CBPF-NF-052/98

Domains of Bosonic Fuctional Integrals

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Abstract

We propose a mathematical framework for Bosonic Euclidean Quantum Field Functional Integrals based on the theory of integration on the dual algebraic Vector Space of Classical Field Sources. We present a generalization of the Minlos-Dao Xing theorem and apply it to determine exactly the Domain of Integration associated to the functional integral representation of the two-dimensional Quantum Electrodynamics Schwinger Generating Functional.

I Introduction

Since the result of R.P. Feynman on representing the initial value solution of Schrödinger Equation by means of an analitycally time continued integration on a infinite – dimensional space of functions ([1] – theorem 6.4), the subject of Euclidean Functional Integrals representations for Quantum Systems has became the correct mathematical – operational framework to analyze Quantum Phenomena ([1],[2], [3]).

One of the most important open problem in the mathematical theory of Euclidean Functional Integrals is that related to implementation of mathematical – numerical approximations to these Infinite-Dimensional Integrals by means of Finite–Dimensional approximations outside the crude and cumbersome often used Space-Time Lattice approximations (see [2], [3] – chap. 9 and section II of this paper). As a first step to study the above cited problem it will be need to characterize mathematically the Functional Domain where these Functional Integrals are defined.

The purpose of this paper is to present in section II, the formulation of Euclidean Quantum Field theories as Functional Fourier Transforms by means of the Bochner-Martin-Kolmogorov theorem for Vector Spaces ([4], [5] – theorem 4.35) and suitable to analyze the Domain of the associated Functional Integrals by means of a proposed generalization of ours of the well-known Minlos-Dao Xing theorem ([5] – theorem 4.312 and [6] – part 2) and presented in section III.

II The Euclidean Schwinger Generating Functional as a Functional Fourier Transform

The basic object in a scalar Euclidean Quantum Field Theory in R^D is the Schwinger Generating Functional

$$Z[j(x)] = \left\langle \Omega_{_{VAC}} | exp\left(i \int d^D x \; j(\vec{x}, it) \phi^{(m)}(\vec{x}, it) \right) |\Omega_{_{VAC}} \right\rangle \tag{1}$$

where $\phi^{(m)}(\vec{x}, it)$ is the supposed Self-Adjoint Minkowski Quantum Field analitically continued to imaginary time and $j(x) = j(\vec{x}, it)$ is a set of functions belonging to a given

Vector Space of functions denoted by E which topology is not specified yet and will be called the Schwinger Classical field source space. It is important to remark that $\{\phi^m(\vec{x}, it)\}$ is a commuting Algebra of Self-Adjoints operators as Symanzik has pointed out ([7]).

In order to write eq. (1) as a Integral over the space E^{alg} of all linear functionals on the Schwinger Source Space E (the called Algebraic Dual of E), we take a procedure different from the usual abstract approach (see the proof of th IV – 11 – [2]) by making the hypothesis that the restriction of the Schwinger Generating Fuctional eq. (1) to any finite-dimensional R^N of E is the Fourier Transform of a positive continuous function, namelly

$$Z\left(\sum_{\alpha=1}^{N} C_\alpha \bar{j}_\alpha(x)\right) = \int_{R^N} exp\left(i\sum_{\alpha=1}^{N} C_\alpha p_\alpha\right) \tilde{g}(P_1, \cdots, P_N) dP_1 dP_N \tag{2}$$

Here $\left\{\vec{j}_\alpha(x)\right\}_{\alpha=1,\cdots,N}$ is a fixed vectorial base of the given finite-dimensional sub-space (isomorphic to R^N) of E.

As a consequence of the above made hypothesis (based physically on the Renormalizability and Unitary of the associated Quantum Field Theory), one can apply the Bocher-Martin-Kolmogorov Theorem ([5] – theorem 4.35) to write eq. (1) as a Functional Fourier Transform on the Space E^{alg} (see appendix)

$$Z[j(x)] = \int_{E^{alg}} exp(ih(j(x))d\mu(h) \tag{3}$$

where $d\mu(h)$ is the Kolmogorov cylindrical measure on $E^{alg} = \prod_{\lambda\in A} (R^\lambda)$ with A denoting the index set of the fixed Hamel Vectorial Basis used in eq. (2) and $h(j(x))$ is the action of the given Linear (algebric) Functional (belonging to E^{alg}) on the element $j(x) \in E$.

At this point, we related the mathematically non-rigorous physicist point of view to the Kolmogorov measure $d\mu(h)$ eq. (3) over the Algebraic Linear Functions on the Schwinger Source Space. It is formally given by the famous Feynman formulae when one identify the action of h on E by means of an "integral" average

$$h(j) = \int_{R^D} dx^D j(x) h(x) \tag{4}$$

Formally we have the equation

$$d\mu(h) = \left(\prod_{x\in R^D} dh(x)\right) exp\{-S(h(x))\} \tag{5}$$

where S is the classical action being of the Classical Field Theory quantized, but with the necessary coupling constant renormalizations need to make the associated Quantum-Field Theory Unitary.

Let us exemplify this point on a $\lambda\phi^4$ – Field Theory on R^4.

At first we will introduced the massive free field theory generating functional directly in the infinite value space R^4

$$Z[j(x)] = exp\left\{-\frac{1}{2}\int d^D x d^D x' j(x)((-\Delta)^\alpha + m^2)^{-1}(x,x')j(x')\right\} \tag{6}$$

where the Free Field Propagator is given by

$$((-\Delta)^\alpha + m^2)^{-1}(x,x') = \int d^4k \; \frac{e^{ik(x-x')}}{k^{2\alpha}+m^2} \tag{7}$$

with α a regularizing parameter.

