Black Hole Entropy in $1+1$ Dimensions from a Quasi-Chern Simons term in a Gravitational Background

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We introduce a "quasi-topological term [1] in $D = 1+1$ dimensions and the entropy for black holes is calculated [2]. The source of entropy in this case is justified by a non-null stress-energy tensor.

Key-words: Black hole; Gravitation; Chern-Simons.

I Introduction

It is well known that higher derivative Chern-Simons extensions in a background of gravitation [1] in $D = 1+2$ dimensions are possible. Such action has some pecularities compared to the usual Chern-Simons gravitational or Abelian Chern-Simons theory. Perhaps the most interesting fact associated with an extension for Chern-Simons theory in $D = 1+2$ is that the energy-momentum tensor is non zero. Considering the action given by Deser-Jackiw [1] it is not difficult to calculate the entropy for black holes in $D = 1+2$ dimensions [6].

Despite the fact that the Chern Simons theory appears only in odd dimensions [5,3] we attempt to write here an analogous action in a background of gravitation in $D = 1+1$ dimensions and compute the entropy of black holes in such a case.

This action in $D = 1+1$ dimension may be called a "quasi-topological action as Chern-Simons" in the same sense as suggested earlier [1,3,6,8].

The quantities such as Hawking's temperature, inverse temperature and entropy correction are shown as in the corresponding case given before [2,6].

The "quasi topological action" is different from the one suggested earlier [2] and depends locally on the potential vector in flat space time.

Let us start by writing the functional integral following the analogy of the Euclidean field theory and statistical mechanics as

$$Z_T = \int \mathcal{D}g \; e^{-(I[g,\varphi] + I[f,g])} \tag{1}$$

where

$$I[g,\varphi] = \frac{1}{4G}\int d^2x \,\sqrt{-g}\; e^{-2\varphi}\left[R + 4(\nabla\varphi)^2 + 4\lambda^2\right] \; . \tag{2}$$

Here $I[g,\varphi]$ is the two dimensional dilaton action motivated by the string theory [7] and

$$I[f,g] = \int d^2x \;\varepsilon^{\mu\nu} f_\mu \Box^2 f_\nu \tag{3}$$

It's a "quasi topological term" in $1+1$ dimensions. In equation (3) f_μ, is a covariant vectors with f_μ given as

$$f_\mu = g^{-1/2} g_{\mu\beta}\varepsilon^{\beta\lambda}A_\lambda \; . \tag{4}$$

The quantities R, φ, g, $\varepsilon^{\beta\lambda}$ and $A_\lambda(x)$ are respectively scalar curvature, dilaton field, determinant of the metric, the Levi-Cevita tensor $\varepsilon^{01} = +1$ and the vector potential respectively.

A general solution of the Einstein's equation is parametrized by one constant [7] and it is given by

$$ds^2 = -A\;dt^2 + \frac{dr^2}{A} \tag{5}$$

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where

$$A = 1 - \frac{M}{\lambda}\, e^{-\alpha\lambda r} \tag{6}$$

and

$$\varphi = -\lambda r \ . \tag{7}$$

The metric describes a static, asymptotically flat two dimensional black hole with event horizon at $r = r_+$ where

$$r_+ = \frac{1}{2\lambda}\ln\left(\frac{M}{\lambda}\right) \ . \tag{8}$$

with the parameter M being identified with the mass of a black hole.

