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On the Critical Behaviour of the 2-point Function in Scalar Field Theories

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Abstract

By the use of a Mellin representation of Feynman integrals, a convergent asymptotic expansion for generic Feynman amplitudes for any set of invariants going to zero or to ∞, may be obtained. In the case of scalar field theories in Euclidean metric, we use this expansion to analyse the behaviour of the two-point function for small values of the mass parameter, for fixed external momentum.

Key-words: Critical behaviour; Infrared divergences.

1 Introduction

Divergences in field theories have been historically considered as an indesirable feature, a kind of "illness" of the theory which should be "cured" at any price. The most striking example is ultraviolet divergences, coming from the non-definiteness of field products at the same spacetime point, and its "cure", the various renormalization procedures. Also, the divergent large distance behaviour of theories containing massless fields, the infrared divergences, received an equivalent amount of attention over the last decades. Infrared divergences may be seen as a special case of a general class of asymptotic behaviours of Feynman amplitudes in a field theory (which includes also UV divergences), as some of the involved masses tend to zero. Actually these divergences appear at different levels. For Green functions in Minkowskian metric it has been shown a long time ago that for some theories (e.g. QED) Green's functions exist at the zero-mass limit for some particles, as distributions on the 4-momenta, i.e., Green's functions are well defined quantities in the infrared limit. [1]. For particles on mass shell Green's functions generally does not have a limit for those theories, even if they are well defined off mass shell Green's functions. The oldest and best known example is infrared divergences in scattering amplitudes in QED. This problem has been investigated exhaustivelly (classical papers on the subject are in refs.([2, 3])), since the celebrated work of Bloch and Nordsieck [4].

Another class of problems arise at the Green's functions level in Euclidean metric, when besides the zero-mass limit, also vanishingly small values for the external momenta are considered. In this case, we speak of the infrared behaviour of correlation functions. These divergences, which are seen as a "pathological" behaviour in the context of the aplications of field theories to particle physics, are associated with the large distance correlations in statistical systems and play a crucial role in the study of critical phenomena and phase transitions. Since the Ginzburg-Landau model was introduced, a half-century ago [5], as a phenomenological model for superfluidity and superconductivity, the idea that Field Theory models can describe statistical systems near criticality becomes well established. Indeed, in its one component version, the Ginzburg-Landau model has been used with remarkable success as a continuous statistical mechanics model for the critical phenomena of systems lying in the same universality class as the Ising model. In its N-component version coupled to Abelian gauge fields it has been used as a model for superconductivity and liquid cristals [6]. Presently the basic field theoretical approach to study critical phenomena is the renormalization group analysis of flows in the neighbourhood of fixed points. In particular, recently some works have been spared to the use of this technique to study the topological Ginzburg-Landau model. For instance in [7] a Ginzburg- Landau model for superconcuctivity with a Chern-Simons term added is considered, and a similar study has also been done in [8]. On more general grounds important works on the subject are in refs. [9], [10], [11], [12]. Another approach, using the finite temperature field theory formalism have also been used recently, as for instance in [13] and [14].

In this note, we adopt a sharper point of view, we fix our attention on a, as precise as we can, study of the two-point function in scalar field theories in the framework of the perturbative method. We start with an study of the asymptotic behaviour of Feynman amplitudes in Euclidean metric. We make use of Mellin transform techniques to represent Feynman integrals, along similar lines as it has been done to study renormalization and asymptotic behaviours of scattering amplitudes in refs. [16, 17, 18] , and to study the heat kernel expansion as in ref. [19]. To fix our framework we consider a theory involving scalar fields $\varphi_i(x)$ having masses m_i, defined on a Euclidean space. For simplicity we may think of a single scalar field $\varphi(x)$ having a mass m. A generic Feynman graph G is a set of I internal lines, L loops, c connected components (a graph is disconnected if $c > 1$) and

v vertices linked by some (polynomial) potential. To each vertex are attributed external momenta $\{p_i\}$ and internal ones $\{k_a\}$. A subgraph $S \subset G$ is a graph such that all the lines vertices and loops belong to G and a quotient graph G/S is a graph obtained from G reducing S to a point.