As the source space, we will consider the vector space of all real sequences on $\prod_{\lambda\in(-\infty,\infty)}(R)^\lambda$, but with only a finite number of non-zero components. Let us define the following family of finite-dimensional Positive Linear Functionals $\{L_{\Lambda_f}\}$ on the Functional Space $C(\prod_{\lambda\in(-\infty,\infty)} R^\lambda; R)$

$$L_{\Lambda_f}\left(e(P_{\lambda_{s_1}};\cdots P_{\lambda_{s_N}})\right) = \int_{(\prod_{\lambda\in\Lambda F} R^\lambda)} e(P_{\lambda_{s_1}},P_{\lambda_{s_N}}) exp\left\{-\frac{1}{2}\sum_{\lambda\in\Lambda_f}(\lambda^{2\alpha}+m^2)(P_\lambda)^2\right\}$$

$$\left(\prod_{\lambda\in\Lambda_f} d(P_\lambda\sqrt{\pi(\lambda^{2\alpha}+m^2})\right) \tag{8}$$

Here $\Lambda_f = \{\lambda_{s_1},\cdots,\lambda_{s_N}\}$ is an ordered sequence of real numbers of the real line which is the index set of the Hamel Basis of the Algebraic Dual of the proposed source space. Note that we have the generalized eingenproblem expansion

$$((-\Delta)^\alpha + m^2)e^{i\lambda x} = (\lambda^{2\alpha}+m^2)e^{i\lambda x} \tag{9}$$

By the Stone-Weirstrass Theorem or the Kolmogoroff Theorem applied to the family of finite dimensional measure in eq. (8), there is a unique extension measure $d\mu(\{P_\lambda\}_{\lambda\in(-\infty,\infty)})$ to the space $\prod_{\lambda\in(-\infty,\infty)} R^\lambda = E^{alg}$ and representing the Infinite-Value Generating Functional on our choosen source space (the usual Riesz-Markov theorem applied to the linear functional $L = \lim_{\{\Lambda_f\}} sup\ L_{\Lambda_f}$ on $C(\prod_{\lambda\in(-\infty,\infty)}, R^\lambda, R)$ leads to this extension measure)

$$Z[j(x)] = Z[\{j_\lambda\}_{\lambda\in\Lambda_f}] = \int_{\prod_{\lambda\in(-\infty,\infty)} R^\lambda} d\mu^{(0),(\alpha)}(\{P_\lambda\}_{\lambda\in(-\infty,\infty)})$$

$$exp\left(i \sum_{\lambda\in(-\infty,\infty)} j_\lambda p_\lambda\right) = exp\left\{-\frac{1}{2}\sum_{\lambda\in\Lambda_f}\frac{(j_\lambda)^2}{\lambda^{2\alpha}+m^2}\right\} \tag{10}$$

At this point it is very important remark that the generating functional has natural extension to any distributions spaces $(S'(R^N), D'(R^N)$, etc.) which contains the continuous functions of compact support as a dense sub-space.

At this point we consider the following Quantum Field interaction functional which is a measurable functional in relation to the above constructed Kolmogoroff measure $d\mu^{(0).(\alpha)}(\{P_\lambda\}_{\lambda\in(-\infty,\infty)})$ for α *non integer* in the original field variable $\phi(x)$

$$\begin{aligned} V^{(\alpha)}(\phi) &= \lambda_R\phi^4 + \frac{1}{2}(Z_\phi^{(\alpha)}(\lambda_R,m)-1)\phi((-\Delta)^\alpha)\phi - \frac{1}{2}[(m^2 Z_\phi^{(\alpha)}(\lambda_R,m)-1 \\ &-(\delta m^2)^{(\alpha)}(\lambda_R)]\phi^2 - [Z_\phi^{(\alpha)}(\lambda_R,m)(\delta^{(\alpha)}\lambda)(\lambda_R,m)]\phi^4 \end{aligned} \tag{11}$$

Here the renormalization constants are given in the usual analitical regularization form for a $\lambda\phi^4$ – Field Theory. It still a open problem in the mathematical–physics of quantum fields to prove the integrability in some distribution space of the cut-off removing $\alpha \to 1$ limit of the interaction lagrangean $exp(-V^{(\alpha)}(\phi))$.

III The Support of Functional Measures

Let us now analyze the measure support of Quantum Field Theories generating functional eq. (3) (the Wiener theorem about Hölder – continuity of the Wiener measure support of the Oscillator Process Functional Integral ([5], Theorem 5.15 is the main result in O-dimensional Quantum Field Theories).

For higher dimensional space-time, the only available result in this direction is when we have a Hilbert structure on E ([4], [5], [6]).

At this point of our paper, we introduce some definitions. Let $e : \mathbb{Z}^+ \to R$ be an increasing function (including the case $e(\infty) = \infty$). Let E be denoted by $\mathcal{H}$ and $\mathcal{H}^Z$ be the sub-space of $\mathcal{H}^{alg} = (\prod_{\lambda \in A=[0,1]} R^{\lambda}$ (with A being the index set of a Hamel basis of $\mathcal{H}$), formed by all sequences $\{x_\lambda\}_{\lambda\in A} \in \mathcal{H}^{alg}$ with coordinates different from zero at most a countable number

$$H^2 = \{(x_\lambda)_{\lambda\in A} | x_\lambda \neq 0 \quad \text{for} \quad \lambda \in \{\lambda_\mu\}_{\mu\in \mathbb{Z}}\} \tag{12}$$

Consider the following weighted sub-set of $\mathcal{H}^{alg}$

$$H^Z_{(e)} = \{\{x_\lambda\}_{\lambda\in A} \in H^Z |$$

and

$$\lim_{N\to\infty} \left\{ \frac{1}{e(N)} \sum_{n=1}^{N} (x_{\lambda\sigma(\mu)})^2 \right\} < \infty \; \}$$

for any $\sigma : N \to N$, a permutation of the natural numbers.

