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Scalar-QED β-functions near Planck's Scale[1]

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Abstract

The Renormalization Group Flow Equations of the Scalar-QED model near Planck's scale are computed within the framework of the average effective action. Exact Flow Equations, corrected by Einstein Gravity, for the running self-interacting scalar coupling parameter and for the running v.e.v. of $\phi^*\phi$, are computed taking into account threshold effects. Analytic solutions are given in the infrared and ultraviolet limits.

Key-words: Quantum gravity; Exact flow equations; Renormalization group equations.

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I Introduction.

Nowadays, the satisfactory state of comprehension of the fundamental interactions of nature based on the gauge principle constitutes a strong appeal to quantize the gravitational field. Also, the search for a grand-unified theory in four dimensions shall demonstrate a complete sense only when the quantization program for the gravitational field has reached the same status of comprehension and consistency of the other three interactions. During the '70s, attempts to verify the renormalizability of Einstein gravity as a perturbative quantum field theory were able to show the casual on-shell finiteness of pure gravity at one-loop [1], the explicit loss of renormalizability at two-loops [2], by the generation of non-trivial higher-order counterterms that can not be absorbed in a redefinition of the physical parameters, and that matter-gravity coupled theories do not renormalize at all. In an attempt to cure the non-renormalizability of Einstein gravity in four dimensions, some higher-derivative models were proposed [3, 4]; these models were shown to be non-unitary, though renormalizable, or unitary but now non-renormalizable again [5].

Within this four dimensional picture, of apparent incompatibility of quantum mechanics and gravity, and having in mind that experiments can only probe a limited range of energies, we are led to take a more realistic position and interpret general relativity as an effective theory [6] up to Planck's scale, where low-energy scales are separated from unknown high-energy physics. The low-energy effective theory, obtained integrating out unknow high-energy physics, does not present problems with renormalizability and/or unitarity. This can be easily understood in the following way : a theory of gravity where high-energy effects are taken into account, presents high-derivative sectors. These are responsible for the renormalizability of the model, if one makes efforts to constrain its coefficients in this direction, but unavoidably show the existence of massive ghosts. As these ghosts have masses proportional to the Planck's mass, they shall not be excited in the low-energy regime [7]. At sufficiently low energies, these high-derivative corrections are not even relevant to Physics phenomena at this lower scale [6].

Owing to the description of physics in a grand-unified picture, the effects due to the gravitational interaction on field theory models become relevant when the energy scale is close enough to Planck's scale. In this scenario of taking into account gravity corrections at the grand-unified range and the need for consistency in a quantum field theory description of general relativity, the viewpoint we shall adopt is that of Wilson's conception of an effective theory [8], where the physical phenomena should be analysed at a characteristic scale where its effects can effectively be verified (in contrast to the "fundamental" thought of taking all scales at a time that has become the orthodoxy in quantum field theory) and the relation between different scales described by "exact renormalization group equations" [9].

Recently, a new effective field theory approach in continuous space [10], improved from Polchinski's presentation of the "flow equations" [9], in the sense it gives the generating functional of 1PI-Green's functions as one runs the characteristic scale κ to zero, was proposed to exhibit infrared properties of theories such as the convexity of the effective action when spontaneous symmetry breaking takes place [11]. This is known by the name of "average effective action".

The average effective action Γ_κ takes averages of fields in the effective action in continuos space [10]-[11] by adding an infrared smooth cutoff ΔS_κ to the action S; so that all contributions to the effective action with momenta $q^2 < \kappa^2$ are effectively suppressed. In the limit $\kappa \to 0$, the cutoff vanishes. ΔS_κ is written as

$$\Delta S_\kappa = \frac{1}{2} \int d^4q \; \phi_{_A}(-q) \; R_\kappa^{[AB]}(q) \; \phi_{_B}(q) \,, \tag{1.1}$$

