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Identifying Conical Singularities.

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Abstract

A method based upon the concept of holonomy of a metric spacetime (M, g), in order to identify the presence of conical singularities in M, is proposed. The validity and usefulness of this so-called holonomy method is proven by applying it to a set of four-dimensional spacetimes and one three-dimensional spacetime. The holonomy method predictions are confirmed by the comparison with the predictions obtained after coordinate transformations which take the metrics g, to a new basis where the global properties of conical singularities are explicitly seen.

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I. INTRODUCTION

Conical singularities have been known to physicists for a long time [1]. During the 60's and 70's, when a great deal of work was done to understand and classify all types of singularities, the properties and geometrical nature of the conical singularities became better understood (for a review on the results in singularities see Ref. [2]).

From these works, we learned that for a spacetime M with a metric tensor g $(M,\, g)$, if there is a conical singularity in M, all scalars formed out of the Riemann tensor are bounded in M, and the presence of this singularity can only be revealed by means of a global procedure [2]. Because of the good behavior of the scalars mentioned above, the conical singularities belong to a type of singularities called: quasiregular singularities [2], [3]. A more recent interest in conical spacetimes developed after two important results, the first due to T. W. B. Kibble, and the second to A. Vilenkin.

Kibble showed in Ref. [4], that the mechanisms of symmetry breaking and symmetry restoration in gauge theories, necessarily, led to the formation of topological defects in the early universe. Among the possible types of defects are the cosmic strings. Inspired by Kibble's ideas Vilenkin demonstrated, among other things in Ref. [5], that the spacetime geometry due to a static, infinite, unidimensional string, with a finite linear energy density, in the weak field approximation, is a conical spacetime.

Cosmic strings have attracted greater attention than the other types of defects not only because the others have some undesirable properties [6], but by their own observable effects [7]. Among these effects one may mention the gravitational lensing of light from distant galaxies or quasars, a significant effect on the motion of massive particles, frequency shifts of photons passing by the string and its possible role in the mechanism of galaxy formation (see Ref. [7] for a complete list of observable effects). From the above discussion one can conclude that conical singularities are very important either on a more formal study of the spacetime structure as well as in processes in the early universe leading to presently observable phenomena. One very proper question, at this stage, would be how to identify the presence of conical singularities in a given metric spacetime. In fact, there is not a general method of determining whether a metric spacetime $(M,\, g)$, has a conical singularity or not.

What one usually does, in order to identify the conical singularity in $(M,\, g)$, is determining whether one of the angular variables, in g, which by definition would vary in the range $[0,\, 2\pi]$ has its range reduced [8]. The amount which is missing to complete 2π is called deficit angle. If a given spacetime has a singular point and all the properties of a conical spacetime, as described above [2], but all the relevant angular variables have their normal range, one may try to find a coordinate transformation which reveals the presence of a deficit angle in one of these angular variables range [9], or construct a particular demonstration of the regularity, or not, of the metric spacetime under investigation [10].

In this paper we would like to propose a method of identifying if a given metric spacetime, singular in one point, and with all the properties of a conical spacetime, is indeed a conical spacetime. This method will use the concept of holonomy group of a spacetime (therefore we shall call it holonomy method), and the global and local properties of a conical singularity.

In the next section, Sec. II, we shall introduce the holonomy method. Sec. III is devoted to the application of the holonomy method to four-dimensional spacetimes which are known to be conical spacetimes. In Sec. IV, we investigate a three-dimensional metric spacetime with a conical singularity which cannot be trivially identified unless we use the holonomy method. Finally, in Sec. V, we conclude by outlining the main results of the paper and commenting on other applications of the holonomy method.

II. DESCRIPTION OF THE HOLONOMY METHOD

Our method will be based in the application of the concept of holonomy [11], to the spacetime being studied. The holonomy group of a spacetime M, equipped with a metric g, $(M,\, g)$ (therefore with an affine connection Γ), is defined as the linear transformations from the tangent space at the point p, T_pM, to itself, constructed by taking a fiducial vector v and parallel transporting it along all closed curves starting and finishing at p. For a spacetime, the set of all linear transformations generated by the parallel transport of v along all possible closed curves from p, is called the holonomy group of M at p. In fact, we shall not be interested in computing the holonomy group of the given spacetime but we shall use this concept in a related way.