We now state our generalization of the Minlos–Dao Xing Theorem.

- **Theorem 3** – Let T be an operator, with Domain $D(T) \subset \mathcal{H}$, and $TD(T) \to \mathcal{H}$ such that for any finite-dimensional space $\mathcal{H}^N \subset \mathcal{H}$, the sum is bounded by the function $e(N)$

$$\left| \sum_{(i,j)=1}^{N} < Te_i, Te_j >^{(0)} \right| \geq e(N) \tag{13}$$

Here $<,>^{(0)}$ is the inner product of $\mathcal{H}$ and $\{e_p\}_{1\leq pN}$ is a vectorial basis of the sub-space $\mathcal{H}^N$ with dimension N.

Suppose that $Z[j(x)]$ is a continuous function an $\tilde{D}(T) = \overline{(D(T), <,>^{(1)}}$ where $<,>^{(1)}$ is a new inner product defined by the operator $T(< j, \bar{j} >^{(1)} = < Tj, T\bar{j} >^{(0)})$ *we have, thus, that the support of the cylindrical measure eq. (3) is the measurable set* $\mathcal{H}^Z_e$.

Proof: following closely references ([1]) - Theorem 2.2., [4]) let us consider the following representation for the characteristic function of the measurable set $\mathcal{H}^Z_e \subset \mathcal{H}^{alg}$

$$\begin{aligned} & X_{H^Z_e}(\{x_\lambda\}_{\lambda\in A}) = \\ & \lim_{\alpha\to 0} \lim_{N\to\infty} \quad exp\left\{ -\frac{\alpha}{e(N)} \sum_{e=1}^{N} x^2_{\lambda_e} \right\} \\ & = 1 \qquad if \qquad \lim_{N\to\infty} \frac{1}{e(N)} \sum_{e=1}^{N} x^2_{\lambda_e} < \infty \\ & 0 \qquad \text{otherwise} \end{aligned} \tag{14}$$

Now its measure satisfies the following inequality

$$\int_{H^{alg}} d\mu(h) = \mu(\mathcal{H}^{alg}) = 1 > \mu(\mathcal{H}_e^Z) \tag{15}$$

but

$$\begin{aligned} \mu(H_e^Z) &= \lim_{\alpha\to 0}\lim_{N\to\infty}\int_{H^{alg}} d\mu(h)\ exp - \left\{\frac{\alpha}{e(N)}\sum_{e=1}^{N} x^2_{\lambda_e}\right\} \\ &\lim_{\alpha\to 0}\lim_{N\to\infty}\left\{\frac{1}{(\frac{2\pi\alpha}{e(N)})^{n/2}}\right\}\int dC_1\cdots dC_N \\ &exp\left(-\frac{1}{2}\left(\frac{e(N)}{\alpha}\right)\sum_{e=1}^{N} j_e^2\right)\tilde{Z}(j_1;\cdots,j_N) \end{aligned} \tag{16}$$

where

$$\tilde{Z}(j_1,\cdots,j_N) = \int_{\substack{\pi R^\lambda \\ \lambda\in A}} d\mu(\{x_\lambda\}) exp\left(i\sum_{e=1}^{N} x_{\lambda_e} j_e\right) \tag{17}$$

Now due to the continuity and positivity of $Z[j]$ in $\tilde{D}(T)$; we have that for any $\varepsilon > 0 \to \exists\delta$ such that the inequality below is true

$$\begin{aligned} &\frac{1}{\left(\frac{2\pi\alpha}{e(N)}\right)^{N/2}}\int_{R^N} dj_1\cdots dj_N\ exp\left(-\frac{1}{2}\frac{e(N)}{\alpha}\sum_{e=1}^{N} j_e^2\right)\tilde{Z}(j_1,\cdots,j_N) \\ &\geq 1-\varepsilon-\frac{2}{\delta^2}\left\{\sum_{(m,n)=1}^{N}\frac{1}{\left(\frac{2\pi\alpha}{e(N)}\right)^{N/2}}\int_{R^N} dj_1\cdots dj_N\ exp\left(-\frac{1}{2}\left(\frac{e(N)}{\alpha}\right)\sum_{m=1}^{N} j_m^2\right)\right\} \\ &j_m j_n < e_m, e_n >^{(1)} \geq 1-\varepsilon-\frac{2}{\delta^2}\left\{\left(\frac{\alpha}{e(N)}\right)\sum_{(m,n)=1}^{N}\delta_{mn} < Te_n, Te_m >^{(0)}\right\} \\ &\geq 1-\varepsilon-\frac{2}{\delta^2}\left(\frac{\alpha}{e(N)}\right)e(N) \geq 1-\varepsilon-\frac{2}{\delta^2}\alpha \end{aligned} \tag{18}$$

By substituting eq. (18) into eq. (15), we get the result

$$1 \geq \mu(\mathcal{H}_e^Z) \geq 1-\varepsilon-\frac{2}{\delta^2}\left(\lim_{\alpha\to 0}\alpha\right) = 1-\varepsilon \tag{19}$$

Since ε was arbitrary we have the validity of our theorem.

As a consequence of this Theorem in the case of $e(N)$ being bounded (so T is an operator of Trace Class), we have that $H_e^Z = \mathcal{H}$ which is the usual Topological Dual of $\mathcal{H}$.

