The Gravitational Energy in Asymptotically Anti-De Sitter Spacetimes

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Abstract

The prescription proposed by Komar in order to calculate the total gravitational mass acting on a test particle in a given stationary asymptotically flat spacetime is generalized to stationary asymptotically anti-de Sitter spacetimes. It is shown that this generalized Komar mass is different from the total gravitational energy calculated by Henneaux and Teitelboim for the Kerr-anti-de Sitter spacetime. This last total energy agrees, however, with the total charge-energy calculated using the Brown-York action formalism.

PACS numbers: 04.20.Cv., 04.20.Me

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1 Introduction

In General Relativity, the total mass-energy of a stationary asymptotically flat spacetime can be calculated in different ways which can be divided in two categories.

The first involves the hamiltonian[1, 2] or action[3] formalism and can be interpreted as the total gravitational energy of the spacetime in question. It is analogous to the evaluation of the energy of an electromagnetic field in a volume using its hamiltonian. Their algebraic expressions coincide, asymptotically, with earlier definitions coming from the Landau-Lifshitz and the Einstein energy-momentum pseudo-tensors[4, 5] in suitably chosen coordinates in the asymptotic region.

The second uses the notion of forces or geodesics[6, 7], and the total gravitational mass is defined by its effect on the motion of a test particle. In fact, what is evaluated in this approach is the source of the gravitational field acting on a test particle (analogous to the charge for the electromagnetic field): the gravitational mass. In the asymptotically flat region, however, this mass must be equal to the total energy contained in the stationary asymptotically flat spacetime. The expression obtained with this method is a generalization of the one derived from the Møller[8] energy-momentum pseudo-tensor.

Although the general expressions for the gravitational mass-energy calculated from these two distinct points of view were indeed completely different (in the former, terms with g_{00} do not appear), it is a mathematical coincidence, physically reasonable as explained above, that these two types of expression give the same final result.

A natural question to ask at this point is whether the two methods described above and used to calculate the total mass-energy of a given asymptotically flat spacetime yields the same result for other asymptotic structures. The aim of this paper is to show that, for certain asymptotically anti-de Sitter spacetimes, this coincidence no longer exists. In particular, we will show that suitable generalizations of these prescriptions give different values for the mass-energy of the gravitational field of the Kerr-anti-de Sitter spacetime.

This paper is organized as follows.

In section 2 we present the Kerr-anti-de Sitter spacetime and the results of Henneaux and Teitelboim[9] for its gravitational energy using the hamiltonian formalism. Afterwards, we show that the same result is obtained using the action formalism of Brown and York[3]. In this way, the calculation of the energy of this gravitational field following the first point of view is completed.

In section 3, the expression of Komar[7] for the gravitational mass of a stationary and asymptotically flat spacetime is generalized to stationary spacetimes which are asymptotically anti-de Sitter. The interpretation of Wald[6] to this mass in terms of forces on test particles can also be suitable generalized. This 'gravitational mass' is calculated for the Kerr-anti-de Sitter spacetime yielding a different result from the one obtained in section 2.

We conclude in section 4 with some comments and discussions.

2 The gravitational energy of the Kerr-anti-de Sitter spacetime in the hamiltonian and action formalisms

The anti-de-Sitter metric is:

$$d\overset{\circ}{s}{}^2 = -[1+(\frac{r}{R})^2]dt^2 + [1+(\frac{r}{R})^2]^{-1}dr^2 + r^2 d\Omega^2 \tag{1}$$

where R is the radius of curvature of such space, related to the cosmological constant Λ by $R = (-3/\Lambda)^{1/2}$.

At spatial infinity, the Kerr-anti-de Sitter spacetime can be written as:

$$ds^2 = d\overset{\circ}{s}{}^2 + h_{\mu\nu} dx^\mu dx^\nu \tag{2}$$

where $h_{\mu\nu}$ are deviations from the background anti-de Sitter metric $d\overset{\circ}{s}{}^2$.

