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Zenithal Distribution of Atmospheric Muons

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Abstract

Hadron diffusion equations are solved using an alternative analytical method based on depth-like ordered exponential operators, similar to that used by Feynman. With this method these equations are solvable for any form of the primary spectrum (an improvement comparing with other ones). The muon fluxes generated by these hadronic showers are then obtained for zenith angles covering 0 to 89 degrees. A comparison of our calculations for the vertical and horizontal muon fluxes with experimental data and with another theoretical calculation is made. The agreement between them are in general very good, greather than 90 percent.

Key-words: Muons; Zenithal distribution; Cosmic-Rays.

1 Introduction

Cosmic ray propagation in the earth's atmosphere has long been studied on the basis of diffusion equations, which depends on the properties of the particles, their interactions and on the structure of the atmosphere [1]. Several different approaches are used to solve these diffusion equations such as analytical as numerical methods [2]. Also, this propagation can be calculated using simulation techniques like Monte Carlo [3]. In our case we think that analytical calculations are still worth to pursue because they are useful for qualitative understanding and to check Monte Carlo results. Besides, the analytical solutions are currently used in order to give accurately the relations among the different fluxes of particles. Muon fluxes play a very important role in the understanding of several cosmic ray and astrophysical issues of current interest. For example, atmospheric neutrino anomalies [4] and prompt muon and prompt neutrino production observed at sea level [5]. Another significant point is the analysis of the angular distribution of cosmic ray muons at sea level to study the problems mentioned above and the primary mass composition. In this paper we calculated the muon flux for different zenith angles originated by a hadronic shower in the earth's atmosphere. We are extending the same technique used in previous papers [6]. It is related with expansional operators similar to those used by Feynman in some Quantum Mechanics and QED problems [7]. This method allow us to investigate the effects of primary spectrum deviations from the power law form and to take into account non-scaling properties of the hadronic cross-section. Also, it is very powerful because permits to obtain the fluxes in the whole energy range in a single solution. In order to check our method of solution we compare our calculations with experimental data [8, 9] and with a solution obtained by Lipari [10]. To do this, we need to perform the same approximations used by this author. This paper is divided as follows: in the next section we solve the hadronic equations analytically using a general form for the primary energy spectrum with Maeda's fit for the atmosphere [11]. In section 3, we calculate the muon flux taking into account energy losses and decay. In section 4, we reduce the expressions obtained in the two last sections for the case where the primary spectrum is taken as a power-like law. In section 5, we present some numerical results and we make a comparison of our muon fluxes with experimental data and Lipari's calculation. Finally, we discuss and make some comentaries on our results.

2 Hadron Diffusion Equations

The diffusion equations for the hadronic components (nucleonic and mesonic) can be written as

$$\frac{\partial N(t,E)}{\partial t} = -\frac{N(t,E)}{\lambda(E)} + \int_0^1 \frac{N(t,E/\eta)}{\lambda(E/\eta)} f_{NN}(\eta)\frac{d\eta}{\eta} \tag{1}$$

and

$$\frac{\partial M(t,E,\theta^*)}{\partial t} = -\frac{M(t,E,\theta^*)}{\lambda_M(E)} - \frac{m_M\, M(t,E,\theta^*)}{c\,\beta\,\tau_M E\,\rho(t,\theta^*)} +$$

$$\int_0^1 \frac{M(t,E/x,\theta^*)}{\lambda_M(E/x)} f_{MM}(x)\frac{dx}{x} + \int_0^1 \frac{N(t,E/x,\theta^*)}{\lambda(E/x)} f_{NM}(x)\frac{dx}{x} \tag{2}$$

where t indicates the slant depth along a given direction of zenith angle θ^*, $\lambda_i(E)$ the interaction length in air and $(c\,\beta\,\tau_M\,\frac{E}{m_M}\,\rho(t,\theta^*))$ the decay length of the meson M in the atmosphere. f_{MM} and f_{NM} are respectively the spectra of the mesons produced in the meson-air nuclei and in the nucleon-air nuclei interactions, η is the elasticity of the collision nucleon-air nuclei and x is the Feynman variable ($x \approx E/E'$) where E' is the primary energy of the nucleon or the meson.

The solutions of equations (1) and (2) are subject to the boundary conditions

$$N(0,E) = G(E) \tag{3}$$

and

$$M(0,E) = 0 \tag{4}$$

where $G(E)$ stands for the differential energy spectrum of nucleons at the top of the atmosphere ($t = 0$). x

In order to solve the diffusion equations (1) and (2) we introduce the operators as we made in ref. [6].

$$\hat{A}_N = -\left(1 - \int_0^1 d\eta\, f_{NN}(\eta)\,\hat{\sigma}\right)\frac{1}{\lambda} \tag{5}$$

$$\hat{A}_M = -\left(1 - \int_0^1 dx\, f_{MM}(x)\,\hat{\sigma}_M\right)\frac{1}{\lambda_m} \tag{6}$$

$$\hat{B}_N = \int_0^1 dx\, f_{NM}(x)\,\hat{\sigma}_N\frac{1}{\lambda} \tag{7}$$

and

$$\hat{G}_M = \hat{g}_M(t,E,\theta^*) \tag{8}$$

where the operator $\hat{\sigma}$ acts on the energy function

$$\hat{\sigma}_i\, H(t,E,\theta^*) = \frac{1}{x}H(t,E/x,\theta^*) \tag{9}$$

for $x \geq x_{min} > 0$, where $i = N$ or M and

$$\hat{G}_M M(t,E,\theta^*) = \frac{1}{\lambda_{M_{decay}}} M(t,E,\theta^*) \tag{10}$$

with $(\lambda_{M_{decay}})^{-1} = (c\,\beta\,\tau_M\frac{E}{m_M}\rho(t,\theta^*))^{-1}$, being the eigenvalue of the operator $\hat{G}_M$ satisfying the eigen-equation (10). The operator $\hat{\sigma}$, in eq.(5), acts in the energy function transforming E to E/η instead of E/x, with $0 < \eta \leq 1$.