2 The Mellin Representation of Feynman Integrals and Asymptotic Expansions

The Feynman amplitude $G(\{a_k\})$ corresponding to G is is a function of the set of invariants $\{a_k\}$ built from external momenta $\sum p^2$ and squared masses m_i^2; it is defined in the Schwinger-Bogoliubov representation by, (see for instance refs. [1], [15]

$$G(a_k) = \int_0^\infty \prod_{i=1}^{I} d\alpha_i U^{-\frac{D}{2}}(\alpha) e^{-\frac{V(\alpha)}{U(\alpha)}}, \tag{1}$$

where D is the space dimension with positive metric.

In the above formula, the Symanzik polynomials $U(\alpha)$ and $W(\alpha)$ are constructed from the graph G by the prescription,

$$U(\alpha) = \sum_{1.T} \prod_{i \notin 1.T} \alpha_i \tag{2}$$

and

$$V(\alpha) = \sum_{2.T} (\sum p_j)^2 (\prod_{i \notin 2.T} \alpha_i) + (\sum_{j \in G} m_j^2 \alpha_j) U(\alpha) \tag{3}$$

where the simbols $\sum_{1.T}$ and $\sum_{2.T}$ means respectively summation over the trees and two-trees (disconnected trees having two connected components) of G passing by all the vertices. The sum $\sum p_j$ is the total external momentum entering one of the two-tree connected components. Notice that $U(\alpha)$ and $W(\alpha)$ are homogeneous polynomials in the α-variables, of degrees L and $L+1$ respectively.

In the following we have in mind as a physical situation, the infared behaviour, but we would like to emphasize that our study is quite general, in the sense that it applies to any asymptotic limit in Euclidean metric (any choice of the subset a_l below), for arbitrarily given external momenta, generic or exceptional, and for arbitrary vanishing or finite masses. If we perform a scale transformation on the subset $\{a_l\}$ of invariants, $a_l \rightarrow \lambda a_l$, the polynomial V is split into two parts,

$$V(\lambda a_m) = \lambda W(a_l, \alpha) + R(a_q, \alpha) \tag{4}$$

where the polynomials $W(a_l, \alpha)$ and $R(a_q, \alpha)$ are also homogeneous of degree $L+1$ in the α-variables.

To be concrete we consider here, a special situation with the external momenta $\{p\}$ fixed and we investigate the limit $\lambda \rightarrow 0$ corresponding to vanishing masses. In this case W is just the second term in Equ. (3). As we have noted above, the method applies along the same lines to any other class of asymptotic behaviour. Incidentally we note that from a dimensional argument,

$$G(\frac{a_l}{\lambda}, a_q) = \lambda^{\omega} G(a_l, \lambda a_q), \tag{5}$$

the study of a given subset going to zero is equivalent to study the $\lambda \rightarrow \infty$ limit on the complementary subset of invariants.

Under the λ-scaling performed in Equ.(4) G becomes a function of λ, $G(\lambda)$, and its Mellin transform, $M(z) = \int_0^\infty d\lambda \lambda^{-z-1} F(\lambda)$ may be written in the form,

$$M(z) = \Gamma(-z) \int_0^\infty \prod_{i=1}^{I} d\alpha_i U^{-\frac{D}{2}} e^{-\frac{R}{U}} (\frac{W}{U})^z. \tag{6}$$

The scaled amplitute associated to the Feynman graph G, $G(\lambda)$, may be obtained by the inverse Mellin transform,

$$G(\lambda) = \frac{1}{2i\pi}\int_{\sigma-i\infty}^{\sigma+i\infty} dz\lambda^z M(z) \tag{7}$$

where $\sigma = Re(z) < 0$ belongs to the analyticity domain of $M(z)$.