The explicit values of these deviations are[9]:

$$\left\{\begin{array}{lcl} h_{tt} &=& \frac{2m}{r}[1-\alpha^2\sin^2\theta]^{-5/2}+O(r^{-3}) \\ h_{t\phi} &=& -\frac{2am\sin^2\theta}{r}[1-\alpha^2\sin^2\theta]^{-5/2}+O(r^{-3}) \\ h_{\phi\phi} &=& \frac{2ma^2\sin^4\theta}{r}[1-\alpha^2\sin^2\theta]^{-5/2}+O(r^{-3}) \\ h_{rr} &=& \frac{2mR^4}{r^5}[1-\alpha^2\sin^2\theta]^{-3/2}+O(r^{-7}) \\ h_{r\theta} &=& -\frac{2mR^2a^2}{r^4}[1-\alpha^2\sin^2\theta]^{-5/2}\sin\theta\cos\theta+O(r^{-6}) \\ h_{\theta\theta} &=& \frac{2ma^4}{r^3}[1-\alpha^2\sin^2\theta]^{-7/2}\sin^2\theta\cos^2\theta+O(r^{-5}) \end{array}\right. \tag{3}$$

where $\alpha = a/R$ and a is related to the angular momentum per unit mass. The non-vanishing components of the gravitational canonical momentum are:

$$\left\{\begin{array}{lcl} \pi^{r\phi} &=& -\frac{3am\sin\theta}{r^2}[1-\alpha^2\sin^2\theta]^{-5/2}+O(r^{-4}) \\ \pi^{\theta\phi} &=& O(r^{-5}) \end{array}\right. \tag{4}$$

Using the hamiltonian formalism, Henneaux and Teitelboim[9] calculated the mass-energy of the Kerr-anti-de Sitter spacetime in the following way. They showed that the hamiltonian must be suplemented by surface terms in order to be consistent with the equations of motion. These surface terms yield conserved charges associated with the Killing vectors of the anti-de Sitter asymptotic geometry. They are:

$$J_A = \frac{1}{16\pi}\int_B d^2S_i[\overset{\circ}{G}{}^{ijkl}(\xi^{\perp}_A\overset{\circ}{\nabla}_j g_{kl} - h_{kl}\overset{\circ}{\nabla}_j\xi^{\perp}_A) + 2\xi^k_A\pi^i_k] \tag{5}$$

where $\overset{\circ}{G}{}^{ijkl} = \frac{1}{2}\overset{\circ}{g}{}^{1/2}(\overset{\circ}{g}{}^{ik}\overset{\circ}{g}{}^{jl} + \overset{\circ}{g}{}^{il}\overset{\circ}{g}{}^{jk} - 2\overset{\circ}{g}{}^{ij}\overset{\circ}{g}{}^{kl})$, $d^2S_i = \frac{1}{2!}\epsilon_{ijk}dx^j \wedge dx^k$, $\epsilon_{ijk} = 1, -1, 0$ being the three-dimensional (metric independent) totally antisymmetric object, B is a 2-sphere at spatial infinity where the integral is performed, $\xi^{\perp}_A$ is the component of the killing vector ξ_A in the direction of the unit normal u^α to the hypersurfaces $t = \text{const}$, h_{kl} is the deviation from the anti-de Sitter metric $\overset{\circ}{g}_{kl}$, π^{kl} is its canonical momentum and $\overset{\circ}{\nabla}_j$ is the covariant derivative with respect to the background metric $\overset{\circ}{g}_{kl}$.

The total energy is the charge associated with the timelike Killing vector field $\frac{\partial}{\partial t}$ of the anti-de Sitter spacetime, and is calculated to be:

$$E = \frac{m}{(1-\alpha^2)^2} \tag{6}$$

Note that, as R goes to infinity (α goes to zero), this energy goes to its value for asymptotically flat spacetime, as should be expected.

In the calculation of the conserved charge by Henneaux and Teitelboim, there is an implicit assumption that the anti-de Sitter background metric is fixed with respect to variations that lead to the equations of motion. This procedure eliminates an infinite term corresponding to the energy of the anti-de Sitter background yielding the finite expression given in equation (6).