By the introduction of these operators in eqs.(1) and (2) we obtain the operator equations

$$\frac{\partial N}{\partial t}(t,E) = \hat{A}_N N(t,E) \tag{11}$$

$$\frac{\partial M}{\partial t}(t,E,\theta^*) = \hat{A}_M M(t,E,\theta^*) + \hat{G}_M M(t,E,\theta^*) + \hat{B}_N N(t,E,\theta^*) \tag{12}$$

The formal solutions of these operator equations which satisfies the initial conditions (3) and (4) are

$$N(t,E) = e^{-tA_N} G(E) \tag{13}$$

and

$$M(t,E,\theta^*) = \int_0^t Exp\left[\int_z^t \left(\hat{A}_M + \hat{G}_M\right) dz'\right] \hat{B}_N N(z,E)\, dz \tag{14}$$

where $Exp\left[\int_z^t \left(\hat{A}_M + \hat{G}_M\right) dz'\right]$ is an expansional defined by a sum of multiple depth-ordered integrals.

3 Muon Diffusion Equations

The one-dimensional diffusion equation of the muons in the atmosphere can be written as

$$\begin{aligned}\frac{\partial \mu}{\partial t}(t,E,\theta^*) = -\hat{G}_\mu(t,E,\theta^*) + \frac{\partial}{\partial E}(\beta(E)\, \mu(t,E,\theta^*)) + \\ + \int_{E_1}^{E_2} dE'\, \hat{G}_M\, M(t,E',\theta^*)\, f_{M\mu}(E,E')\end{aligned} \tag{15}$$

where

$$\hat{G}_\mu\, \mu(t,E,\theta^*) = \frac{1}{\lambda_{\mu_{decay}}}\, \mu(t,E,\theta^*) \tag{16}$$

The first term on the right-hand side of eq.(15) represents the muon decay, the second describes energy loss of the muon in the atmosphere and the last term the sources of the muons.

$\hat{G}_M$ and $\hat{G}_\mu$ represent the decay operator of the meson and of the muon in the earth atmosphere defined in eq.(10) and eq.(16), respectively. The decay lenght of the muon is

$$\lambda_{\mu_{decay}} \approx \frac{c\,\tau_\mu}{m_\mu} \epsilon(E, t-z)\, \cos\theta^*(z)\, \rho(z) \tag{17}$$

The solution of the eq.(15) satisfies the boundary condition

$$\mu(0,E,\theta^*) = 0 \tag{18}$$

We shall assume that the losses occur continuously, *i.e.*, fluctuations can be neglected. In this case a simple approximation expression for the muon energy loss is

$$-\frac{dE}{dt} = \beta(E) = a + b\,E \tag{19}$$

with "a" representing ionization and excitation losses and "b" the *bremsstrahlung*, pair-production and nuclear interaction losses. In this case the solution of the equation (15) can be written as a sum of a homogeneous part plus a particular case of the inhomogeneous one

$$\mu(t,E,\theta) = \mu_{hom}(t,E,\theta) + \mu_{part}(t,E,\theta)\cdot \tag{20}$$

As $\mu_{hom}(t,E,\theta)$ is equal to zero, due to the initial condition (18) the solution of the equation (15) becomes

$$\mu(t,E,\theta) = \int_0^t exp\left[b(t-z_1) - \int_{z_1}^t \frac{1}{\lambda_{\mu_{decay}}(z,\epsilon,\theta^*(z))}dz\right]\cdot$$

$$\cdot H(z,\epsilon(E,t-z_1),\theta^*(z_1))\,dz_1 \tag{21}$$

where,

$$\epsilon(E,t-z_1) = E^{b(t-z_1)} + \frac{a}{b}(e^{b(t-z_1)} - 1) \tag{22}$$

represents the muon energy at depth z_1 in order to arrive at depth t with energy E, $\frac{1}{\lambda_{\mu_{decay}}}$ is the eigenvalue of the operator $\hat{G}_\mu$ satisfying eq.(16) and

$$H(z,\epsilon,\theta^*(z_1)) = \int_{E_1}^{E_2} (BR)_M\,\hat{G}_M\,M(z_1,E',\theta^*(z_1))\,f_{M\mu}(E,E')\,dE'\cdot \tag{23}$$

$f_{M\mu}$, E_1 and E_2 are obtained from the relativistic kynematics of two or three bodies in the final state, $(BR)_M$ is the branching ratio of the meson M and $\theta^*(z_1)$ is the zenith angle at the muon production point.