Suppose that the metric g of M is apparently singular at a point s, and all scalars formed out of the Riemann curvature are well behaved in M. Then, for a fiducial vector v in M, we shall evaluate the transformation generated by the parallel transport of v around closed loops. This transformation is a matrix which we call holonomy matrix (H), its elements are functions of the coordinates. For a point p belonging to a closed loop C, both in M, the holonomy matrix at p, and around C, relates the initial value of v at T_p with the value of v, again at T_p, but now after it has completed a full loop around C. The closed loops are defined in order to converge to the apparently singular point s, of M. The convergence of the loops to s is specified by a certain limit of the coordinates describing these loops.

Now, after having computed the holonomy matrix we apply to it the same limit of the coordinates which cause the closed loops to converge to the apparently singular point s. Then, if the point s is regular, the limit of the holonomy matrix defined by the parallel transport of the fiducial vector v around closed loops must be the identity matrix. In other words, the limit of the initial and final values of the vector v at s must be the same. If this is not the case, the spacetime is non-regular at s, or it has a conical singularity.

Let us now apply this method to two different types of spacetimes: The first is composed of spacetimes where one may trivially identify the presence of conical singularities; the second is a spacetime with a conical singularity which cannot be trivially identified unless we use the holonomy method.

III. APPLICATION TO CONICAL SPACETIMES

We shall consider here a set of three four-dimensional spacetimes with a conical singularity. Although we shall treat them together because of their similar metric expressions, they have distinct topologies and curvature scalars $\mathbf{R}$. The general metric expression is

$$ds^2 = \eta dt^2 + dz^2 + dr^2 + \mu^2 F^2(r) d\theta^2, \tag{1}$$

where $-\infty < t < \infty$, $-\infty < z < \infty$, $r_{\rm min} < r < r_{\rm max}$, $0 \leq \theta \leq 2\pi$; μ is a number which is in the range, $0 < \mu \leq 1$; $F(r)$ is a function of r without numerical coefficients with the property that, $F(r = r_0) = 0$ for a $r_0 \in [r_{\rm min}, r_{\rm max}]$; η may be either -1, which means a Lorentzian signature, or $+1$, which means an Euclidean signature; and the range of r depends on which spacetime we are considering. Some remarks are now in order.

It is important to note that the angular variable must vary over the full range $[0, 2\pi]$, otherwise we would identify the presence of a conical singularity by inspection. If μ was allowed to be greater than one, after a coordinate transformation of the type

$$\mu\,\theta \rightarrow \tilde{\theta}, \tag{2}$$

the new angular variable $\tilde{\theta}$ would vary in the range

$$0 < \tilde{\theta} < 2\pi\mu. \tag{3}$$

This means that we would not be able to talk about deficit angle in a natural way, besides that for this case the space changes character and the description becomes more complicated [9].

The three distinct spacetimes will be characterized by different values of the function $F(r)$ [12]. We shall compute for each of them the curvature scalar $\mathbf{R}$, which for the generic form of the metric (1) is

$$\mathbf{R} = -\frac{F''(r)}{F(r)}, \tag{4}$$

where $'$ means derivation in respect to r.

For each spacetime we may list the relevant properties in the following way:

Spacetime (a): Topology $\Re^2 \times S^2$; $F(r) = \sin r$; $0 < r < 2\pi$; $\mathbf{R} = 1$,

Spacetime (b): Topology $\Re^3 \times S^1$; $F(r) = r$; $0 < r < \infty$; $\mathbf{R} = 0$,

Spacetime (c): Topology $\Re^2 \times H^2$; $F(r) = \sinh r$; $-\infty < r < \infty$; $\mathbf{R} = -1$.

One may note by the above properties of (a), (b), and (c), that they have well behaved scalar curvatures $\mathbf{R}$ (it is easy to verify that the same is true for other scalars formed out of the Riemann tensor) and an apparent singularity at $r = 0$. Indeed, by means of the transformation (2) it is very easy to identify the presence of conical singularities in these spacetimes. By eq. (3) it is clear that, if $\mu = 1$ the spacetimes are regular, otherwise they have conical singularities with deficit angles $2\pi(1-\mu)$. Let us see, now, how to apply the holonomy method to the above spacetimes (1).