At this point, a simple proof may be given to the usual Minlos–Dao Xing Theorem on Schwartz Spaces ([5], [6], [6]).

Let us consider $S(R^D)$ represented as the countable normed spaces of sequences ([8])

$$S(R^D) \bigcap_{m=0}^{\infty} \ell^2_\mu \tag{20}$$

where

$$\ell^2_m = \{(x_n)_{n\in \mathbb{Z}}, x_n \in R \Big| \sum_{n=0}^{\infty} (x_n)^2 n^m < \infty\} \tag{21}$$

The Topological Dual is given by the nuclear structure sum ([8])

$$S'(R^D) = \bigcup_{n=0}^{\infty} \ell^2_{-n} = \bigcup_{n}^{\infty} (\ell^2_n)^* \tag{22}$$

We, thus, consider $E = S(R^D)$ in eq. (3) and $Z[j(x)] = Z[\{j_n\}_{n\in\mathbb{Z}}]$ as a continuous on $\bigcup_n^\infty \ell_n$. Since $Z[\{j_n\}_{n\in\mathbb{Z}}] \in C(\bigcup_n^\infty \ell^2_n, R)$ we have that for any fixed integer $P, Z[\{j_n\}_{n\in\mathbb{Z}}]$ is continuous on the Hilbert Space ℓ^2_p which, by its turn, may be considered as the Domain of the following operator

$$\begin{aligned} T_p : \ell^2_p c \ell_0 &\to \ell_0 \\ \{j_n\} &\to \{n^{p/2} j_n\} \end{aligned} \tag{23}$$

It is straightforward to have the estimative

$$\left| \sum_{(m,n)=1}^{N} < T_p e_i . T_p \ell_j >^{(0)} \right| \le N^{(Bp)} \tag{24}$$

for some positive integer B and $\{e_i\}$ being the canonical orthonormal basis of ℓ^2_0. By an application of our theorem for each fixed p; we get that the support of measure is given by the union of weighted spaces

$$\text{supp} d\mu(h) = \bigcup_{p=0}^{\infty} (\ell^2)^2 = \bigcup_{p=0}^{\infty} \ell^2_{-p} = S'(R^D). \tag{25}$$

At this point we can give a straightforward (non topological) generalization of the Minlos–Dao Xing Theorem.

- **Theorem 4** - Let $\{T_\beta\}_{\beta\in C}$ be a family of operators satisfying the hypothesis of Theorem 3. Let us consider the Locally Convex space $\bigcup_{\beta\in C} \overline{Don(T_\beta)}$ (supposed non-empty) with The Family of norms $||\psi||_\beta =< T_\beta\psi, T_\beta\psi >^{1/2}$

If the Functional Fourier Transform is continous on this Locally Convex Space, the support of the Kolnogaroff measure eq. (3) is given by the following sub-set of $[\bigcup_{\beta\in C}\overline{Don(T_\beta)}]^{olg}$, namely

$$\text{supp} d\mu(h) = \bigcup_{\beta\in C} H^2_{e_\beta} \tag{26}$$

where e_β are the functions given by Theorem 3.

Let us now proceed to apply the above displayed results by considering the Schwinger Generating Functionals for two-dimensional Euclidean Quantum Electrodynamics in Bosonized Parametrization ([9])

$$Z[j(x)] = exp\left\{-\frac{1}{2}\int_{R^2} d^2x \int_{R^2} d^2y\ j(x)((-\Delta)^2 + \frac{e^2}{\pi}(-\Delta))^{-1}(x,y)j(y)\right\} \tag{27}$$

where in eq. (27), the electromagnetic field has the decomposition in Landau Gauge

$$A_\mu(x) = (\varepsilon_{\mu\nu}\partial_\nu\phi)(x) \tag{28}$$

and $j(x)$ is, thus, the Schwinger Source for the $\phi(x)$ field taken as a basic dynamical variable ([9]).

Since eq. (27) is continuous in $L^2(R^2)$ with the inner product defined by the trace class operator $((-\Delta)^2 + \frac{e^2}{\pi}(-\Delta))^{-1}$, we conclude on basis of theorem 3 that the associated Kolnogoroff measure in eq. (3) has its support in $L^2(R^2)$ with the usual inner product. As a consequence, the Quantum Observable Algebra will be given by the Functional Space $L^1(L^2(R^2), d\mu(h))$ and usual orthonormal Finite – Dimensional approximations in Hilbert Spaces may be used safely i.e if one consider the basis expansion $h(x) = \sum_{n=1}^{\infty} h_n e_n(x)$ with $e_n(x)$ denoting the eigenfunctions of the operator in Eq. (27) we get the result

$$\bigcup_{n=1}^{\infty} L^1(R^N, d\mu(h_1,\cdots,h_N)) = L^1(L^2(R^2), d\mu(h)) \tag{29}$$

It is worth mentioning that if one uses the Gauge Field Parametrization for the $(\text{Q.E.D})_2$ – Schwinger Functional

$$Z[j_1(x), j_2(x)] = exp\left\{-\frac{1}{2}\int_{R^2} d^2x \int_{R^2} d^2y\ j_i(x)\left(-\Delta + \frac{e^2}{\pi}\right)^{-1}(x,y)\delta_{i\ell}j_\ell(x)\right\} \tag{30}$$

the associated measure support will now be the Schwinger Space $S'(R^2)$ since the operator $(-\Delta + \frac{e^2}{\pi})^{-1}$ is an application of $S(R^2)$ to $S'(R^2)$. As a consequence it will be very cumbersome to use Hilbertinian Finite Dimensional approximations ([8]) as in eq. (29).