Substituing the expression (17) in the muon flux (21) we obtain

$$\mu(t,E,\theta) = \int_0^t exp\left[b(t-z_1) - \int_{z_1}^t \frac{m_\mu\ \sec\theta^*(z)}{c\,\tau_\mu\,\epsilon(E,t-z)\,\rho(z)}\,dz\right]\cdot$$

$$\cdot H(z_1,\epsilon(t-z,E),\theta^*(z_1))\,dz_1 \tag{24}$$

4 Particular Case

In order to test our method we consider the cosmic ray primary spectrum in the usual form,

$$N(0,E) = N_0\,E^{-(\gamma+1)}. \tag{25}$$

As the muon flux presents a small variation due to the decrease with energy of the hadron interaction lenghts [12], we assume a constant value for them.

The nucleon flux takes the well-known form

$$N(t,E) = N_0 \, E^{-(\gamma+1)} \, e^{-t/L} \tag{26}$$

where

$$L = \frac{\lambda}{1 - Z_{NN}} \tag{27}$$

is the nucleon absorption mean free path with

$$Z_{NN} = \int_0^1 \eta^\gamma \, f_{NN}(\eta) \, d_\eta \cdot \tag{28}$$

The meson fluxes are simplified by use of the properties of the Expansional.

As the operators $\hat{A}_M$ and $\hat{G}_M$ are not commutative, we decompose the expansional operator from eq.(14) in the following way (see appendix).

$$M(t,E,\theta^*) = \int_0^t Exp\left(\int_z^t dz' \hat{A}_M\right) Exp\left[\int_z^t dz' Exp\left(\int_{z'}^t dz" \hat{A}_M\right) \cdot \right.$$
$$\left. \cdot \hat{G}_M Exp\left(\int_t^{z'} dz" \hat{A}_M\right)\right] \hat{B}_N \, N(z,E) \, dz \tag{29}$$

As the operator $\hat{A}_M$ is depth-independent we have

$$Exp\left[\int_a^b dz \hat{A}_M\right] = e^{(b-a)\hat{A}_M} \tag{30}$$

and

$$\hat{B}_N N(z,E) = \frac{Z_{NM}}{\lambda} N(z,E) \tag{31}$$

where Z_{NM} is the energy spectrum of the secondary mesons from the nucleon-air interaction,

$$Z_{NM} = \int_0^1 x^\gamma \, f_{NM}(x) \, dx \cdot \tag{32}$$

Therefore, using eq.(30) and eq.(31) the expression (29) can be written as

$$M(t,E,\theta^*) \;=\; \int_0^t Exp\left[\int_z^t e^{(t-z')\hat{A}_M} \hat{G}_M \, e^{-(t-z')\hat{A}_M} \, dz'\right] \cdot$$
$$\cdot e^{\frac{-(t-z)}{L_M(\gamma)}} \, \frac{Z_{NM}}{\lambda} \, e^{-\frac{Z}{L(\gamma)}} \, N_0 \, E^{-(\gamma+1)} \, dz \tag{33}$$

where, $-\frac{1}{L_M(\gamma)}$ and $-\frac{1}{L(\gamma)}$ are the eigenvalues of the operators $\hat{A}_M$ and $\hat{A}_N$ acting on the eigenfunction $N_0 E^{-(\gamma+1)}$, with

$$L_M(\gamma) = \frac{\lambda_M(\gamma)}{1 - Z_{MM}(\gamma)} \tag{34}$$

and

$$Z_{MM}(\gamma) = \int_0^1 x^\gamma \, f_{MM}(x) \, dx \cdot \tag{35}$$

where MM stands for meson production due to the meson-air interaction.

Using the eq.(49) of the appendix the eq.(33) takes the form

$$\begin{aligned} M(t, E, \theta^*) \;\; &= \;\; \int_0^t Exp\left[\int_z^t dz' \hat{G}_M \, e^{(t-z')\left(\hat{A}_M(\gamma) - \hat{A}_M(\gamma+1)\right)}\right] \cdot \\ &\cdot e^{\frac{-(t-z)}{L_M(\gamma)}} \, \frac{Z_{NM}}{\lambda} \, e^{-\frac{Z}{L(\gamma)}} \, N_0 \, E^{-(\gamma+1)} \, dz \end{aligned} \tag{36}$$

where $\hat{A}_M(\gamma)$ and $\hat{A}_M(\gamma+1)$ are operators with eigenvalues $(-\frac{1}{L_M(\gamma)})$ and $(-\frac{1}{L_M(\gamma+1)})$ respectively.

Using the definition of the expansional operator (see appendix) and taking into account the first two terms of the sum of multiple depth-ordered integrals we obtain

$$\begin{aligned} M(t, E, \theta^*) = \int_0^t e^{\frac{-(t-z)}{L_M(\gamma)}} \, \frac{Z_{NM}}{\lambda} \, e^{-\frac{z}{L(\gamma)}} \, Exp\Big\{\int_z^t \Big[\hat{G}_M(z') + \\ + \hat{G}_M(z')(t - z')\left(\hat{A}_M(\gamma) - \hat{A}_M(\gamma+1)\right)\Big] \, dz'\Big\} \, N_0 \, E^{-(\gamma+1)} \, dz \end{aligned} \tag{37}$$