Since the integrand of Equ.(7) vanishes exponentially at $\sigma \pm i\infty$ due to the behaviour of $\Gamma(z)$ at large values of Imz, the integration contour may be displaced to the right by Cauchy's theorem, picking up successivelly the poles of the integrand, provided we can desingularize the integral in Equ.(6). Such a problem has been studied by an appropriate choice of local coordinates in [20] and also in [16] using Hepp sectors and a Multiple Mellin representation. In these works it has been possible to show that the meromorphic structure of $M(z)$ has the form,

$$M(z) = \sum_{n,q} \frac{A_{nq}q!}{(z-n)^{q+1}}. \tag{8}$$

It results from the displacement of the integration contour in the inverse Mellin transform, an expansion for small values of λ, of the form,

$$G(\lambda) = \sum_{n=n_0}^{N} \lambda^n \sum_{q=0}^{q_{max}(n)} A_{nq} ln^q(\lambda) + R_N(\lambda) \tag{9}$$

where the coefficients $A_n(\{p\})$ and the powers of logarithms come from the residues at the poles $z = n$.

The rest of the expansion $R_N(\lambda)$ is given by

$$R_N(\lambda) = \int_{-\infty}^{+\infty} \frac{d(Imz)}{2i\pi}\lambda^z\Gamma(-z)F(z), \tag{10}$$

with

$$N < Re(z) < N+1, Re(z) = N+\eta, 0 < \eta < 1 \tag{11}$$

and where,

$$F(z) = \int_0^\infty \prod_{i=1}^{I} d\alpha_i U^{-\frac{D}{2}} e^{-\frac{R}{U}} (\frac{W}{U})^z \tag{12}$$

3 Analysis of the Rest of the Expansion

Introducing the notation $z = \sigma + i\beta$ we have a first bound to the rest R_N in the truncated expansion above,

$$|R_N(\lambda)| \leq \lambda^{N+\eta} Q_N \tag{13}$$

with

$$Q_N = \int_{-\infty}^{+\infty} \frac{d\beta}{2\pi} |\Gamma(-\sigma - i\beta) F(\sigma + i\beta)|. \tag{14}$$

Using recurrence formulas we may relate $\Gamma(-\sigma - i\beta)$ to a gamma function which has positive real part of the argument. We get, remembering Equ. (11) above,

$$\Gamma(-\sigma - i\beta) = \Gamma(2 - \eta - i\beta) \prod_{j=0}^{N+1} \frac{1}{(-N - \eta + j) - i\beta} \tag{15}$$

Now, it may be shown [21] that for $c > 0$ the gamma function $\Gamma(c - i\beta)$ is bounded in absolute value,

$$|\Gamma(c - i\beta)| \leq e^{-\epsilon|\beta|} \int_{-\infty}^{\infty} du e^{cu - e^u cos\epsilon}, \tag{16}$$

where $\epsilon < \frac{\pi}{2}$ is a positive constant and $c = 2 - \eta$ is also a positive constant. Thus the bound has the form,

$$|\Gamma(c - i\beta)| < c' e^{-\epsilon|\beta|} \tag{17}$$

From Equs. (14), (15), and (17) we have,

$$Q_N < c' \int_{-\infty}^{\infty} \frac{d\beta}{2\pi} |F(N + \eta + i\beta))| \prod_{j=0}^{N+1} \frac{1}{[(-N - \eta + j)^2 + \beta^2]^{\frac{1}{2}}} \tag{18}$$

Also we have the inequalities,

$$\prod_{j=0}^{N+1} \frac{1}{[(-N - \eta + j)^2 + \beta^2]^{\frac{1}{2}}} \leq \prod_{j=0}^{N+1} \frac{1}{|(-N - \eta + j)|} < \frac{1}{N!\eta|(1 - \eta)|} \tag{19}$$

The first one is obvious, since $\beta^2 \geq 0$. To see the second one, let us recall the notation $\sigma = N + \eta$, and write,

$$\prod_{j=0}^{N+1} \frac{1}{|-\sigma + j|} = \prod_{j=0}^{N+1} \frac{1}{|\sigma - j|} = \frac{1}{\sigma(\sigma - 1)...(\eta + 1)\eta|\eta - 1|}. \tag{20}$$

