^{\dagger})^n}{\sqrt{[n]_a!}}|0\rangle \tag{4}$$

where

$$[n]_a! = [n]_a[n-1]_a...[1]_a, \quad [n]_a = \frac{r^n - r^{-n}}{r - r^{-1}}. \tag{5}$$

From the above equations one can infer that

$$aa^{\dagger} = [N+1] \tag{6}$$

and

$$a^{\dagger}a = [N] \tag{7}$$

and the number operator N is such that

$$N|n\rangle = n|n\rangle. \tag{8}$$

So, one have the following commutation relations:

$$[N, a^{\dagger}] = a^{\dagger} \tag{9}$$

$$[N, a] = -a \tag{10}$$

$$[a, a^{\dagger}]_r = aa^{\dagger} - ra^{\dagger}a = r^{-N} \tag{11}$$

For our purposes, a more appropriate basis is given by [14]

$$A = r^{N/2}a, \quad A^{\dagger} = a^{\dagger}r^{N/2} \tag{12}$$

which have the following commutation relation:

$$[A, A^\dagger]_q = AA^\dagger - r^2 A^\dagger A = 1 \tag{13}$$

In terms of these new operators, the Fock representation is

$$A|0\rangle = 0 \tag{14}$$

and

$$|n\rangle = \frac{(A^\dagger)^n}{\sqrt{[n]_A!}}|0\rangle \tag{15}$$

which are eigenstates of the number operator N

$$N|n\rangle = n|n\rangle. \tag{16}$$

We also have

$$AA^\dagger = [N+1]_A, \quad A^\dagger A = [N]_A \tag{17}$$

The change of operators, from A to a implies a change in the definition of $[n]$

$$[n]_A = \frac{r^{2n} - 1}{r^2 - 1} = r^{n-1}[n]_a \tag{18}$$

We shall investigate the Planck radiation law in the framework of the generalized statistics, making use of an r-ideal deformed system described by the Hamiltonian [14] :

$$H = h\nu[n]_A, \tag{19}$$

where ν is the photon fequency. Note that in the limit $r \to 1$, $[n]$ aproaches n, the bosonic particle number operator. To obtain the photon energy density per unit volume, i.e., the Planck radiation law, we must know the generalized canonical partition function. From the definition of the Tsallis entropy, we have that the density operator is [3]

$$\rho = \frac{[1-(1-q)\beta H]^{1/(1-q)}}{Z_q}, \tag{20}$$

where Z_q is the generalized canonical partition function, given by

$$Z_q = Tr[1-(1-q)\beta H]^{1/(1-q)}. \tag{21}$$

When the departure from extensivity is small ($q \to 1$), the mean value of an observable O is given by [3, 16]

$$< O >_q = Tr\rho^q O = < \rho^{q-1} O >, \tag{22}$$

and, in the limit $\beta(1-q)\rightarrow 0$, becomes

$$Z_q \simeq Z\{1-\frac{1}{2}(1-q)\beta^2 < H^2 >\}, \tag{23}$$

where Z is the canonical partition function ($q=1$). Now, using equations (20) and (23) in (22) the mean value of O is obtained in terms of the usual Boltzmann-Gibbs one:

$$< O >_q \simeq Z^{1-q} < O > \left\{1+(1-q)\beta\left[\frac{< OH >}{< O >}+\frac{\beta}{2}\left(< H^2 > - \frac{< OH^2 >}{< O >}\right)\right]\right\} \tag{24}$$

We also want a slightly deformed system, i.e., $1-r\rightarrow 0$. Doing so, the Hamiltonian becomes

$$H = h\nu\{n+n(n-1)(1-r)\}. \tag{25}$$

Retaining first order terms and making $O=H$, we arrive at the generalized Planck radiation law:

$$\begin{aligned} D_{q,\epsilon} \simeq D(\nu)(1-e^{-x})^{q-1}&\left\{1+(1-q)x\left[\frac{1+e^{-x}}{1-e^{-x}}-\frac{x}{2}\frac{1+3e^{-x}}{(1-e^{-x})^2}\right]\right\} \\ &-D(\nu)\frac{4xe^{-x}}{(1-e^{-x})^2}(1-e^{-x})^{q-1}(r-1), \end{aligned} \tag{26}$$

where $x=h\nu/kT$ and

$$D(\nu)=\frac{8\pi h\nu^3}{c^3(e^{h\nu/kT}-1)} \tag{27}$$

is the usual Planck radiation law. Normalizing the expression, we have

$$\begin{aligned} \frac{D_{q,\epsilon}(\nu)h^2c^3}{8\pi(k_BT)^3} \simeq \frac{x^3}{e^x-1}D(\nu)(1-e^{-x})^{q-1}&\left\{1+(1-q)x\left[\frac{1+e^{-x}}{1-e^{-x}}-\frac{x}{2}\frac{1+3e^{-x}}{(1-e^{-x})^2}\right]\right\} \\ &-\frac{x^3}{e^x-1}D(\nu)\frac{4xe^{-x}}{(1-e^{-x})^2}(1-e^{-x})^{q-1}(r-1) \end{aligned} \tag{28}$$