Introducing the operators

$$\hat{T}_M(t, z) = Exp\left(\int_z^t dz' \, \hat{G}_M(z')\right) \tag{38}$$

and using the decomposition expansional properties the expression (37) takes the form

$$\begin{aligned} M(t, E, \theta^*) = \int_0^t dz \, e^{\frac{-(t-z)}{L_M(\gamma)}} \, \frac{Z_{NM}}{\lambda} \, e^{-\frac{t}{L(\gamma)}} \, \hat{T}_M(t, z) \cdot \\ \cdot Exp\Big[\int_z^t dz' \, \hat{T}_M(t, z') \, \hat{G}_M(z') \, (t - z') \cdot \\ \cdot \left(\hat{A}_M(\gamma) - \hat{A}_M(\gamma+1)\right) \, \hat{T}_M^{-1}(t, z')\Big] \, N_0 \, E^{-(\gamma+1)} \end{aligned} \tag{39}$$

Expanding the expansional in a sum of multiple depth-ordered integrals we obtain

$$\begin{aligned} M(t, E, \theta^*) = \int_0^t dz \, \frac{Z_{NM}}{\lambda} \, e^{\frac{-(t-z)}{L_M(\gamma)}} \, e^{-\frac{z}{L(\gamma)}} \, \Big[\hat{T}_M(t, z) + \\ + \int_z^t dz' \Big(\, \hat{T}_M(z', z) \;\, \hat{G}_M(z') \, (t - z') \, (\hat{A}_M(\gamma) - \\ - \hat{A}_M(\gamma+1)) \, \hat{T}_M(t, z')\Big) + \ldots\Big] \, N_0 \, E^{-(\gamma+1)} \end{aligned} \tag{40}$$

The eigenvalue of the operator $\hat{G}_M(z')$ is a function of the atmospheric density, $\rho(t,\theta^*)$ and of the factor $\frac{1}{E}$. As the meson production angle θ^* has a dependence with the slant depth, we realize that only in special cases the expansional $\hat{T}_M(t,z)$ can be performed exactly. For example, if we have a linear isothermal atmosphere the expansional $\hat{T}_M(t,z)$ can be written in a closed form

$$\hat{T}_M(t,z) = \left(\frac{t}{z}\right)^{\frac{b_M}{E\ \cos\theta}} \tag{41}$$

and the meson flux (40) becomes

$$M(t,E,\theta^*) = N_0\ E^{-(\gamma+1)}\ \frac{Z_{NM}}{\lambda} \int_0^t e^{\frac{-(t-z)}{L_M(\gamma)}}\ e^{-\frac{z}{L(\gamma)}} \left(\frac{t}{z}\right)^{\frac{b_M}{E\cos\theta}} dz \tag{42}$$

This last expression is the usual muon flux for a zenith angle $\theta \leq 60^o$ and for a primary cosmic ray with a power law spectrum. This flux was obtained considering only the first term on the right-hand side of the equation (40).

Substituting (40) in the expression (23) we obtain the function $H(z,\epsilon,\theta^*(z))$ that describes the source of muons from the decay of mesons M. So the flux of muons reaching an observation point of slant depth t with zenith angle θ can be calculated from the source $H(z,\epsilon,\theta^*(z))$ using the expression (24).

5 Comparison With Data

In order to make a comparison with the horizontal muon fluxes measured at sea level, we need to take into account several factors such as the primary cosmic ray spectrum, the hadronic Z-factors, the energy losses and decays and the interaction lengths λ, λ_π and λ_K.

As our intention is to estimate the sensibility of the Lipari's solution in comparison with our results, we used the same parameters and distributions, mentioned above, as suggested by Gaisser [13] and by Lipari [10].

The vertical column density as a function of height, $x(h)$, used in our calculation are taken from the fit of Maeda [11] for the average U.S. Standard Atmosphere. This fit corresponds to choosing a constant temperature in the stratosphere ($h \geq 11$ Km) and a linear dependence in the troposphere ($h < 11$ Km).

Figures 1 and 2 show a comparison of our calculations with the vertical and horizontal muon fluxes measured at sea level (ref.[8] for vertical and ref.[9] for horizontal muon data) and with the Lipari's analitycal solution [10]. The agreement is in general very good. Our vertical and horizontal muon fluxes are, respectively, 3 and 5% greater than Lipari's results, in the energy region from 2 to 10000 GeV and approximately 8% smaller than the Butkevich's calculation [1]. Figure 3 show the contributions of the $K^\pm$ to $\pi^\pm$ decay ratio to the vertical muon fluxes at sea level. This figure also shows a comparison with other ratios obtained by Lipari [10] and Volkova [14].

From this figure we note that the kaon decay accounts for approximately 10% of the muon flux at 100 GeV and this kaon contribution raises to $\approx 35\%$ at 10^4 GeV. At high energy, the Volkova's solution becomes larger than ours and Lipari's calculation. The disagreement can be traced through different choices of hadronic Z-factors (the factors used by Volkova are larger than ours) and the power index of the primary spectrum (we used a larger value). Figure 4 shows the variation of the horizontal muon fluxes at sea level for three different values of zenith angles $\cos\theta = 0$, 0.4 and 1. At low energies the vertical muon fluxes are much greater than the horizontal fluxes. At high energy, however, the former always gives smaller intensity.

6 Conclusions

We have solved the hadron diffusion equations analytically by means of Feynman-like procedure of ordered exponential operators. Then, we derived the zenith-angle spectrum of atmospheric muons from these hadronic cascades. One solution is showed to be valid in all energy interval from GeV to PeV and allows to include non-isothermal atmosphere.