An alternative to approximate tempered distributions is the use of its Hermite expansion in $S'(R)$ space associated to the eigenfunctions of the Harmonic-oscillator $V(x) \in L^\infty(R) \bigcap L^2(R)$ potential perturbation (see ref. [3] for details)

$$(-\frac{d^2}{dx^2} + x^2 + V(x))H_\mu(x) = E_\mu H_\mu(x) \tag{31}$$

Another important class of Bosonic Functionals Integrals are those associated to an Elliptic Positive Self-Adjoint Operator A on $L^2(\Omega)$ with suitable Boundary conditions. Here Ω denotes a D-dimensional compact manifold of R^D with volume element $d\nu(x)$

$$Z[j(x)] = exp\left\{ -\frac{1}{2} \int_\Omega d\nu(x) \int_\Omega d\nu(y)\ j(x)A^{-1}(x,y)j(y) \right\} \tag{32}$$

If A is an operator of trace class on $(L(\Omega), d\nu)$ we have, thus, the validity of the usual eigenvalue Functional Representation

$$Z[\{j_n\}_{\mu\in\mathbb{Z}}] = \int (\prod_{e=1}^{\infty} d(c_e\sqrt{\lambda_e}))exp\left(-\frac{1}{2}\sum_{e=1}^{\infty}\lambda_e c_e^2\right) \chi_{\ell^2}(\{e_n\}_{n\in\mathbb{Z}})\ exp(i\sum_{e=1}^{N} c_e j_e)$$

with the spectral set

$$\begin{aligned} A\sigma_e &= \lambda_e\sigma_e \\ j_e &= < j, \sigma_e > \\ c_e &= < \sigma, \sigma_e > \end{aligned} \tag{33}$$

and the characteristic function set

$$\chi_{\ell^2}(\{c_n\}_{n\in\mathbb{Z}}) = \begin{cases} 1 & \text{if} \quad \sum_{e=0}^{\infty} c_e < \infty \\ 0 & \text{otherwise} \end{cases} \tag{34}$$

It is instructive point out the usual Hermite functional basis (see 5.4 – [5]) are a complete set in $L^2(E^{clg}, d\mu(h))$, only if the Gaussian Kolnogoroff measure $d\mu(h)$ is of the class above studied.

Finally, in the always used framework of Finite-Dimensional Space-Time Lattice approximations for Functional Integrals, the convergence proofs may be not well – defined, since the basic used result that the sub-space $L^2(\Omega)$ is isomeytrically imbedding in $L^2(R^2)$ is true only if Ω is a set of non zero measure. As a consequence, the sequence of approximations lattice functions $f_n(x) = f(x)x_{\mathbb{Z}_\delta}(x)$ of a given $f(x) \in L^2(R^2)$ do not converge in $L^2(R^2)$ for $\sigma = \sigma_n \to 0$ where $\mathbb{Z}_\sigma$ is a Lattice of spacing σ of R^2 (see the Theorem VII.5 – ref. [2]; and proposition 9.55 – [3] for similar claims in Functional Spaces case).

A further criticism to the usual framework to construct Euclidean Field Theories is that is very cumbersome to analyze the infinite volume limit from the Schwinger Generating Functional defined originally on Compact Space-Times. In two dimensions the use of the result that the massive Scalar Field Theory Generating Functional

$$exp\left\{-\frac{1}{2}\int_{R^2} d^2x \int_{R^2} d^2y \; j(x)(-\Delta + m^2)^{-1}(x,y)j(y)\right\} \tag{35}$$

with $j(x) \in S(R^2)$; is given by the limit of *Finite Volume* Dirichlet Field Theories

$$\lim_{\substack{L\to\infty \\ T\to\infty}} exp\left\{-\frac{1}{2}\int_{-L}^{L} dx^0 \int_{-T}^{T} dx^1 \int_{-L}^{L} dy^0 \right.$$
$$\left. \int_{-T}^{T} dy^1 j(x^0,x^1)(-\Delta_D + m^2)^{-1}(x^1,y^8,x^0,y^0)j(y^0,y^1)\right\} \tag{36}$$

may be considered in our opinion as the similar claim made that is possible to deduce the Fourier Transforms from Fourier Series.

Finally, let us comment of the functional integral associated to Feynman propagation of fields configurations used in geometrodynamical theories in the scalar case

$$G[\beta^{in}(x);\beta^{out}(x),T](j) = \int_{\substack{\phi(x,0)=\beta^{in}(x) \\ \phi(x,t)=\beta^{out}(x)}}$$
$$exp\left\{-\frac{1}{2}\int_0^T dt \int_{-\infty}^{+\infty} d^Dx \left(\phi\left(-\frac{d^2}{dt^2}+A\right)\phi\right)(x,A)\right\}$$
$$exp\left(i\int_0^T dt \int_{-\infty}^{+\infty} d^\nu x \; j(x)t)\phi(x,t)\right) \tag{37}$$