From $\sigma > N$, $\sigma - 1 > N - 1$,..., $\eta + 1 > 1$, we find,

$$\prod_{j=0}^{N+1} \frac{1}{|-\sigma + j|} < \frac{1}{N!\eta(1-\eta)} \tag{21}$$

Combining Equs. (19), (18) and (13) we obtain a bound for the rest of the truncated expansion for the λ-scaled amplitude $G(\lambda)$,

$$|R_N(\lambda)| < \frac{\lambda^N.c'}{N!\eta|(\eta-1)|} \int_{-\infty}^{+\infty} \frac{d\beta}{2\pi} e^{-\epsilon|\beta|} |F(N+\eta+i\beta)| \tag{22}$$

Displacing indefinitely the integration path would generate instead of the truncated expansion, a series, provided the rest R_N have an appropriate behaviour as $N \to \infty$. Let us particularize to the limit of all masses going to zero. In this case the function $F(z)$ in Equ. (12) has the form,

$$F(z) = (\mu^2)^z \int_0^\infty \prod_{i=1}^{I} d\alpha_i U^{-\frac{D}{2}} e^{-\frac{R}{U}} (\frac{W'}{U})^z \tag{23}$$

where μ is a constant mass parameter and $W' = (\sum_{j\in G} \alpha_j)U(\alpha)$. For the absolute value of $F(z)$ we obtain a bound,

$$|F(z)| \leq (\mu^2)^N h(N, \{p\}) \tag{24}$$

with

$$h(N, \{p\}) = \int_0^\infty \prod_{i=1}^{I} d\alpha_i U^{-\frac{D}{2}} e^{-\frac{R}{U}} (W'/U)^{N+\eta} \tag{25}$$

where $\{p\}$ stands for the external momenta.

Taking I-dimensional spherical coordinates, the radial integration may be explicitly performed taking into account the homogeneity properties of the polynomials R, W' and U in the α variables ($R(\alpha)$ and $V'(\alpha)$ are homogeneous of degree $L+1$ and $U(\alpha)$ is homogeneous of degree L). We obtain an expression in terms of an integral over the I-dimensional angular variables Ω,

$$h(N, \{p\}) = \Gamma(I + N + \eta - \frac{DL}{2} + 1) \int d\Omega a(\Omega)[b(\Omega)]^{N+\eta}, \tag{26}$$

$a(\Omega)$ and $b(\Omega)$ being regular functions, and since the above integral in over angular variables, it has an upper bound $K_N(\{p\})$, K_N being a positive quantity. From Equs. (22), (24) and (26) we see that,

$$|R_N(\lambda)| < \frac{\Gamma(N+\eta+I-\frac{DL}{2}+1)}{\Gamma(N+1)} \frac{c'}{\eta(1-\eta)} (\mu^2)^{N+\eta} K_N(\{p\}) \lambda^{N+\eta}. \tag{27}$$

For $I - \frac{DL}{2} > 0$ (which is just the condition for UV convergence for the graph), the ratio $\kappa = \frac{\Gamma(N+\eta+I-(DL/2)+1)}{\Gamma(N+1)}$ is clearly a finite positive quantity. Thus renaming the various constants appearing in the expressions above, the rest of the asymptotic expansion may be written in the form,

$$|R_N(\lambda)| < K_1 K_N(\{p\})(\mu^2)^N \lambda^N. \tag{28}$$

The scaling parameter λ is arbitrarily small in the limit of the masses going to zero. Under the assumption that the constant K_N does not grow too fast as N becomes larger (for instance a growth slower than $N!$), the factor $(\lambda\mu^2)^N$ in the bound above makes the sequence of the remainders $R_N(\lambda)$ converge to zero as $N \to \infty$, which would be a condition for convergence of the asymptotic expansion. Although we think that the above is a reasonable assumption (it can be verified on several examples of Feynman amplitudes), we have not found a rigorous proof of that assumption for a general Feynman amplitude, so, from a mathematical viewpoint, the question of the convergence of the asymptotic expansion will remain as a conjecture.