As we showed this method is very powerful because permits solutions with a quite general form for the primary spectrum used as initial condition. Besides, it allows to get solutions with an energy dependent mean free path.

Our calculated muon fluxes (horizontal and vertical) fit very well the experimental data as well Lipari's calculation with a difference less than 10%. In particular, we think that the small difference between our calculation and Lipari's one could come from analytical method itself, since we have used the same parameters and distributions for the numerical calculations.

We investigate the muon fluxes in three zenithal angles, $\cos\theta = 0$, 0.4 and 1. We observe a well-known suppression effect for low energy muon as $\cos\theta$ goes to 1. This effect is due to the fact that they travel a very long path and, therefore, loose much more energy by ionization consequently increasing their decays.

Finally, we showed the contributions of the $K^{\pm}$ to $\pi^{\pm}$ decay ratio to the vertical muon flux. In our calculation the muon flux originated from kaons is 10% from those originated by pions, at 100 GeV. Instead, when the energy rises to 100 TeV we observe an enhancement of the kaon decay contribution, around 40%. This is a reflection of the increase of kaons production at higher energies. Our result is compared with Lipari and Volkova ones. At low energies we notice a good agreement among the three results. But, as the energy increases the Volkova's solution becomes higher than the others. This effect can be explained as a mixture of the following factors: a lower power index for the primary spectrum and a higher values of the secondary kaon distributions used by Volkova.

7 appendix

Let the operator function $H(x)$ with x a real parameter and $[H(x_i), H(x_j)] \neq 0$ for $x_i \neq x_j$.

The expansional operator is defined as the transformation operator generated by $H(x)$ corresponding to the variation of the parameter x between a and b. This transformation operator is an infinite product of the infinitesimal transformation operator $(\mathbf{1} + dxH(x))$

arranged from right to left corresponding to the sucession of ordering parameter x from a to b. So,

$$T(b,a) = Exp\left(\int_a^b dx H(x)\right) = \mathbf{1} + \int_a^b dx H(x) + \\ + \int_a^b dx_1 \int_a^{x_1} dx_2 H(x_2) + ... \tag{43}$$

Here Exp is an indication to discriminate our expansional operator from the usual exponential operator. As, $[H(x_i), H(x_j)] \neq 0$ for $x_i \neq x_j$ so the concept of ordering is of fundamental relevance.

Properties

a) $T^{-1}(b,a) = T(a,b)$, inverse operator
b) $T(b,a) \cdot T(a,b) = \mathbf{1}$, unitary operator
c) $T(b,a) \cdot T(a,c) = T(b,c)$
d) Composition and Decomposition Rules.

If $H(x) = H_1(x) + H_2(x)$, for $[H_1(x), H_2(x)] \neq 0$ then the expansional can be factorized.

d.1) Composition Rule

$$\begin{aligned} T_1(b,a) \cdot T_2(b,a) &= Exp\{\int_a^b dx\, [H_1(x) + T_1(x,a) \cdot H_2(x) \cdot T_1(a,x)]\} \\ &= Exp\{\int_a^b dx\, [H_2(x) + T_2(b,x) \cdot H_1(x) \cdot T_2(x,b)]\} \end{aligned} \tag{44}$$

where

$$T_i(b,a) = Exp\left[\int_a^b H_i(x)\, dx\right] \qquad \text{for i = 1 or 2} \tag{45}$$

d.2) Decomposition Rule

$$\begin{aligned} T(b,a) &= T_1(b,a) \cdot Exp\{\int_a^b dx\, [T_1(a,a_1) \cdot H_2(x) \cdot T_1(a_1,a)]\} \\ &= Exp\{\int_a^b dx\, [T_2(b,a_1) \cdot H_1(x) \cdot T_2(a_1,b)]\} \cdot T_2(b,a) \end{aligned} \tag{46}$$

with

$$T(b,a) = Exp\left[\int_a^b dx\, (H_1(x) + H_2(x))\right] \tag{47}$$

If, $[H(x_i), H(x_j)] = 0$ for $x_i \neq x_j$ the concept of ordering is irrelevant, so the expansional reduces to the usual exponential operators. Therefore, the capital letter E must be turned into its small letter and the integration in the exponent must be performed according to its usual definition.

Particulary, the most familiar type of exponential operator is when $H(x)$ is constant. Assuming $a = 0$ and $b = 1$, then