In the action formalism of Brown and York[3], a quasi-local energy is defined in terms of a surface energy-momentum tensor. They considered a manifold M foliated by spacelike hypersurfaces Σ. The manifold is spatially bounded by a three-dimensional timelike surface ${}^3B$ whose intersection with Σ gives the two-dimensional surface B where this surface energy-momentum tensor is defined. This definition is based in an analogy with the definition of energy in classical mechanics using the action and the Hamilton-Jacobi equation. If the spacetime has a Killing vector field, a total conserved charge can be defined and it is given by:

$$Q^A = \int_B d^2x\sqrt{\sigma}(\epsilon u^i + j^i)\xi_i^A \tag{7}$$

where u^i is the unit normal to the hypersurfaces Σ (defined by t = const), ξ_i^A a killing vector field and $d^2x\sqrt{\sigma}$ is the proper surface element of the 2-boundary B. Also $\epsilon = \frac{1}{8\pi}(k - \overset{\circ}{k})$ is the expression for the energy surface density, where $k = -\sigma^{\alpha\beta}D_\alpha n_\beta$ is the trace of the extrinsic curvature of the boundary surface B with unit normal n_β as embedded in the hypersurface t = const, $\sigma^{\alpha\beta}$ being the induced metric on this surface and D_α the covariant derivative on this hypersurface. The quantity $\overset{\circ}{k}$ is the trace of the extrinsic curvature of a two-dimensional surface with the same induced metric $\sigma^{\alpha\beta}$ but which is now immersed in a fixed spacelike hypersurface of a fixed spacetime, called the reference space. It comes from the arbitrariness in the definition of the action and, if we choose the anti-de Sitter spacetime as the reference space, it serves to eliminate the infinite charge-energy coming from the anti-de Sitter background[1]. In the terminology of Brown and York, this corresponds to the reference space subtraction. In general, a superscript ($\circ$) denotes a geometrical quantity of this reference space. Finally, $j^i = -2(\sigma^i_k n_l \pi^{kl}/\sqrt{h} - \overset{\circ}{\sigma}{}^i_k \overset{\circ}{n}_l \overset{\circ}{\pi}{}^{kl}/\sqrt{\overset{\circ}{h}})$ is the corresponding momentum surface density.

[1]In the case of the total quasi-local energy itself, given by the surface integral $\int_B d^2x\sqrt{\sigma}\epsilon$, this subtraction woud give a null total energy, like in the Schwarzschild-anti-de Sitter spacetime[10].

For the timelike Killing vector field $\frac{\partial}{\partial t}$ of the Kerr-anti-de Sitter spacetime, the above conserved charge evaluated on a surface at spatial infinity[2] defines the Brown-York total mass-energy of this gravitational field. We have chosen the surface defined by $r =$ const. Its unit normal 1-form components n_μ are $n_\mu = \delta^r_\mu \frac{1}{\sqrt{g^{rr}}} = \delta^r_\mu \frac{1}{\sqrt{\overset{\circ}{g}{}^{rr} + h^{rr}}}$. For the reference space we have $\overset{\circ}{n}_\mu = \delta^r_\mu \frac{1}{\sqrt{\overset{\circ}{g}{}^{rr}}}$.

Asymptotically, the value of the energy surface density is:

$$\begin{aligned} \epsilon = \frac{1}{8\pi}(k - \overset{\circ}{k}) &= \frac{1}{8\pi}[(1 + (\frac{r}{R})^2)^{3/2}\,\frac{2mR^4}{r^6}(1 - \alpha^2 \sin^2\theta)^{-3/2} + \\ &+ (1 + (\frac{r}{R})^2)^{1/2}\,\frac{3ma^2}{r^4}\sin^2\theta(1 - \alpha^2\sin^2\theta)^{-5/2}] \end{aligned} \tag{8}$$

The total charge-energy turns out to be:

$$E \equiv Q_E = -\int_B d^2x N\sqrt{\sigma}\epsilon \tag{9}$$

where $N = u_\mu \xi^\mu$, yielding the result

$$E = \frac{m}{(1 - \alpha^2)^2} \tag{10}$$

which agrees with the result obtained by Henneaux and Teitelboim given in equation (6). We remark that equation (9) is exactly the hamiltonian energy in the Brown-York formalism.

It may also be checked[12] that this value coincides with the total gravitational energy calculated from the Einstein and Landau-Lifshitz pseudo-tensors calculated on a 2-sphere at spatial infinity.

In sum, the Henneaux-Teitelboim energy (obtained from the hamiltonian formalism), the Brown-York energy (obtained from the action formulation), and the Einstein and Landau-Lifshitz energy all agree for the asymptotically non-flat Kerr-anti-de Sitter spacetime. In the next section, we will perform analogous calculations for a generalization of the Komar definition of energy to asymptotically anti-de Sitter spacetimes based on the notion of geodesics and 'forces' on test particles.