If one define the formal functional integral by means of the eigenfunctions of the self-adjoint Eliptic operator A, namelly:

$$\phi(x,t) = \sum_{\{k\}} \phi_k(t)\psi_k(x) \tag{38}$$

where

$$A\psi_k(x) = (\lambda_k)^2\psi_k(x) \tag{39}$$

it is straightforward to see that eq. (36) is formally exactly evaluated in terms of a infinite product of usual Feynman-path measures

$$\begin{aligned}
&G[\beta^{in}(x);\beta^{out}(x),T](j) = \\
&= \prod_{\{k\}} \int_{\substack{c_k(0)=\phi_k(0)\\ c_k(T)=\phi_k(T)}} D^F[c_k(t)]\ exp\left\{-\frac{1}{2}\int_0^t \left(c_k\left(-\frac{d^2}{dt^2}+\lambda_k\right)c_k\right)(t)dt\ exp\left(i\int_0^T j_k(t)c_k(t)\right)\right. \\
&= \prod_{\{k\}}\left\{\sqrt{-\frac{\lambda_k}{sin(\lambda_k T)}}\ exp\left\{-\frac{\lambda_k}{2sin(\lambda_k T)}\ \Big[(\phi_k^2(T)+\phi_k^2(0))cos(\lambda_k t) - 2\phi_k(0)\phi_k(T)\right.\right. \\
&-\frac{2\phi_k(T)}{\lambda_k}\int_0^T dt\ j_k(t)sin(\lambda_k t) - \frac{2\phi_k(0)}{\sqrt{\lambda_k}}\int_0^T dt\ j_k(t)sin(\lambda_k T - t) \\
&\left.\left. +\frac{2}{(\lambda_k)^2}\int_0^T dt\int_0^T ds j_k(t)j_k(s)sin(\lambda_k t)sin(\lambda_k s)\Big]\right\}\right.
\end{aligned} \tag{40}$$

Unfortunatelly, our theorems do not apply to infinite (continuum) measure product of Wiener measures in eq. (40) to produce a sensible measures theory on the functional space of the infinite product of Wiener trajectories $\{c_k(t)\}$ (Note that for each x fixed, a field sample configuration $\phi(t,0)$ in eq. (36) is a Hölder continuous function, result opposite to the usual functional integral representation for the Schwinger generating functional eq. (1)-eq.(5)).

Let us call attention that there is a formal definition of the above Feynman Path propagator for fields eq. (37) which at large time $T \to +\infty$ gives formally the Quantum Field functional integral eq. (5) associated to the Schwinger Generating Functional.

We thus consider the functional domain of eq. (37) as composed of field configurations which has a classical piece added with another fluctuating component to be functionally integrated out, namelly

$$\sigma(x,t) = \sigma_{CL}(x,t) + \sigma_q(x,t) \tag{41}$$

Here the classical field configuration problem (added with all zero modes of the theory)

$$\left(-\frac{d^2}{dt^2} + \mathcal{L}\right)\sigma^{CL}(x,t) = j(x,t) \tag{42}$$

with

$$\sigma^{CL}(x,-T)=\beta_1(x) \quad ; \quad \sigma^{CL}(x,t)=\beta_2(x) \tag{43}$$

namelly

$$\sigma_{CL}(x,t)=\left(-\frac{d^2}{dt^2}+\mathcal{L}\right)^{-1} j(x,t)+(\text{all zero modes}) \tag{44}$$

As a consequence of the decomposition eq. (41), the formal geometrical propagator with a external source below

$$\begin{aligned} &G[\beta_1(x),\beta_2(x),T,[j]] \\ &= \int_{\substack{\sigma(x,-T)=\beta_1(x)\\ \sigma(x,+T)=\beta_2(x)}} D[\sigma(x,t)] \;\; exp\left(-\frac{1}{2}\int_{-T}^{T} dt\; d^Dx\; \sigma(x,t)\left(-\frac{d^2}{dt^2}+\mathcal{L}\right)\sigma(x,t)\right) \\ &exp(i\int_{-T}^{T} dt \;\int d^Dx\; j(x,t)\sigma(x,t)) \end{aligned} \tag{45}$$

is exactly given by the following mathematically well defined Gaussian functional measure

$$\begin{aligned} &exp\left\{-\frac{1}{2}\int_{-T}^{T} dt\int d^Dx\; j(x,t)\;\sigma^{CL}(x,t)\right\} \\ &\int_{\substack{\sigma_q(x,-T)=0\\ \sigma_q(x,+T)}} D\sigma_q(x,t)\; exp\left\{-\frac{1}{2}\int_{-T}^{T} dt\int d^Dx\;\sigma_q(x,t)\left(-\frac{d^2}{dt^2}+\mathcal{L}\right)\sigma_q(x,t)\right\} \end{aligned} \tag{46}$$

The above claim is a consequence of the result below

$$\begin{aligned} &\int_{\substack{\sigma_q(x,-T)=0\\ \sigma_q(x,T)}} D[\sigma_q(x,t)]\; exp\left\{-\frac{1}{2}\int_{-T}^{T} dt\int d^Dx\;\sigma_q(x,t)\left(-\frac{d^2}{dt^2}+\mathcal{L}\right)\sigma_q(x,t)\right\} \\ &= det^{-\frac{1}{2}}\left[-\frac{d^2}{dt^2}+\mathcal{L}\right] \end{aligned} \tag{47}$$

where the sub-script Dirichlet on the functional determinant means that one must impose formally the Dirichlet condition on the domain of the operator $\left(-\frac{d^2}{dt^2}+\mathcal{L}\right)$ on $D'(R^D\times[-T,T])$ (or $L^2(R^D\times[-T,T])$ if $\mathcal{L}$ belongs to trace class. Note that the operator in eq. (46) does not have zero modes by the construction of eq. (41).

At this point, we remark that at the limit $T\to+\infty$ is exactly the Quantum Field functional eq. (5) if one takes $\beta_1(x)=\beta_2(x)=0$ (Note that the classical vacuum the limit $T\to\infty$ on Wiener measures are mathematically ill-defined (see theorem 5.1. of ref. [ ]).

It is important point out that $\sigma_{_{CL}}(x,t)$ is a regular $(C^\infty([-T,T]\times\Omega)$ solution of the Eliptic problem eq. (42) and the fluctuating $\sigma_q(x,t)$ will be a Schwartz distribution since the Eliptic operator $-\frac{d^2}{dt^2}+A$ in eq. (47) acts now on $D'([P-T,T]\times\Omega))$ with range $D([-T,T]\times\Omega)$, which by its turn shows the difference between this framework and the previous one related to the infinite product of Wiener measures since these objects are functional measures in *different* Functional Spaces.