$$Exp\left[\int_0^1 dx H\right] = e^H \tag{48}$$

For completeness we include the formula frequently used without proof

$$e^{-A} B e^{A} = \sum_{n=0}^{\infty} B_n \tag{49}$$

where

$$B_0 = B \tag{50}$$

and

$$B_n = [B_{n-1}, A] \tag{51}$$

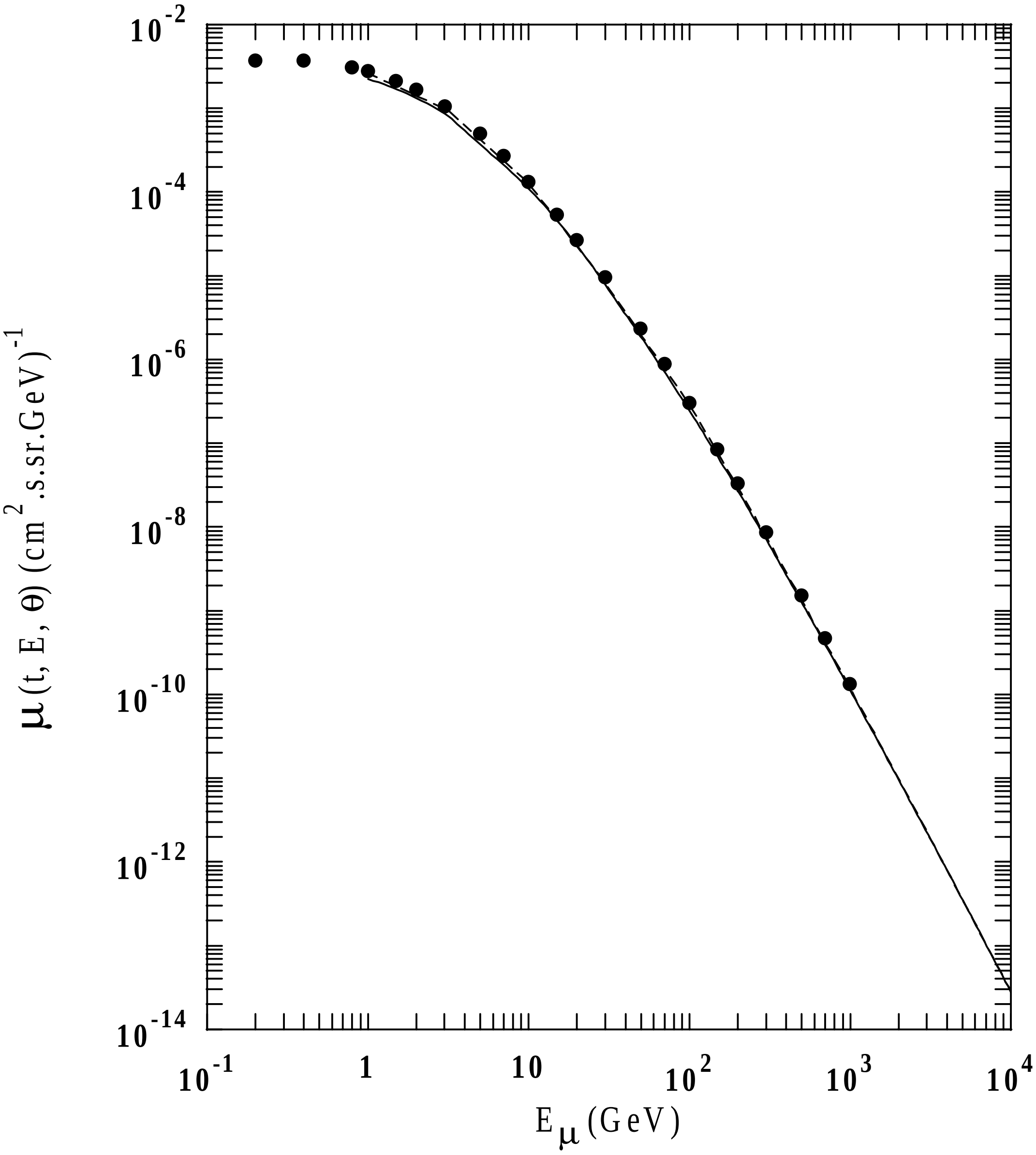
Figure Captions

Figure 1 - Vertical muon fluxes at sea level. Experimental data taken from Allkofer *et al.* (black dots); Muon flux calculated by P. Lipari (solid line); Muon flux calculated in this work (dashed line).