[2]For finite regions of Kerr-like spaces, this calculation would be more involved. As an example, see reference [11].

3 The generalized Komar mass of the Kerr-anti-de Sitter spacetime

When an asymptotically flat spacetime has a timelike Killing vector field ξ^{μ}, the total Komar[7] mass is defined as:

$$M_K = -\frac{1}{8\pi}\int_B * d\xi \tag{11}$$

where $*d\xi$ is the dual to the two-form $d\xi$, ξ being the timelike Killing one-form $\xi = \xi_{\mu}dx^{\mu}$, and B is a closed two-dimensional spacelike surface where this integral is performed.

The exterior derivative of this two-form is given by:

$$d * d\xi = \frac{2}{3}R^{\alpha}_{\beta}\xi^{\beta}d^3\Sigma_{\alpha} \tag{12}$$

where $d^3\Sigma_{\alpha} = \frac{1}{3!}\eta_{\alpha\rho\mu\nu}dx^{\rho}\wedge dx^{\mu}\wedge dx^{\nu}$ is the three-form volume element[3] and R^{α}_{β} is the spacetime Ricci-tensor. Thus, in empty space and assuming Einstein's equations, the two-form $*d\xi$ is closed and, using Stokes's theorem, it follows that the Komar mass M_K is independent of the two-surface B.

In a coordinate frame we have:

$$M_K = -\frac{1}{8\pi}\int_B (\nabla^{\nu}\xi^{\mu} - \nabla^{\mu}\xi^{\nu})d^2S_{\mu\nu} \tag{13}$$

where $d^2S_{\mu\nu} = \frac{1}{2!}\eta_{\mu\nu\alpha\beta}dx^{\alpha}\wedge dx^{\beta}$, $\nabla^{\mu} = g^{\mu\nu}\nabla_{\nu}$ and ∇_{ν} is the covariant derivative with respect to the metric $g_{\mu\nu}$.

This mass can be interpreted as the total force needed to keep in place a unit surface mass density distributed over B[6]. Also, using again Stokes's theorem and Einstein's equations, the Komar mass can be written as:

$$M_K = 2\int_{\Sigma}(T_{\alpha\beta} - \frac{1}{2}Tg_{\alpha\beta})u^{\alpha}\xi^{\beta}dv \tag{14}$$

which is the total Whitakker's effective mass[13] for static spacetimes.

When the time coordinate is chosen such that $\xi^{\mu} = \delta^{\mu}_{0}$ the above expression reduces to the Møller energy

$$E_M = \int_B g^{i\mu}g^{0\nu}(g_{\nu 0,\mu} - g_{\mu 0,\nu})\sqrt{-g}d^2S_i \tag{15}$$

[3] $\eta_{\alpha\rho\mu\nu} = \sqrt{-g}\epsilon_{\alpha\rho\mu\nu}$ where $\epsilon_{\alpha\rho\mu\nu}$ is the four-dimensional (metric independent) completely antisymmetric object.

This expression also coincides asymptotically with the gravitational mass experienced by a test particle (via a Newtonian potential) following a geodesic $\frac{d^2x^i}{ds^2} + \Gamma^i_{00}\frac{dx^0}{ds}\frac{dx^0}{ds} = 0$ (supposing that $\frac{dx^i}{ds} << 1$) in the asymptotic region and given by:

$$M_g = -\frac{1}{4\pi}\int_B \frac{d^2x^i}{ds^2} d^2S_i \tag{16}$$

For asymptotically flat spacetimes, this Komar mass agrees with the gravitational energy calculations described in section 2, even being a completely different algebraic expression, which depends on g_{00}. Is this also true for asymptotically non-flat spacetimes?