We start by computing the holonomy matrix H, which can only be done when we have the parallel transport equations. The easiest way to derive them, from eq. (1), is by working on the orthonormal basis, defined by the transformations

$$w^{\hat{i}} = dx_i; \qquad w^{\hat{\theta}} = \mu F(r) d\theta, \tag{5}$$

where $x_{0,1,2} = t, z, r$. The non-vanishing connection coefficient components in this basis are

$$-\Gamma^{\hat{r}}_{\hat{\theta}\hat{\theta}} = \frac{F'(r)}{F(r)} = \Gamma^{\hat{\theta}}_{\hat{r}\hat{\theta}}. \tag{6}$$

The general expression for the parallel transport equations of a fiducial vector v, along a given curve is [13]

$$\frac{dv^\alpha}{d\lambda} + v^\beta \Gamma^\alpha_{\beta\gamma} \frac{dx^\gamma}{d\lambda} = 0, \tag{7}$$

where λ is an affine parameter describing the curve; and the Greek indices vary over all the coordinates.

We must define, now, the closed loops for these cases, as well as the limit which will make them collapsing to the apparently singular point $r = 0$. Since the t and z directions do not play important roles in the present cases (the singular point is in the r axis and θ is the unique angular variable), we set them to be constants, independent of λ. For an apparently singular point in r, the simplest closed loops have constant and different values of r, and θ is a non-constant function of λ. These loops will collapse to $r = 0$ when we take the limit $r \to 0$. As a matter of simplicity, we choose θ to depend linearly on λ.

Our curves may be written as

$$x_i \; = \; C_i, \qquad \theta \; = \; \lambda \; + \; \lambda_0, \tag{8}$$

where λ_0 is a constant and λ varies in the range $[-\lambda_0, -\lambda_0 + 2\pi]$; C_i are constants and i varies as in eq. (5).

A final remark before writing down the parallel transport equations, is that the set of eqs. (7), for the non-coordinate basis, must be modified such that we may use the closed loops (8) given in the coordinate basis. Then, the new set of equations is written

$$\frac{dv^\alpha}{d\lambda} \; + \; v^\beta \Gamma^\alpha_{\beta\gamma} \Lambda^\gamma_\delta \frac{dx^\delta}{d\lambda} \; = \; 0, \tag{9}$$

where Λ is the basis one-form transformation matrix, also responsible for the transformation of vector components.

We may now write the parallel transport equations (9), for the fiducial vector v, around the closed loop (8), with the aid of eq. (6). They are

$$\frac{dv^{\hat{t}}}{d\lambda} \; = \; 0 \; = \; \frac{dv^{\hat{z}}}{d\lambda}, \tag{10}$$

$$\frac{dv^{\hat{r}}}{d\lambda} \; - \; \mu F' v^{\hat{\theta}} \; = \; 0, \tag{11}$$

$$\frac{dv^{\hat{\theta}}}{d\lambda} \; + \; \mu F' v^{\hat{r}} \; = \; 0. \tag{12}$$

The general solution for the system of coupled, linear, homogeneous, first order differential equations (10) - (12), is given by

$$v^j(\lambda) = \tilde{C}_j, \tag{13}$$

$$v^{\hat{r}}(\lambda) = A \sin(\alpha\lambda \; + \; \delta_0), \tag{14}$$

$$v^{\hat{\theta}}(\lambda) = A \cos(\alpha\lambda \; + \; \delta_0), \tag{15}$$

where $j = \hat{t}, \hat{z}$; $\tilde{C}_j$, A, δ_0 are constants to be determined by the initial conditions; and $\alpha \equiv \mu F'(r)$.