4 The 2-point Function Critical Behaviour

We have shown in the preceeding section that we obtain a convergent series from Equ.(9) as $N \to \infty$. Let us particularize, as in the preceeding section, to the limit of all masses going to zero, and consider for simplicity the case of a single field having mass m. The analysis below can be generalized without difficulty to the case of several fields having different masses.

In the following, we consider dimensionally regularized amplitudes, that is we take the Euclidean space dimension D to be such that the amplitudes are formally defined as convergent integrals, divergences appearing later as singularities for some diagrams.

For the 2-point function $G^{(2)}(p^2, m^2)$, the only non-zero invariant of the type $(\sum p)^2$ contributing to the construction of the Symansik polynomial $V(\alpha)$ in Equ(3) is p^2. This may be seen if we note that for any diagram G contributing to the two-point function, the whole set of two-trees in the definition of $V(\alpha)$ in Equ.(3) divide into two classes, in which the total external momentum entering one of its connected components is either p^2 or zero (named respectively *relevant* and *irrelevant* two-trees). In this case, introducing a fixed mass scale μ, is easy to see that the inverse Mellin transform Equ(7) may be rewritten in terms of the variable p^2/m^2. Then the small mass behaviour of a graph G contributing to the two-point function, $G^{(2)}(p^2, m^2)$, has the form,

$$G(p^2, m^2) = \sum_{n=-\omega}^{\infty} (\frac{m^2}{p^2})^n \sum_{q=0}^{q_{max}(n)} A_{nq}(\mu^2) \left[- \ln(\frac{m^2}{p^2}) \right]^q , \qquad (29)$$

and we remember using Equs.(6) and (7), that the expansion above comes from the inverse Mellin transform,

$$G(\frac{m^2}{p^2}) = \frac{1}{2i\pi} \int_{\sigma - i\infty}^{\sigma + i\infty} dz (\frac{p^2}{m^2})^z \Gamma(-z) \int_0^\infty \prod_{i=1}^{I} d\alpha_i U^{-\frac{D}{2}} e^{-\mu^2 \sum_{i \in G} \alpha_i} (\frac{W'}{U})^z , \qquad (30)$$

where $W' = \mu^2 (\sum_{j \in G} \alpha_j) U(\alpha)$, $R'(\alpha) = \sum'_{2.T} \prod'_{\notin 2.T} \alpha_i$ and the notations $\sum'$ and $\prod'$ indicate respectively summation and product over *relevant* two-trees.

The coefficients $A_{nq}(\mu^2)$ in Equ.(29) come from the meromorphic structure of the Mellin transform displayed in Equ.(8). It is an extremelly hard task to determine explicitly all these coefficients, which is equivalent to completelly desingularize respective to z the integral over the α-variables in Equ.(30). In refs. [15] and [16] those coefficients $A_{\omega q}$ corresponding to the leading poles of the Mellin transform have been studied in the case of the behaviour of Feynman amplitudes at large momenta, which is mathematically equivalent to the case of vanishing masses we study here. We adapt the method used in the above mentioned works to get an expression for the leading coefficients in the case of the small mass behaviour. In the following we give only the general lines of the method, the results we have obtained and the definitions of the basic objects. Calculations are very involved and the full mathematical details in the case of the large momenta behaviour are in the above quoted references. These calculations can be adapted to our case without major difficulties. The main tool used to perform the analytic continuation of the Mellin transform is the generalized Taylor operator, a generalization of the operators used in Field theory for the purpose of renormalization. It is defined as follows: given a function $f(x)$ such that $x^{-\nu}f(x)$ is infinitely differentiable at $x = 0$, we define the generalized Taylor operator τ^n as,