where $\phi_{_A}$ are generic fields whose Lorentz character is described by the labels A and B. R_κ is defined in such a way as to imply that the 1PI-effective action is recovered from Γ_κ when the limit $\kappa \to 0$ is taken, independently of the form of R_κ, and $\Gamma_{_{\kappa\to\infty}}[\phi] = S[\phi]$ in the opposite limit [11]; showing that a flow equation can be written with solutions interpolating from the classical action to the effective action. Introducing sources to the action $S + \Delta S_\kappa$, and defining the generating functional of 1PI-Green's functions by a Legendre transformation of the κ-dependent generating functional of connected Green's functions, one easily gets the exact evolution equations, or flow equations, for the average effective action [11]

$$\kappa \frac{\partial}{\partial \kappa} \Gamma_\kappa[\phi] = \frac{1}{2} Tr \; \left[\left(\Gamma_\kappa^{(2)}[\phi] + R_\kappa \right)^{-1} \; \kappa \frac{\partial}{\partial \kappa} R_\kappa \right] \,, \tag{1.2}$$

that relates phenomena at different scales. $\Gamma_\kappa^{(2)}$ is the second functional derivative of the average effective action with respect to the classical field ϕ; it is interpreted as the inverse full propagator.

In order to avoid some possible problems discussed in [7], that may occur when explicitly stating R_κ, we are going to use here a further consideration: the squared momentum contributions, q^2, that we find in all inverse propagator coefficients obtained from the linearized form of the action $S + \Delta S_\kappa$, shall be replaced by the function

$$P_\kappa(q^2) = \frac{q^2}{1 - f_\kappa^2(q)} \,, \tag{1.3}$$

where

$$f_\kappa(q) = exp \left[-a \left(\frac{q^2}{\kappa^2} \right)^b \right] \,, \tag{1.4}$$

with a and b constants, as the only effect that the cutoff term ΔS_κ has over the effective action [7].

For $q^2 > \kappa^2$, $P_\kappa(q^2)$ goes exponentially fast to q^2, while for $q^2 < \kappa^2$ the low-energy modes are suppressed. The function $f_\kappa(q)$ behaves like a modulating function acting on q^2

varying from a Gaussian function, when $a = \frac{1}{2}$ and $b = 1$, for example, where $P_\kappa(q^2)$ tends to κ^2 for $q^2 \to 0$, to a step function, when $b \to \infty$, forbidding the modes with $q^2 < \kappa^2$ from propagation. So, thanks to this behavior and the κ-derivative of $P_\kappa(q^2)$ (see below), that suppress the propagation of large momentum fluctuations, the momentum integration of the flow equations is ultraviolet and infrared finite. Thus, no regularization scheme is needed in order to deal with high-energy quantum fluctuations; provided an appropriate non-trivial cutoff term modulate the asymptotic behavior of effective propagators and vertices in such a way that the loop integrals of the flow equations are performed, effectively, over a finite number of degrees of freedom. In this sense, the method is close to what is called non-perturbative field theory.

The average effective action, as proposed above, might turn the exact solution of the flow equations impossible to be found out due to an infinite number of effective insertion terms that can be added to the effective action. Using some symmetry requirements along with the understanding that experiments go over a limited range of energies, one is then led to consider only a finite and small number of relevant parameters that characterize physical phenomena up to a given energy scale by performing a truncation scheme at a given scale κ_1. The integration of the flow equations of the truncated effective action down to $\kappa_2 < \kappa_1$ is, in principle, possible and so is the description of the flow of the relevant parameters.

In this work, we follow the approach of [12] when describing gauge fields by the background field method, where it is shown that the gauge symmetry is preserved with respect to the gauge transformation of the background.

The paper is planned as follows : in section 2, we define the average effective potential at one-loop and extract the flow of relevant parameters in a general form. In the $3^{rd.}$ and $4^{th.}$ sections, we compute the flow equations for the Scalar-QED model with and without gravity interaction. In section 5, we present our concluding remarks.

II Flow Equations.

Given the general form of the exact evolution equations (1.2) above, we can now introduce a set of prescriptions by imposing some definitions, or parametrizations, for the first few parameters appearing in the truncated form of the average effective action Γ_κ. Remembering that we want to describe the flow of the relevant parameters in a spontaneously broken regime, let us parametrize the action by the minimum of its potential and the quartic self-interacting coupling at the minimum. In this phase, the average effective potential V_κ has its minimum at $\rho_\kappa = (\varphi^*\varphi)_\kappa$. So,

$$V'_\kappa(\rho_\kappa) \;=\; 0 \quad , \quad \lambda_\kappa = V''_\kappa(\rho_\kappa) \quad ; \tag{2.1}$$

where each prime denotes a derivative with respect to ρ.