The initial conditions will be given by

$$v^{\alpha}(\lambda = -\lambda_0) \; = \; v_0^{\alpha}, \tag{16}$$

where α varies as in eq. (7); and v_0^{α} are constants. If one imposes these conditions (16) in the eqs. (13) - (15), after some calculations one finds

$$v^{j}(\lambda) = v_0^{j}, \tag{17}$$

$$v^{\hat{r}}(\lambda) = \cos[\alpha(\lambda + \lambda_0)]\, v_0^{\hat{r}} \; + \; \sin[\alpha(\lambda + \lambda_0)]\, v_0^{\hat{\theta}}, \tag{18}$$

$$v^{\hat{\theta}}(\lambda) = - \; \sin[\alpha(\lambda + \lambda_0)]\, v_0^{\hat{r}} \; + \; \cos[\alpha(\lambda + \lambda_0)]\, v_0^{\hat{\theta}}, \tag{19}$$

where j varies as in eq. (13). We may now write the holonomy matrix which relates the initial value of the fiducial vector v, with its value after a complete loop around the closed loop. It is achieved by setting $\lambda = -\lambda_0 + 2\pi$ in eqs. (17) - (19):

$$v(-\lambda_0 + 2\pi, \alpha) \; = \; H(\alpha)\, v_0, \tag{20}$$

where

$$H(\alpha) \; = \; \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \cos(2\pi\alpha) & \sin(2\pi\alpha) \\ 0 & 0 & -\sin(2\pi\alpha) & \cos(2\pi\alpha) \end{pmatrix}. \tag{21}$$

Before we proceed, it is important to note that H is independent of η, or in other words, it is independent of the metric signature. One may use, then, this method for Lorentzian or Euclidean spacetimes. It is clear from eq. (21) that $H(\alpha)$ is not, in general, the identity transformation, not even in the case of a regular spacetime. This fact is only a verification that the spacetime has a non-vanishing curvature. For spacetimes (a) and (c) one may verify this fact, for they have constant positive and negative scalars of curvature, but not for spacetime (b) which is flat. The final step in our present analysis, is the evaluation of $H(\alpha)$ (21) in the limit when r goes to 0.

The relevant components of the holonomy matrix to be analyzed are

$$\cos(2\pi\alpha) \qquad \text{and} \qquad \sin(2\pi\alpha). \tag{22}$$

We must, now, take the limit of these quantities (22), when r goes to 0. It is not difficult because for all three spacetimes (a), (b) and (c), the desired limit of the first derivative of $F(r)$ is the same

$$\lim_{r\to 0} F'(r) \ = \ 1. \tag{23}$$

So, once $\alpha = \mu F'(r)$, we obtain the following limit as $r \to 0$ to the relevant quantities (22)

$$\cos(2\pi\mu) \qquad \text{and} \qquad \sin(2\pi\mu). \tag{24}$$

We may conclude that the spacetime is regular, or in other words the holonomy matrix goes to the 4×4 identity in this limit, if and only if $\mu = 1$, otherwise there is a conical singularity in these spacetimes. This is the same result stated above and derived by a trivial coordinate transformation in the angular coordinate θ.

IV. APPLICATION TO THE DJH CONICAL SPACETIME

This spacetime was introduced by S. Deser, R. Jackiw and G. 't Hooft (DJH), as a solution of Einstein's equations for a single, massive, spinless point-particle, at rest, at the origin of the coordinate system, in three-dimensions (time + 2 spatial dimensions) [9]. Its metric is given by

$$ds^2 \ = \ -\, dt^2 \ + \ \frac{1}{R^{2\rho}} \left(\, dR^2 \ + \ R^2 d\Theta^2 \, \right) \tag{25}$$

where $0 \leq t \leq \infty$, $0 \leq R \leq \infty$, $0 \leq \Theta \leq 2\pi$; and ρ is proportional to the mass of the particle and varies in the range $0 \leq \rho < 1$.

It is easy to see from eq. (25) that $R = 0$ is an apparent singularity of this spacetime, and after a brief calculation one finds that its scalar of curvature $\mathbf{R}$ is nil. Gathering together these two results we conclude that it is very likely that the apparent singularity at $R = 0$ is of a conical type. How can we know beyond any doubt? In what follows we shall apply the holonomy method to this spacetime (25), and compare its result with the one derived in Ref. [9] from a non-trivial coordinate transformation.