The FIRAS (far infrared absolute spectrophotmeter) instrument on the COBE (Cosmic Background Explorer) satellite has provided the most accurate data on the cosmic background microwave radiation [19]. The FIRAS was designed to measure the spectrum of the radiation to high precision, making differential measurements, which means that it measures the difference between the cosmic background and an internal reference, whose temperature is adjusted to be about 2.7 K. In Fig. 1, we have a plot of the brightness, with the FIRAS-measured CMBR residuals. Fitting these data with a Planck spectrum gives a χ^2 a factor of 4 greater than the number of degrees of freedom [16]. This deviation, as pointed out by Mather *et al* [19], may be due entirely to instrumental effects, which are large at low frequencies. Upper limits on the distortions can be found taking into account additional input of energy into the cosmic background [19].

We assume here [16] that the internal reference of the FIRAS has a Planck spectrum, with a temperature of (2.72584 ± 0.005)K and, using the residuals shown in figure 2 and Eq. (29), we find the best fit values for $q-1$, $r-1$ and δT, the temperature shift. Note that we can only find differences in the parameters between the cosmic background and the internal reference spectrum, because the data we have came from differential measurements. The internal reference spectrum has $q = r = 1$.

To fit the data for the parameters above, we must first remove the factor of 2 increase in the errors Mather *et al* applied to account for unexplained errors. Then, the point at 10.76 cm^{-1} is removed, because of a vibration in the instrument [19]. We also removed the last point, at 20.95 cm^{-1}, which left us with 32 degrees of freedom. We studied three cases: $r = 1$, fixed, and $q-1$ and δT varying, which means nonextensivity but no deformed oscillators; $q = 1$, fixed, and $r-1$ and δT varying, which means deformation and extensivity; and, finally, we let all parameters vary. In the first case, we got

$$\begin{aligned} q-1 &= (-0.82 \pm 1.22) \times 10^{-5} \\ \delta T &= (-0.27 \pm 1.78) \times 10^{-5} K \end{aligned} \tag{29}$$

The χ^2 is 109 with 32 degrees of freedom. To find an upper limit for $q-1$ it is necessary that the χ^2 be equal to the number of degrees of freedom. This can be achieved by inflating the errors bars by a factor of 1.85, and the result is

$$|q-1| < 0.89 \times 10^{-4}. \tag{30}$$

In the second case, we got

$$\begin{aligned} r-1 &= (-8.83 \pm 3.10) \times 10^{-5} \\ \delta T &= (-0.27 \pm 0.88) \times 10^{-5} K \end{aligned} \tag{31}$$

The χ^2 is again 109 for 32 degrees of freedom. The upper limit for $r-1$ is found after multiplying the errors bars by 1.85 :

$$|r-1| < 4.49 \times 10^{-4}. \tag{32}$$

In the third case, we let both parameters vary, and we got

$$q-1 = (-4.84 \pm 1.68) \times 10^{-5}$$

$$
\begin{aligned}
r - 1 &= (-1.82 \pm 0.48) \times 10^{-4} \\
\delta T &= (2.18 \pm 0.28) \times 10^{-4}
\end{aligned}
\tag{33}
$$

The χ^2 is again 109 for 32 degrees of freedom. To find the upper limits, again we make the χ^2 equal to the number of degrees of freedom by multiplying the errors bars by 1.85 and the results are

$$
\begin{aligned}
|q - 1| &< 3.56 \times 10^{-5} \\
|r - 1| &< 2.54 \times 10^{-4}
\end{aligned}
\tag{34}
$$

As we saw, the fit of the FIRAS' data taking into account deformed oscillators give excellent results. If we compare with the previous atempts [16, 21], we see that the introduction of deformed oscillators makes a better fit, mainly in the low frequency region (see Fig. 1). The temperature shift, which in the case without deformed oscillators is a quantity with comparable errors, also gets a better estimative, since the introduction of deformed oscillators shortens its errors. Then, if one assume, within the systematic errors, that the FIRAS-COBE data used here [19] are precise enough, the generalized Planck spectrum proposed really can account for these data, which gives an interesting insight on the structure of space-time and the interaction between radiation and matter.