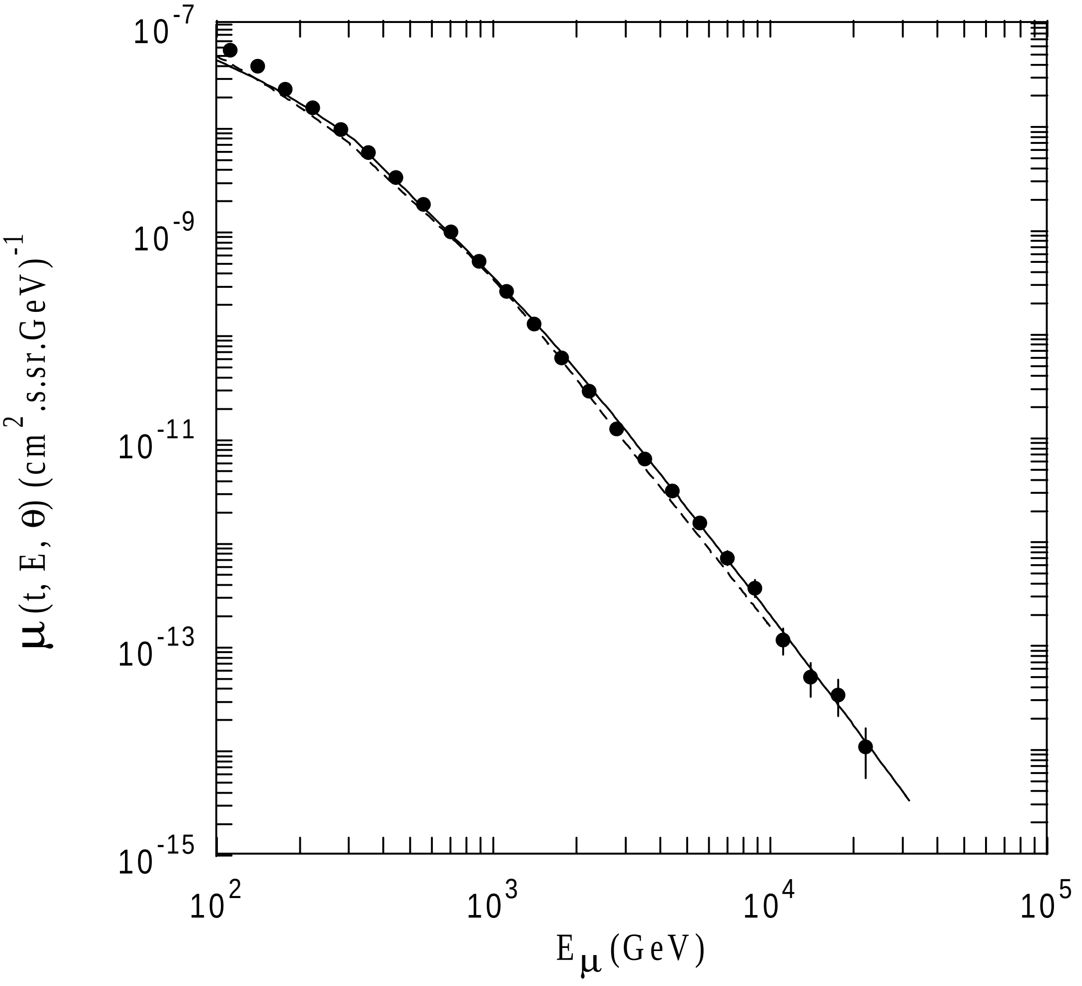
Figure 2 - Horizontal muon fluxes at sea level. Experimental data taken from Matsumo *et al.* (black dots); Muon flux calculated by P. Lipari (solid line); Muon flux calculated in this work (dashed line).

Figure 3 - Contribution of the $K^{\pm}$ to $\pi^{\pm}$ decay ratio to the muon fluxes, at sea level, for $\theta = 0$. L. Volkova [14] (crosses); P. Lipari [10] (white dots); Our calculations (black dots).

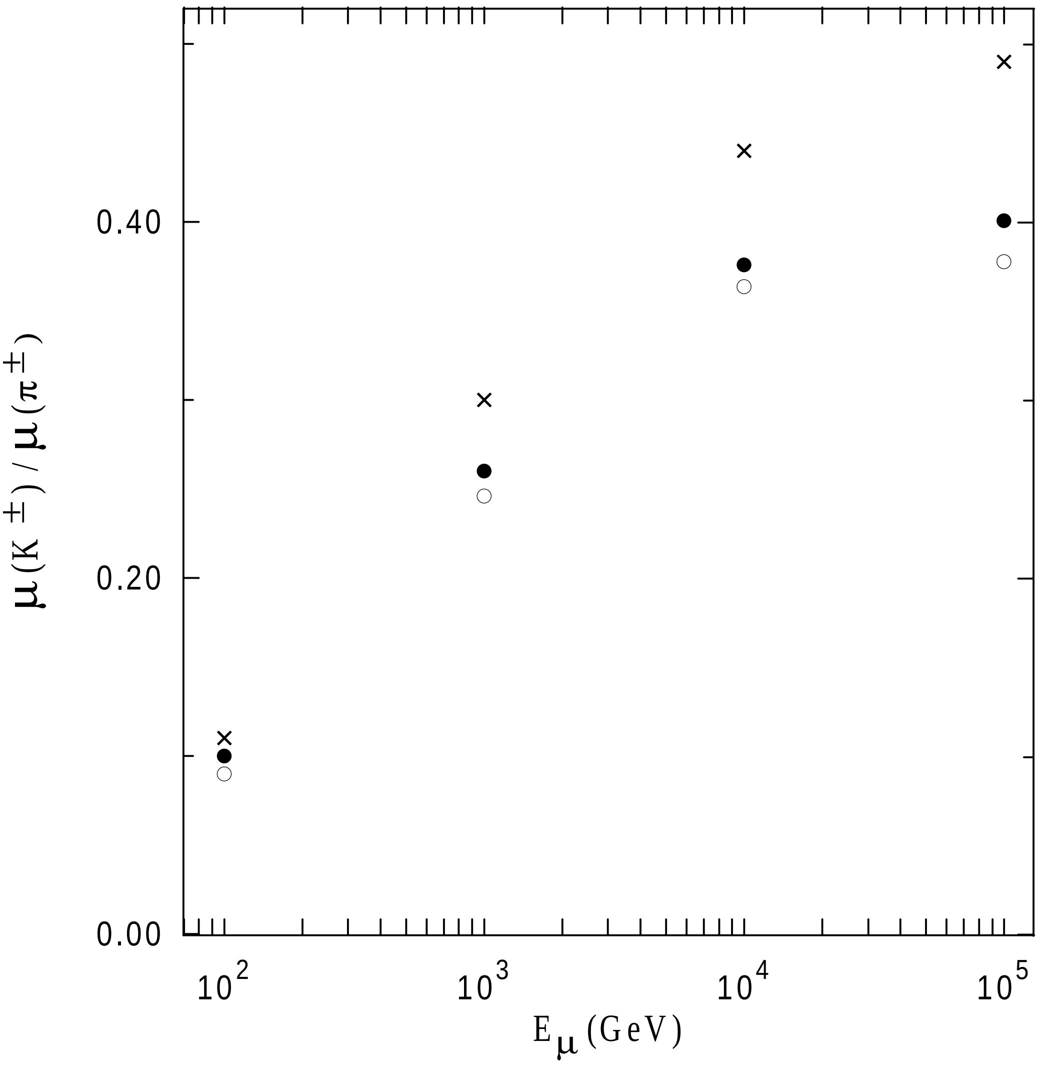
Figure 4 - Horizontal muon fluxes at sea level for three different values of zenith angles. $\cos\theta = 0$ (black dots); $\cos\theta = 0.4$ (crosses); $\cos\theta = 1$ (white dots).