We start the application of the holonomy method by rewriting eq. (25) in a non-coordinate basis, which coordinate transformations read

$$w^{\hat{t}} \; = \; dt; \qquad w^{\hat{R}} \; = \; R^{-\rho} dR; \qquad w^{\hat{\theta}} \; = \; R^{1-\rho} d\Theta. \tag{26}$$

The non-vanishing connection coefficient components in this basis are

$$\Gamma^{\hat{t}}_{\hat{\phi}\hat{\phi}} \; = \; \frac{-(1 \, - \, \rho)}{t^{1-\rho}} \; = \; - \, \Gamma^{\hat{\phi}}_{\hat{t}\hat{\phi}}. \tag{27}$$

From here onward the calculations will be a repetition of the previous case, discounting the obvious fact that we have, now, one less spatial coordinate. The closed loops will be defined by constant values of t and R and a linear function of Θ, which may be written as

$$x_i \; = \; C_i, \qquad \Theta \; = \; \lambda \, + \, \lambda_0, \tag{28}$$

where $x_{0,1} \; = \; t, R$; C_i and λ_0 are constants; and λ is an affine parameter describing the closed loop, which varies in the range $[-\lambda_0, -\lambda_0 + 2\pi]$. These loops will collapse to the apparently singular point when we take the limit $R \to 0$.

The initial conditions for the fidutial vector v, which will be parallel transported around the closed loops, are given by eq. (16) (where here $\alpha \, = \, t, R, \Theta$). With these conditions (16), and the aid of eq. (28), we may write and solve the parallel transport equations for v, derived from eq. (9). After some calculations we may write the holonomy matrix $H(\rho)$, for the present case, in the following way:

$$H(\rho) \; = \; \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos[2\pi(1-\rho)] & \sin[2\pi(1-\rho)] \\ 0 & -\sin[2\pi(1-\rho)] & \cos[2\pi(1-\rho)] \end{pmatrix} . \tag{29}$$

Observing eq. (29) carefully, we note that it is already the final result we were looking for because $H(\rho)$ does not depend upon R. Therefore, the limit of $H(\rho)$ when R goes to zero does not modify eq. (29).

The holonomy method predicts, then, that if $\rho = 0$ the spacetime (25) is regular, otherwise there is a conical singularity in it. Let us compare, now, this prediction with the one derived from a non-trivial coordinate transformation.

In ref. [9], Deser, Jackiw, and 't Hooft, proposed the following coordinate transformation for the metric (25)

$$r \;=\; \frac{R^{1-\rho}}{1\,-\,\rho}, \qquad \theta \;=\; (1\,-\,\rho)\,\Theta. \tag{30}$$

In terms of r and θ the metric (25) becomes flat

$$ds^2 \;=\; -\,dt^2 \,+\, dr^2 \,+\, r^2 d\theta^2. \tag{31}$$

The ranges of the new variables may be derived by eq. (30), with the aid of the ranges of the old ones, which were given after eq. (25). They are

$$0 \,\leq\, r \,\leq \infty, \qquad 0 \,\leq \theta \,\leq\, 2\pi(1\,-\,\rho). \tag{32}$$

V. CONCLUSIONS

In the present paper we have introduced what we have called the holonomy method. It is a systematic way, based upon the concept of holonomy, of determining the presence of conical singularities in a given metric spacetime $(M,\, g)$. In Sec. II, we have introduced the holonomy method. Subsequently, in Secs. III and IV, we have demonstrated the validity and usefulness of the method by means of its application in two examples.

It is important to mention that the results derived from the application of the holonomy method for a given metric spacetime $(M,\, g)$, may be used to analyze another metric spacetime $(\tilde{M}, \tilde{g})$. This can be done only and if only g and $\tilde{g}$ are related by a non-singular basis one-form transformation S, because the old and new holonomy matrices (H and $\tilde{H}$ respectively) will have the following transformation relation

$$\tilde{H} \;=\; S\, H\, S^{-1}. \tag{33}$$

Therefore, knowing the value of H for a given one-form basis one has only to determine the values of the transformation S, and its inverse S^{-1}, in order to derive the holonomy matrix in the new one-form basis $\tilde{H}$.

Finally, there is yet another type of metric spacetimes where the use of the holonomy method greatly simplifies the identification of conical singularities. These d-dimensional spacetimes have the topology $\Re$ (time) times a flat or negatively curved, $d-1$-dimensional, compact spatial sector. The compactification of the spatial sector is done via the identification of opposite $d-2$-dimensional "sides" of a chosen figure [10], [12] and [14]. It is possible that such identifications introduce a singularity of a conical, or even more complicate nature in the resulting spacetime [10] and [14]. Therefore, applying the holonomy method to spacetimes constructed in this way, one may identify the presence of conical singularities there.

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