Following [2] we can find the Hawking's temperature α, inverse temperature as well the Euclidean time period β, and entropy S as

$$\alpha = \left.\frac{df(r)}{dr}\right|_{r=r_+} , \tag{9}$$

$$\beta = \frac{2\pi}{\alpha} , \tag{10}$$

$$S = \frac{\tilde{A}_0}{4} \tag{11}$$

with $f(r)$ being equal to A and $\tilde{A}_0$ meaning the area of the event horizon in $D = 1+1$ dimensions and $\tilde{A}_0 = r_+$ respectively. These quantities are then given as

$$\begin{aligned} \alpha &= 2\lambda \ , \\ \beta &= \frac{\pi}{\lambda} \ , \\ S &= \frac{1}{8\lambda}\ \ln\left(\frac{M}{\lambda}\right) \ . \end{aligned} \tag{12}$$

It is usual to redefine the Hawking's temperature as

$$T_H = \frac{1}{\beta} , \tag{13}$$

thus, in this case the temperature is given as

$$T = \frac{\lambda}{\pi} \ . \tag{14}$$

The equation (3) may be expressed as

$$I[f,g] = \int d^2x (f_0 \Box^2 f_1 - f_1 \Box^2 f_0) \tag{15}$$

using equation (4), the two contributions f_0 and f_1 are given as

$$\begin{aligned} f_0 &= \left(1 - \frac{M}{\lambda}\, e^{-2\lambda r}\right)\ A_1(x) \\ f_1 &= -\left(1 - \frac{M}{\lambda}\, e^{-2\lambda r}\right)^{-1}\ A_0(x) \ . \end{aligned} \tag{16}$$

It is important to remember that equation (15) was obtained considering only the static configuration for f_0 and f_1 fields.

Then we have

$$Z_T \cong e^{f_0 \Box^2 f_1 - f_1 \Box^2 f_0} \int \mathcal{D}g\ e^{-(I[g,\varphi] + I[f,g])} \tag{17}$$

where only two terms in integrand (15) are given as

$$f_0 \Box^2 f_1 - f_1 \Box^2 f_0 \cong \Box \frac{4M\ e^{-2\lambda r}}{(1 - \frac{M}{\lambda}\ e^{-2\lambda r})}\ F(A_1, A_0) \tag{18}$$

and

$$F(A_1, A_0) = A_1\left(\frac{\partial A_0}{\partial r}\right) + A_0\left(\frac{\partial A_1}{\partial r}\right) \ . \tag{19}$$

Considering now the following approximation: $f_0\Box^2 f_1 - f_1\Box^2 f_0 \simeq \Box(f_0\Box f_1 - f_1\Box f_0)$ where we wish to imply that the operator $\Box$ will act only on the second part of each term of the expression $f_0\Box f_1 - f_1\Box f_0$, i.e, it acts on $\Box f_1$ and $\Box f_0$ only.

We approximate the partition function associated with the "quasi topological term" as

$$Z^{Chern}_{Simons} \simeq e^{f_0 \Box^2 f_1 - f_1 \Box^2 f_0} \tag{20}$$

$$Z^{Chern}_{Simons} = e^{\Box\left(\frac{4\pi F(A_1, A_0)}{\beta e^{2\lambda r} - \pi/\lambda}\right)} \tag{21}$$

The contribution to entropy is found from

$$S = \ln\ Z - \beta\ \frac{\partial}{\partial\beta}\ \ln Z \tag{22}$$

and we get

$$S \approx \Box\left[\frac{4\pi\ F(A_0, A_1)}{(\beta\ e^{2\lambda r} - \frac{\pi}{\lambda})} + \frac{4\pi\beta\ F(A_0, A_1)e^{2\lambda r}}{(\beta\ e^{2\lambda r} - \frac{\pi}{\lambda})^2}\right] \tag{23}$$

The average energy is obtained from

$$M = -\frac{\partial}{\partial\beta}\ln\ Z \tag{24}$$

and is

$$M \cong \Box\left[\frac{4\pi\ F(A_1, A_0)e^{2\lambda r}}{(\beta\ e^{2\lambda r} - \frac{\pi}{\lambda})}\right] \tag{25}$$

It is assumed that all fields at infinity go to zero i.e. $F(A_0, A_1) \to 0$ when $r \to \infty$.