Finally we comment that Functional Schrödinger equation, may be mathematically defined for the above displayed field propagators eq. (37) only in the situation of eq. (40). For instance, with $\mathcal{L}=-\Delta$ (the Laplacean), we have the validity of the Euclidean field wave equation

$$
\begin{aligned}
&\frac{\partial}{\partial T}G[\beta_1(x),\beta_2(x),T,[j]] = \\
&= \int_\Omega d^\nu x \left[+\frac{\delta^2}{\delta^2\beta_2(x)}+|\nabla\beta_2(x)|^2+j(x,T)\right] G[\beta_1(x),\beta_2(x),T,[j]]
\end{aligned} \tag{48}
$$

with

$$
\lim_{T\to 0^+} G[\beta_1(x),\beta_2(x),T] = \delta^{(F)}(\beta_1(x)-\beta_2(x)) \tag{49}
$$

Acknowledgments

This work was supported by CNPq-Brasil.

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Appendix

In this appendix we give new functional analytic proofs of the Bochner–Martin–Kolmogorov Theorem of section II.

Theorem of Bochner-Martin-Kolmogorov (Version 1) let $f : E \to R$ be a given real function with domain being a vector space E and satisfying the following properties

1) $f(0) = 1$

2) The restriction of f to any finite-dimensional vector sub-space of E is the Fourier Transform of a real continuous function of compact support.

Then there is a measure $d\mu(h)$ on a σ-álgebra containing the Barelians of the Space of Linear Functionals of E with the topology of pontual convergence denoted by E^{alg} such that for any $g \in E$

$$f(g) = \int_{E^{alg}} exp(ih(g))d\mu(h) \tag{A.1}$$

Proof: Let $\{\hat{e}_{\lambda \in A}\}$ be a Hamel (Vectorial) basis of E and $E^{(N)}$ a given sub-space of E of finite-dimension. By the hypothesis of the Theorem, we have that the restriction of the function to $E^{(N)}$ (generated by the elements of the Hamel basis $\{\hat{e}_{\lambda_1}, \cdots, \hat{e}_{\lambda_N}\} = \{e_\lambda\}_{\lambda \in \Lambda_F}$ is given by the Fourier Transform

$$f\left(\sum_{\ell=1}^{N} \sigma_{\lambda_1} \hat{e}_{\lambda_1}\right) = \int_{\prod\limits_{\lambda \in \Lambda_F} R^\lambda} (dP_{\lambda_1} \cdots dP_{\lambda_N})\ exp[\sum_{e=1}^{N} a_{\lambda_i} P_{\lambda_i}] \hat{g}(P_{\lambda_1}, \cdots, P_{\lambda_N}) \tag{A.2}$$

with $\hat{g}(P_{\lambda_1}, \cdots, P_{\lambda_N}) \in C_0\left(\prod\limits_{\lambda \in \Lambda_F} R^\lambda\right)$

As a consequence of the above written result we consider the following well-defined family of linear positive functionals on the space of continuous function on the product space of the Alexandrov Compactifications of R denoted by R^w:

$$L_{\Lambda_F} \in \left[C\left(\prod_{\lambda \in \Lambda_F} (R^w)^\lambda ; R\right)\right]^{Dual} \tag{A.3}$$

with

$$L_{\Lambda_F}[\sigma(P_{\lambda_1}, \cdots, P_{\lambda_N})] = \int_{\prod\limits_{\lambda \in \Lambda_F} (R^w)^\lambda} \int \hat{g}(P_{\lambda_1}, \cdots, P_{\lambda_N}) \sigma(P_{\lambda_1}, \cdots P_{\lambda_N})(dP_{\lambda_1} \cdots dP_{\lambda_N}) \tag{A.4}$$

Here $\hat{g}(P_{\lambda_1}, \cdots, P_{\lambda_N})$ still denotes the unique extension of eq. (A-2) to the Alexandrov Compactifications R^w.

We remark now that the above family of linear continuos functionals have the following properties:

1) The norm of L_{λ_F} is always the unity since

$$||L_{\lambda_F}|| = \int_{\prod_{\lambda \in \Lambda_F}(R^w)^\lambda} \hat{g}(P_{\lambda_1}, \cdots, P_{\lambda_N}) dP_{\lambda_1} \cdots dP_{\lambda_N} = 1 \tag{A.5}$$

2) If the index set Λ_F, contains Λ_F the restriction of the associated linear functional L_{Λ_F}, to the space $C\left(\prod_{\lambda \in \Lambda_F} (R^w)^\lambda, R\right)$ coincides with L_{Λ_F}.

Now a simple application of the Stone–Weisrtress Theorem show us that the topological closure of the union of the sub-spaces of functions of finite variable is the space $C\left(\prod_{\lambda \in A} (R^w)^\lambda, R\right)$, namelly

$$\overline{\bigcup_{\Lambda_F \subset A} C\left(\prod_{\lambda \in \Lambda_F} (R^w)^\lambda, R\right)} = C(\prod_{\lambda \in A} (R^w)^\lambda, R) \tag{A.6}$$

where the union is taken over all family of sub-sets of finite elements of the index set A.