Let us take the Kerr-Anti-de Sitter spacetime. It is clear that if we take naively the above expression as the Komar mass of this spacetime, two problems will arise: the mass will be infinite and surface dependent. Infinite because the anti-de Sitter background will give an infinite contribution. Surface dependent because the exterior derivative of the 2-form which is integrated in equation 13 will no longer be zero in the vacuum region:

$$d * d\xi = \frac{2}{3}\Lambda\xi^\alpha d^3\Sigma_\alpha \tag{17}$$

where Λ is the cosmological constant, $d^3\Sigma_\alpha$ the volume element of the Kerr-anti-de Sitter metric and ξ^μ is its Killing timelike vector field. The solution to these two problems is to use a procedure analogous to the one used by Brown and York[3] described in the previous section, and subtract a term corresponding to the Komar mass of a reference spacetime. The metric of the reference spacetime is the one which characterizes the asymptotic region, namely, the anti-de Sitter metric. For operational simplicity we choose the coordinate system of the anti-de Sitter reference spacetime by taking the limit $m = 0$ of the Kerr-anti-de Sitter metric in a fixed coordinate system, just like in equations (1), (2) and (3).

We define the generalized Komar mass as:

$$M_K = -\frac{1}{8\pi}\int_B * d(\xi - \overset{\circ}{\xi}) \tag{18}$$

where $\overset{\circ}{\xi} = \overset{\circ}{\xi}_\mu dx^\mu$ is the timelike Killing one-form of the reference spacetime, assuming that the surface B can be immersed in the reference spacetime.

In a coordinate frame, equation (18) is written as:

$$M_K = -\frac{1}{8\pi}\int_B [(\nabla^\nu\xi^\mu - \nabla^\mu\xi^\nu) - (\overset{\circ}{\nabla}{}^\nu\xi^\mu - \overset{\circ}{\nabla}{}^\mu\xi^\nu)] d^2S_{\mu\nu} \tag{19}$$

We used that

$$\begin{cases} \xi^\mu = \overset{\circ}{\xi}{}^\mu \\ \sqrt{-g} = \sqrt{-\overset{\circ}{g}} \end{cases} \tag{20}$$

for the Kerr-anti-de Sitter and the anti-de Sitter metrics.

Thus, equation (19) says that the generalized Komar mass M_K is the usual Komar mass contained in a surface B in the Kerr-anti-de Sitter spacetime minus the Komar mass contained in the same surface B in the anti-de Sitter spacetime.

Now, due to equations (17) and (20), the exterior derivative of the quantity which is integrated in equation (18) is zero in the region without matter: $d * d(\xi - \overset{\circ}{\xi}) = 0$. Thus, we can use Stokes's theorem to show that the generalized Komar mass given in equation (19) is surface independent in this region. It is clearly a conserved quantity because it is just the usual Komar mass, which is conserved, minus a constant term, which is the 'Komar mass' of the anti-de Sitter background.

Using Wald's interpretation[6], this can be viewed as a total generalized 'force' (between quotes because we no longer have a Newtonian limit) needed to keep in place a unit surface mass density distributed over B in the Kerr-anti-de Sitter spacetime minus the same 'force' in the anti-de Sitter background.

It can also be viewed as an Whitaker's effective mass as given in equation (14).

In the coordinate system where $\xi^\mu = \delta^\mu_0$, this generalized Komar mass reduces to the Møller energy of the Kerr-anti-de Sitter spacetime contained in the surface B minus the Møller energy of the anti-de Sitter background contained in the same surface B. In the asymptotic limit, this generalized Møller energy can be interpreted as a 'mass' obtained by 'measuring' the deviation of the coordinate acceleration of a test particle following a geodesic of the Kerr-anti-de Sitter spacetime from the coordinate acceleration of the same test particle following a geodesic of the anti-de Sitter reference spacetime in the Newtonian like approximation $\frac{dx^i}{ds} << 1$ and $\frac{dx^i}{d\overset{\circ}{s}} << 1$:

$$M = -\frac{1}{4\pi}\int_B \left(\frac{d^2x^i}{ds^2} - \frac{d^2x^i}{d\overset{\circ}{s}{}^2}\right) d^2S^i \tag{21}$$

with (keeping only first order terms in $h_{\mu\nu}$, $\frac{dx^i}{ds}$ and $\frac{dx^i}{d\overset{\circ}{s}}$)

$$\frac{d^2x^i}{ds^2} - \frac{d^2x^i}{d\overset{\circ}{s}{}^2} \approx -\Gamma^i_{00}\left(\frac{dx^0}{ds}\right)^2 + \overset{\circ}{\Gamma}{}^i_{00}\left(\frac{dx^i}{d\overset{\circ}{s}}\right)^2 \approx$$