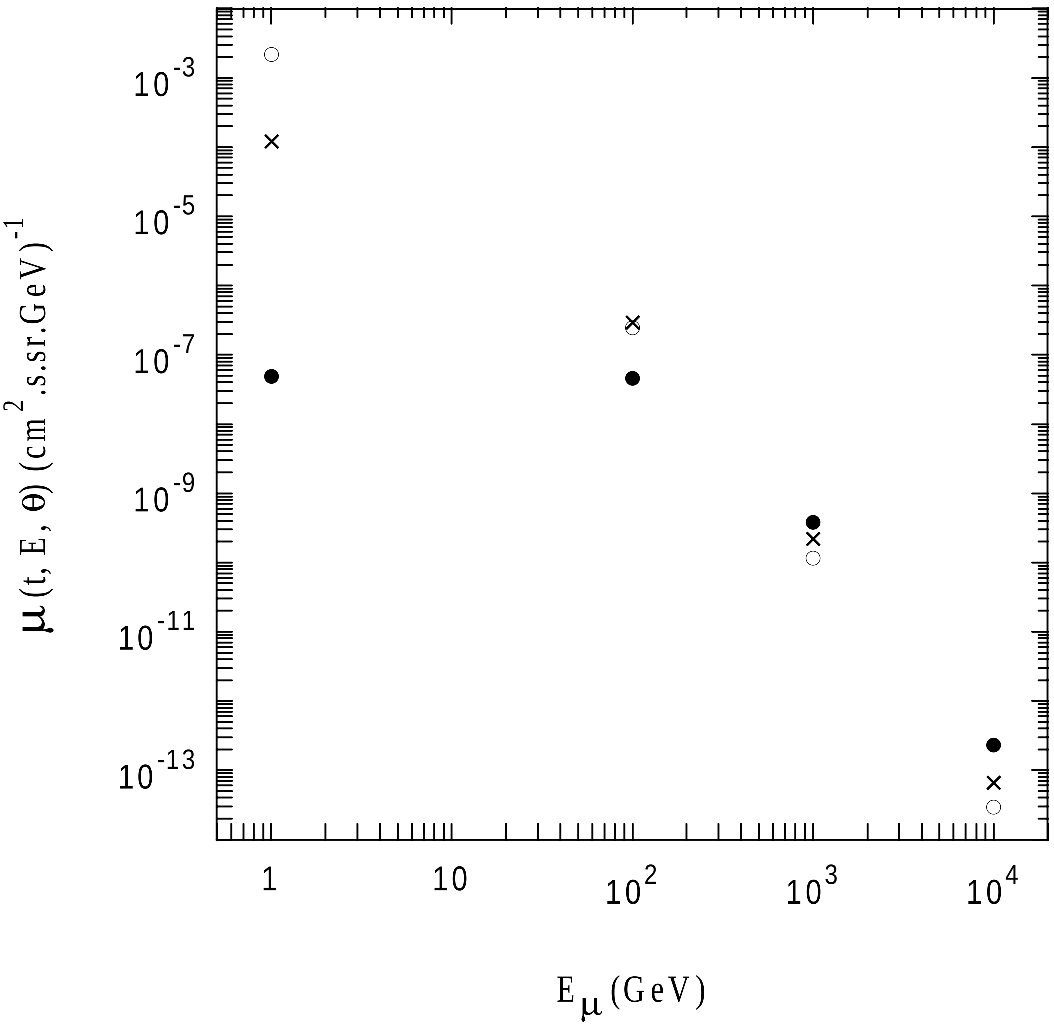
10^{-2}
10^{-4}
10^{-6}
10^{-8}
10^{-10}
10^{-12}
10^{-14}
$\mu(t, E, \theta)$ $(cm^2.s.sr.GeV)^{-1}$
10^{-1}
1
10
10^2
10^3
10^4
E_μ (GeV)



10^{-7}
10^{-9}
10^{-11}
10^{-13}
10^{-15}
10^2
10^3
10^4
10^5
$\mu(t, E, \theta)\ (cm^2.s.sr.GeV)^{-1}$
E_μ (GeV)



0.40
0.20
0.00
$\mu(K^{\pm}) / \mu(\pi^{\pm})$
10^2
10^3
10^4
10^5
E_μ (GeV)



10^-3
10^-5
10^-7
10^-9
10^-11
10^-13
μ (t, E, θ) (cm^2.s.sr.GeV)^-1
1
10
10^2
10^3
10^4
E_μ (GeV)

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