$$\tau_x^n f(x) = x^{-\lambda-\epsilon}T^{n+\lambda}[x^{\lambda+\epsilon}f(x)], \tag{31}$$

where T is the usual Taylor operator, $\lambda \geq -E'(\nu)$ is an integer, $E'(\nu)$ is the smallest integer $\geq Re(\nu)$ and $\epsilon = E'(\nu) - \nu$. For any subdiagram $S \subseteq G$ it corresponds a generalized Taylor operator defined by,

$$\tau_S^n f(\{\alpha\}) = [\tau_\varrho^n f(\{\alpha\})|_{\alpha_i=\varrho^2\alpha_i, i\in S}]_{\varrho=1}, \tag{32}$$

and a basic quantity playing a role in the desingularization procedure has the form,

$$\prod_{S\subseteq G}(1 - \tau_S^{I(S)})(U^{\frac{D}{2}}(\alpha)\left[\frac{W'(\alpha)}{U(\alpha)}\right]^z), \tag{33}$$

where the product runs over subdiagrams S of the graph G, including G itself. Although the τ operators do not commute, it can be shown that the complete product $\prod_{S\subseteq G}(1 -$

$\tau_S^{I(S)}[.]$ is independent of the order of application upon the function between the brackets $[.]$.

The procedure follows along lines parallel as is done in [15] and in [16] for the large momenta behaviour. We obtain for the leading coefficients $A_{\omega q}$ the expression,

$$A_{\omega q} = \frac{1}{q!} \sum_{\{S_1,...,S_k\}} \frac{2^k}{(k-q-1)!} \xi_{G/S_1} \tilde{\xi}_{S_1/S_2} ... \tilde{\xi}_{S_{k-1}/S_k} \left[\frac{d^{k-q-1}}{dz^{k-q-1}} [\Gamma(-z) \tilde{\gamma}_{S_k}(z)] \right]_{z=-n_0} . \quad (34)$$

In the above equation, ω is the Weinberg leading power,

$$\omega = Sup_G[\omega(S)], \quad (35)$$

where Sup_G runs over the superficial degree of divergence of all essential subdiagrams of G, $\omega(S) = L(S)D - 2I(S)$ and $q_{max}(\omega) = Q - 1$, Q being the number of elements in the largest set of nested leading subdiagrams. The sum runs over all forests $\{S_1, ..., S_k\}$ of $k(> q)$ nested *leading* subdiagrams $S_1 \supset S_2 \supset ... \supset S_k$ (we remember that *leading* subdiagrams are those whose superficial degree of divergence equals ω); the quantities ξ, $\tilde{\xi}$, $\tilde{\gamma}$ are obtained from the subdiagrams $S \subseteq G$, by the formulas,

$$\xi_S = \int_\kappa^\infty \prod_{i \in S} d\alpha_i e^{-\mu^2 \sum_{i \in S} \alpha_i} \prod_{S' \subseteq S} (1 - \tau_{S'}^{-2I(S')}) U_S^{\frac{D}{2}}(\alpha), \quad (36)$$

$$\tilde{\xi}_S = -\frac{d\xi_S}{d\mu^2} = \mu^2 \sum_{i \in S} \xi_{S_i}, \quad (37)$$

$$\tilde{\gamma_{S_k}} = \frac{\Gamma(-\frac{z}{2})}{\Gamma(-z)} \mu^2 \int_0^\infty \prod_{i \in S} d\alpha_i e^{-\mu^2 \sum_{j \in S} \alpha_j} \prod_{S' \subseteq S} (1 - \tau_{S'}^{-2I(S'}) \left[\frac{W_S'}{U_S}(\alpha) \right]^{\frac{z}{2}} (\sum_{i \in S} \alpha_i) U_S^{-\frac{D}{2}}(\alpha), \quad (38)$$

where S_i is the diagram obtained from S by inserting a two leg vertex (a mass insertion μ^2) on the line i; particular cases for the quantities in the above equations are $\varrho_{G/G} = 1$ and $\varrho_{G/S} = 0$ if G itself is leading. The Feynman amplitude corresponding to S_i is simply given by,

$$G_{S_i} = \int_0^\infty \prod_{i \in S} d\alpha_i U^{-\frac{D}{2}}(\alpha)(\alpha_i) e^{-\frac{V(\alpha)}{U(\alpha)}}. \quad (39)$$