$$\approx \quad -(\Gamma^i_{00} - \overset{\circ}{\Gamma}{}^i_{00})\frac{1}{\overset{\circ}{g}_{00}} + \overset{\circ}{\Gamma}{}^i_{00}\frac{h_{00}}{\overset{\circ}{g}{}^2_{00}} \tag{22}$$

where s and $\overset{\circ}{s}$ are, respectively, the proper time along a geodesic of the Kerr-anti-de Sitter metric and the proper time along a geodesic of the anti-de Sitter reference spacetime. It can be verified that, in the asymptotic limit, the integrand in equation (19) reduces to the right-hand-side of equation (22). This is a generalization of the mass calculated in terms of geodesic motion in asymptotically flat spacetimes given in equation (16).

Thus, the generalized Komar mass defined in equation (19) is a coordinate and surface independent conserved quantity which can be interpreted as a suitable generalization of the concept of mass given in asymptotically flat spacetimes in terms of forces and geodesics of test particles. When the cosmological constant is zero, it agrees with the usual Komar mass. It is a mass as good as the ones defined in section 2 and can be used to calculate the total mass-energy of the Kerr-anti-de Sitter spacetime.

Taking the surface B to be the two sphere $r =$ const., using equations (1), (2), (3) and (19), and taking the limit $r \to \infty$, we obtain for the generalized Komar mass:

$$M_K = \frac{m(1+\alpha^2)}{(1-\alpha^2)^2} \tag{23}$$

which is a different result from the one obtained in section 2 and given in equation (10) (or (6)).

Thus, for an asymptotically non-flat spacetime, the notion of a total mass-energy of a given gravitational field is not unique.

4 Conclusion

In the last two sections, we have shown that the total hamiltonian energy of the gravitational field of the Kerr-anti-de Sitter spacetime is different from a suitable extention of the total gravitational mass concept based on a generalization of the Komar mass. These two values are, respectively:

$$E = \frac{m}{(1-\alpha^2)^2} \tag{24}$$

$$M_K = \frac{m(1+\alpha^2)}{(1-\alpha^2)^2} \tag{25}$$

Thus, the equivalence between the total gravitational mass and the total gravitational energy may not be necessarily true in an asymptotically non-flat spacetime (where there is no asymptotic Poincaré group).

We can also calculate the Casimir invariant of the anti-de Sitter group (which is $O(3,2)$) related to the mass, the Casimir mass[9]. It is given by:

$$M_c = \sqrt{E^2 + (\frac{J}{R})^2} \tag{26}$$

where J is the total angular momentum of the Kerr-anti-de Sitter metric. This angular momentum can be calculated from equations (5) or (7) by taking $\xi_A = \frac{\partial}{\partial\phi}$. It is given by:

$$J = -\frac{ma}{(1-\alpha^2)^2} \tag{27}$$

yielding the following value for the Casimir mass:

$$M_C = \frac{m\sqrt{1+\alpha^2}}{(1-\alpha^2)^2} \tag{28}$$

which is also different from the other two masses given in equations (24) and (25).

Note that, when $\alpha = \frac{a}{R} = 0$, the three masses are equal (like in the Schwarzschild-anti-de Sitter spacetime, an example showing that a non-flat asymptotic structure is necessary but not sufficient to yield the aforementioned ambiguity among masses). Thus, it is the combination of the anti-de Sitter asymptotic structure and the angular momentum per unit mass a of the Kerr-anti-de Sitter spacetime that gives rise to the differences in the masses. A clue for a physical interpretation of the above results may be given by expanding the Komar mass in terms of the Casimir mass for small values of the parameter a. We obtain that $M_K \approx M_C + \frac{1}{2}M_C\frac{a^2}{R^2}$. This means that the total gravitational mass is, for small a, the Casimir mass M_C plus the rotational energy of a ring of mass M_C, radius R and angular momentum aM_C. If we had expressed M_K in terms of the energy E, we would have obtained a different result with a similar interpretation.

Certainly this issue deserves further study. The understanding of this ambiguity in the definition of the total mass-energy of the Kerr-anti-de Sitter spacetime will give us a better comprehension of the cosmological constant and its consequences to the most basic concepts we have in General Relativity. One good step forward would be to study these definitions of mass for other asymptotically anti-de Sitter spacetimes with angular momentum, including $2+1$ dimensional spacetimes.

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