As a consequence of the remark 2 and eq. (A-6) there is a unique extension of the family of linear functionals $\{L_{\Lambda_F}\}$ to the whole space $C\left(\prod_{\lambda \in A} (R^w)^\lambda, R\right)$ and denoted by L_∞. The RieszMarkov Theorem give us a unique measure $d\bar{\mu}(h)$ on $\prod_{\lambda \in A} (R^w)^\lambda$ representing the action of this functional on $C\left(\prod_{\lambda \in A} (R^w)^\lambda, R\right)$.

We have, thus, the following functional integral representation for the function $f(g)$:

$$f(g) : \int_{\prod_{\lambda \in A} R^P \lambda} (R^w)^\lambda \; exp \; \bar{h}(g) d\bar{\mu}(\bar{h}) \tag{A.7}$$

Or equivalently (since $\bar{h}(g) = \sum_{i=1}^{N} p_i a_i$ for some $\{P_i\}_{i \in N} < \infty$), we have the result

$$f(g) = \int_{\prod_{\lambda \in A} R^\lambda} R^\lambda(exp \; ih(g)) \; d\mu(h) \tag{A.8}$$

which is the proposed theorem with $h \in (\prod_{\lambda\in\lambda_F} R^\lambda)$ being the element which has as the image of $\bar{h}$ on the Alexandrov Compactification $\prod_{\lambda\in\lambda_F}(R^w)^\lambda$.

The practical use of the Bochner–Martin Kolmogorov Theorem is difficulted by the present day non existence of an algorithm generating explicitly a Hamel (Vectorial) Basis on Function of Spaces. However, if one is able to apply the theorems of section III one can construct explicitly the functional measure by only considering Topological Basis as in the Gaussian Functional Integral eq. (32).

Theorem of Bochner-Martin-Kolmogorov (Version 2)

We have now the same hypothesis and results of theorem version 1 but with the more general condition.

3) The restriction of f to any finite-dimensional vector sub-space of E is the Fourier Transform of a real continuous function vanishing at "infinite".

For the proof of the theorem under this more ample mathematical condition, we will need two lemmas and some definitions.

Definition 1 – Let X be a normal Space, locally compact and satisfying the following σ-compacity condition

$$X = \bigcup_{n=0}^{\infty} K_n \tag{A.9}$$

with

$$K_n \subset \, int(K_{n+1}) \subset K_{n+1} \tag{A.10}$$

we define the following space of continuous function "vanishing" at infinite

$$\tilde{C}_0(X,R) = \left\{ f(x) \in C(X,R) | \lim_{n\to\infty} \ \sup_{x\in(K_n)^c} |f(x)| = 0 \right\} \tag{A.11}$$

We have, thus, the following lemma.

Lemma 1 – The Topological closure of the functions of compact support contains $\tilde{C}_0(X,R)$ in the topology of uniform convergence.

Proof: Let $f(x) \in \tilde{C}_0(X,R)$ and $g_\mu \in C(X,R)$, the (Uryhson) functions associated to the closed disjoints sets $\bar{K}_n$ and (K^c_{n+1}). Now it is straightforwardly to see that $(f.g_n)(x) \in C_i(X,R)$ and converges uniformilly to $f(x)$ due to the definition (A-11).

At this point, we consider a linear positive continuous functional L on $\tilde{C}_0(X,R)$. Since the restriction of L to each sub-space $C(K_n,R)$ satisfy the conditions of the Riesz-Markov Theorem, there is a unique measure $\mu^{(n)}$ on K_n containing the Berelians on K_n and representing this linear functional restriction. We now use the hypothesis eq. (A-10) to have a well defined measure on a σ-algebra containing the Barelians of X

$$\bar{\mu}(A) = \lim \ sup \ \mu^{(n)}(A \bigcap K_n) \tag{A.12}$$

for A in this σ-algebra and representing the functional L on $\tilde{C}_0(X,R)$

$$L(f) = \int_X f(x) d\bar{\mu}(x) \tag{A.13}$$

Note that the normally of the Topological Space X is a fundamental hypothesis used in this proof by means of the Uryhson lemma.

Unfortunatelly, the non-countable product space $\prod_{\lambda \in A} R^\lambda$ is not a Normal Topological Space (the famous Stone counter example) and we can not, thus, apply the above lemma to our Vectorial case eq. (A-8). However, we can overcame the use of the Stone Weirstrass Theorem in the Proof of the Bochner-Martin-Kolmogorov Theorem by considering directly a certain Functional Space instead of that given by eq. (A-6).

We define, thus, the following Space of Infinite-Dimensional functions vanishing at finite

$$C_0(R^\infty, R) \equiv C_0\left(\prod_{\lambda \in A} R^\lambda, R\right) = \overline{\bigcup_{\Lambda_F \subset A} \tilde{C}_0\left(\prod_{\lambda \in \Lambda_F} R^\lambda, R\right)} \tag{A.14}$$

where the closure is taken in the topology of uniform convergence.

If we consider a given continuous linear functional L on $C_0\left(\prod_{\lambda \in A} R^\lambda, R\right)$ there is a unique measure μ^∞ on the union of the Borelians of $\prod_{\lambda \in \Lambda_F} R^\lambda$ representing the action of L on $C_0(R^\infty, R)$.

Conversely, given a family of consistent measures $\{\mu_{\Lambda_F}\}$ on the finite-dimensional spaces ($\prod_{\lambda\in\Lambda_F} R^\lambda$) satisfying the property of $\mu_{\Lambda_F}\left(\prod_{\lambda\in\Lambda_F} R^\lambda\right) = 1$, there is a unique measure on the cylinders of $\prod_{\lambda\in A} R^\lambda$ associated to the functional L on $C_0\prod_{\lambda\in A} R^\lambda, R)$.

Collecting the results of the above written lemmas we get the Proof of eq. (A-8) in this more general case ([4], [5], [6]).