The various factors in Equs. (34), (36), (37) and (38) can be reorganized to write the leading coefficients in the more convenient form,

$$A_{\omega q} = \frac{1}{q!} \sum_{\{S_1,...,S_k\}} \frac{2^k}{(k-q-1)!} \frac{d^{k-q-1}}{dz^{k-q-1}} \left[\int_0^\infty \prod_{i \in G} d\alpha_i e^{-\mu^2 \sum_{i\in G} \alpha_i} A(\alpha; z) \right]_{z=\omega}, \qquad (40)$$

where the function $A(\alpha; z)$ is defined by,

$$A(\alpha; z) = \Gamma(-\frac{z}{2}) R'_G \left[U_{G/S_1}^{-\frac{D}{2}} U_{S_1/S_2}^{-\frac{D}{2}} ... U_{S_{k-1}/S_k}^{-\frac{D}{2}} U_{S_k}^{-\frac{D}{2}} (\sum_{S_1/S_2} \alpha_i) ... (\sum_{S_{k-1}/S_k} \alpha_i)(\sum_{S_k} \alpha_i)(W'_{S_k}/U_{S_k})^{\frac{z}{2}} \right], \qquad (41)$$

and, taking the convention $S_0 \equiv G$, R'_G is the operator (the order $-2I(T)$ is understood for each τ operator corresponding to a subdiagram T),

$$R'_G = \prod_{l=1}^{k} \left[\prod_{T_l \subseteq S_{l-1}/S_l} (1 - \tau_{T_l}) \right] \prod_{T \subseteq S_k} (1 - \tau). \qquad (42)$$

The operator R'_G does not change the homogeneity properties of of the functions upon which it acts. So, remembering the homogeneity properties of the polynomials U and W', and noting that $L(G/S_1) + L(S_1/S_2) + ... + L(S_{k-1}/S_k) + L(S_k) = L(G)$, we see from Equ.(41) that $A(\alpha; z)$ is a homogeneous function in the α-variables of degree $\frac{L(G)D}{2} + k + \frac{z}{2}$. Taking spherical coordinates in α-space we may write from the preceeding equations,

$$A_{\omega q} = \frac{1}{q!} \sum_{\{S_1,...,S_k\}} \frac{2^{k-1}}{(k-q-1)!} \frac{d^{k-q-1}}{dz^{k-q-1}} \left[\Gamma(-\frac{z}{2}) \int d\Omega \int_0^\infty d\varrho e^{\varrho\mu^2 f(\Omega)} \varrho^{I - \frac{L(G)D}{2} + k + \frac{z}{2} - 1} g(\Omega; z) \right]_{z=\omega}. \qquad (43)$$

The integral over ϱ in the equation above may be expressed in terms of the Γ-function, and the functions $f(\Omega)$, $g(\Omega; z)$ are functions of the angular variables Ω depending on the specific topological characteristics of the graph G considered. We get for the leading coefficients $A_{\omega q}$,

$$\begin{aligned} A_{\omega q} &= \frac{1}{q!} \sum_{\{S_1,...,S_k\}} \frac{2^{k-1}}{(k-q-1)!} \frac{d^{k-q-1}}{dz^{k-q-1}} \left[\Gamma(-\frac{z}{2}) \int d\Omega g(\Omega; z) (\mu^2 f(\Omega))^{-(I - \frac{L(G)D}{2} + k + \frac{z}{2})} \right. \\ &\quad \left. \Gamma(I - \frac{L(G)D}{2} + k + \frac{z}{2}) \right]_{z=\omega}. \end{aligned} \qquad (44)$$