The entropy, (23), and mass, (25), are zero in that case and we can recover the results (12) for Einstein-Hilbert theory only. For $r = r_+$ the equations (23) and (25) diverge then we have a mechanism for generation of mass and entropy. Finally, for $0 < r < r_+$ a positive value for S is obtained.

In the general case, however, the total entropy is

$$S_T \sim S^{Einstein}_{Hilbert} \oplus S^{Chern}_{Simons} \tag{26}$$

with $S^{Einstein}_{Hilbert}$ given by (12) and S^{Chern}_{Simons} given by eq. (23).

Again the source of entropy from action (3) is traced to the fact that the energy momentum tensor is not zero but is given by

$$T^{\mu\lambda} = \frac{2\Box}{g^2}\ \varepsilon^{\nu k}\varepsilon^{\theta\gamma}\varepsilon^{\xi\delta} g^{\mu\lambda} g_{k\theta} g_{\nu\xi} A_\gamma A_\delta \tag{27}$$

or using (4) it can be written as

$$T^{\mu\lambda} = \frac{2\Box}{g}\, g^{\mu\lambda} f_k f^k$$

Then $T^{00} \sim g^{00}$ and at $r = r_+$, the stress energy momentum tensor diverges. On the other hand, in the limit $r \to \infty$, $T^{\mu\lambda}$ is null due to our assumption that all fields are zero at infinity; so the metric (5) reduces to the flat space time with eq. (4) going to zero along with eq. (3).

The conservation of $T^{\mu\lambda}$ on-shell is easily checked with the equation of motion given as

$$\varepsilon^{k\nu}\varepsilon^{\theta\gamma}\varepsilon^{\xi\delta} g_{k\theta} g_{\nu\varepsilon} A_\gamma A_\delta = 0 \quad (28)$$

or still yet

$$\varepsilon^{\nu}_{\theta}\varepsilon^{\theta\gamma}\varepsilon^{\delta}_{\nu} A_\gamma A_\delta = 0 \quad (29)$$

and finally using again (4) it is given as

$$\varepsilon^{k\nu} f_k f_\nu = 0$$

and thus, independent of the metric. The complete metric dependence is on the field f_μ given by eq, (4).

Hence, the action (3) or (15) is clearly not completly a topological action as in [5,4] but is a "quasi-topological in the sense that the metric dependence is entirely contained in f_μ in the same way as [1,6,8]. It is only with this meaning that we are calling the action (3) "A quasi-chern-simons action" or "quasi-topological term". The action (3) has the same form as that of Jackiw's action in [1], but there taking the limit of $g_{\mu\nu}$ for $\eta_{\mu\nu}$, flat space time, the higher derivative Chern-Simons extension can be written as $I_{ECS} \sim \int d^3x\ f_\alpha \partial_\beta f_\gamma \varepsilon^{\alpha\beta\gamma}$ and $f^\alpha = \frac{1}{2}\varepsilon^{\alpha\mu\nu} F_{\mu\nu}$ where the extension depends locally on the field strength and not on the vector potential. In our present case, the equation (4) is reduced to $f^\alpha = \varepsilon^{\lambda\beta} A_\beta$ and then the action (3) depends locally on the vector potential directly, while the equation of motion is independent of metric, thus, our case being different than that of [1]. However, the best analogy with $D = 2+1$ we can do is through considering the equation (3) and some quasi-topological aspects such in [1,3,8].

Hence, for $D = 1+1$ dimension we can't write an exact topological Chern-Simons action but one can evaluate some contribution for entropy of black holes by constructing an action analogously to [1] and improving the same approach as in [2].

Conclusions and Comments

We had introduced a "quasi topological action" as in [1] and with an appropriate definition of a vector in $D = 1+1$ dimension. The contribution to entropy of black holes in two dimensions is found.

The correction for average energy has been written as a function of $F(A_0, A_1)$ with the assumption that all fields at infinite are zero.

The contribution to entropy is attributed to a non-null energy momentum tensor (27). We justify the source of entropy as the stress-energy tensor.

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