The results we find here are consistent with the results from Tsallis *et al* and Plastino *et al* [16, 21]. They found, respectively, upper limits of $|q - 1| < 3.6 \times 10^{(} - 5)$ and $|q - 1| < 5.3 \times 10^{-5}$ from the FIRAS-COBE data. Plastino *et al* also calculated the upper limit of $|q - 1|$ from the experimental value of the Stefan-Boltzmann constant: 0.67×10^{-4}. Tsallis used 31 degrees of freedom, removing the point at $10.76 cm^{-1}$ and the last two points. Nevertheless, our result, $|q - 1| < 8.9 \times 10^{-5}$, is in pretty good agreement with the others, showing that the effect of nonextensivity, as well as deformation, on the cosmic background radiation, although small ($\sim 10^{-4}$), may prove to be non-negligible and possible to be detected via experiments.

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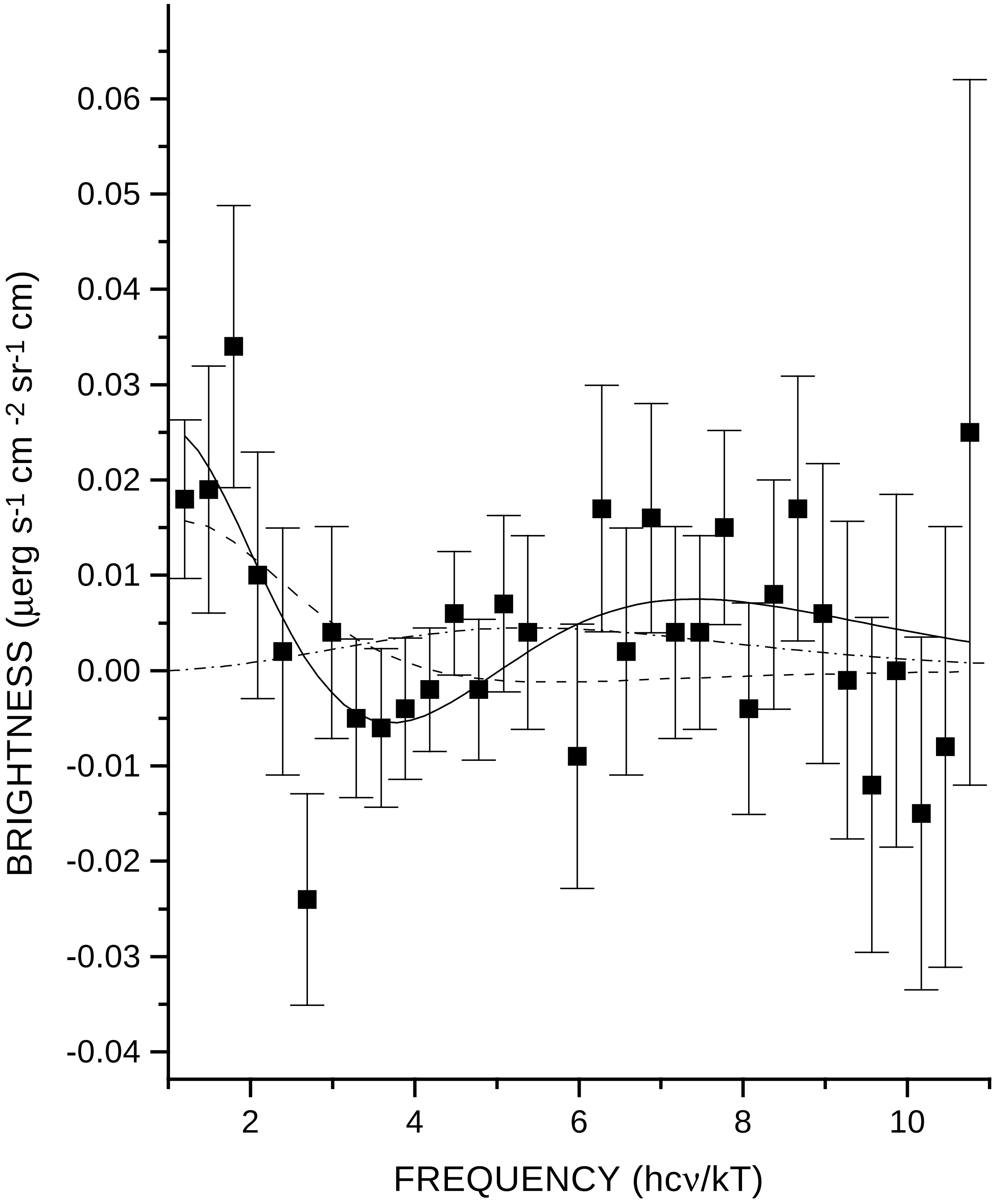
BRIGHTNESS (μerg s-1 cm -2 sr-1 cm)
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0.05
0.04
0.03
0.02
0.01
0.00
-0.01
-0.02
-0.03
-0.04
2
4
6
8
10
FREQUENCY (hcν/kT)

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