In the very neighbourhood of criticality the contribution from the amplitude G is given by the leading term in the expansion (29) which corresponds to the highest powers of m^2/p^2 and of $-log(m^2/p^2)$, that is, for very small values of m^2 we have,

$$G(p^2,m^2) \approx A_{\omega Q}(\frac{m^2}{p^2})^{-\omega}\left[-\ln(\frac{m^2}{p^2})\right]^{Q-1}, \tag{45}$$

where we remember that ω is the Weinberg leading power, Equ.(35), and Q is the number of elements in the largest set of leading subdiagrams of G. To get the coefficient $A_{\omega Q}$ in Equ.(45) from Equ.(44), we note that the sum has only one term, corresponding to the nest $\{S_1,...,S_Q\}$ and a zero-th order derivative. We obtain,

$$A_{\omega Q} = \frac{2^{Q-1}}{(Q-1)!}\Gamma(-\frac{\omega}{2})g(\Omega;z)\Gamma(I-\frac{L(G)D}{2}+k+\frac{z}{2})\int d\Omega g(\Omega;\omega)\left[\mu^2 f(\Omega)\right]^{-(I-\frac{L(G)D}{2}+k+\frac{z}{2})}. \tag{46}$$

Now, if G itself is leading it does not contribute to the expansion, since in this case $\varrho_{G/S}=0$ for every $S\subset G$. If G is not leading, $\omega(G)=L(G)-2I(G)<\omega$, and there exists a $\delta>0$ such that $\omega=L(G)D-2I(G)+\delta$. The arguments of the Γ-functions in Equ.(46) above are respectively $-\frac{\omega}{2}=\frac{2I(G)-L(G)D-\delta}{2}$, and $\delta+Q$. Since $\delta>0$ and $Q\geq 1$, singularities in $A_{\omega Q}$ come from the factor $\Gamma(-\frac{\omega}{2})$. Thus at fixed space dimension Δ (for instance $\Delta=3$), if the diagram G has a topological structure such that $L(G)\Delta-2I(G)+\delta=2n$, n an integer ≥ 0, the corresponding singularity of the Γ-function above, $\Gamma(-\frac{\omega}{2})$, should be removed by a renormalization procedure. This may be done, as usual, taking $D=\Delta-\epsilon$, and subtracting the pole at $\epsilon=0$, leaving some regular function $\Gamma_{Ren}(\Delta)$. The result in space dimension Δ for the coefficient $A_{\omega Q}$ reads,

$$A_{\omega Q}(G) = \frac{2^{Q(G)-1}}{(Q(G)-1)!}\Gamma_{Ren;G}(\Delta)(\mu^2)^{-[\delta+Q(G)]}\Gamma(\delta+Q(G))\int d\Omega g_G(\Omega;\omega)\left[f_G(\Omega)\right]^{-[\delta+Q(G)]}. \tag{47}$$

In the above equation we have displayed explicitly the dependence of the various quantities on the single Feynman amplitude G we have considered.

Thus the behaviour of the two-point function near criticality is described by an expression having the form,

$$G^{(2)}(p^2, m^2) \approx \sum_G A_{\omega Q}(G)(\frac{m^2}{p^2})^{-\omega(G)} \left[-\ln(\frac{m^2}{p^2}) \right]^{Q(G)-1} \quad (48)$$

where the symbol $\sum_G$ means summation over the whole set of Feynman diagrams contributing to the two-point function. The quantities under the summation symbol can be obtained by explicit calculation for each Feynman diagram G. This result holds for any scalar field theory without derivative couplings.

Concluding Remark The present study of the critical behaviour of the two-point function concerns mainly a detailed analysis of the behaviour of the Feynman amplitudes contributing to it in the perturbative expansion. In applications of field theory to critical phenomena, the examples of models of field theory that have been found to give relevant information, are controled by the free field fixed point, or by fixed points that approach the free field fixed point in some limit. This means that Feynman diagram approach to field theory plays an important role in understanding physical situations in critical phenomena. This is one of the reasons why we hope the analysis presented in this work could be interesting.

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