The flow equations for ρ_κ and λ_κ can be read off by taking derivatives of (2.1) with respect to κ as below:

$$\kappa \frac{\partial}{\partial \kappa} \rho_\kappa = -\frac{1}{\lambda_\kappa} \left(\kappa \frac{\partial}{\partial \kappa} V'_\kappa \right)_{\rho \,=\, \rho_\kappa} \equiv \kappa^2 \gamma(\kappa) \ , \tag{2.2}$$

$$\kappa \frac{\partial}{\partial \kappa} \lambda_\kappa = \left(\kappa \frac{\partial}{\partial \kappa} V''_\kappa \right)_{\rho \,=\, \rho_\kappa} \equiv \beta(\kappa) \ . \tag{2.3}$$

In the definition of $\beta(\kappa)$ we neglected a third derivative term which accounts for the variation in the point of definition of λ_κ; we consider it an irrelevant contribution to the truncated action.

As it can be seen, the evolution equation of ρ_κ and λ_κ can be computed using the definitions above directly from the average effective potential V_κ. This potential shall be given only at one-loop order due to truncations, although it can be represented by one-loop Feynman diagrams of full propagators [11]. To this end, we consider the effective action at a scale κ as the classical action at a lower scale [7]. Using the background field method, we write down the linearized form of the Euclidean classical action, expand the effective action in powers of momentum around the position of vanishing external momenta, choose a specific configuration of the fields, so that only translationally invariant vacuum expectation values are taken into account [13], and properly define the determinant of the operators of small fluctuations (see below) as to account for the volume of the spacetime Ω.

The linearized quadratic action, including Faddeev-Popov ghosts, gauge-fixing sectors and potential terms, can be written as

$$S^{(2)} = \int d^4q \ \phi_A(-q) \ a^{AB}_{ij}(J^P) \ P^{AB}_{ij}(J^P) \ \phi_B(q) \,, \tag{2.4}$$

where $a^{AB}_{ij}(J^P)$ are coefficient matrices that represent inverse propagators with definite spin and parity; $P^{AB}_{ij}(J^P)$ are spin-projection operators [5, 7, 14, 15] listed in the appendix and the indices A, B label field fluctuations above the background.

The one-loop average effective action is obtained through a Legendre transformation of the generating functional of connected Green's functions and an appropriate choice of ΔS_κ so as to make possible the substitution of q^2 by $P_\kappa(q^2)$ in the operator of small fluctuations $\mathcal{O}_{AB}$ defined below:

$$S^{(2)} = \int d^4q \ \sum_{A,B} \ \phi_A(-q) \ \mathcal{O}_{AB} \ \phi_B(q) \,. \tag{2.5}$$

So, provided an ultraviolet cutoff Λ is taken, we define the one-loop average effective action as

$$\Gamma_\kappa(\rho) = \frac{1}{2} \ln \left(\frac{\det_\kappa \mathcal{O}(\rho)}{\det_\kappa \mathcal{O}(\rho_\kappa)} \right) , \tag{2.6}$$

where ρ_κ is the minimum of V_κ and $\det_\kappa$ is the determinant with momentum integrations modified as described above. With this definition for Γ_κ, with the normalization denominator $\det_\kappa \mathcal{O}(\rho_\kappa)$ included, the minimum of the average effective potential is zero for all scales,

$$V_\kappa(\rho_\kappa) = 0 ; \tag{2.7}$$

this is consistent with the set of parametrizations adopted in (2.1).

Now, the average effective potential at one-loop can be written as

$$V_\kappa(\rho) = \frac{1}{2} \sum_{J,P} (2J+1) \int_{|q^2|=0}^{\Lambda} \frac{d^4q}{(2\pi)^4} \ln \left(\frac{\det_\kappa a_\kappa(J^P)(\rho)}{\det_\kappa a_\kappa(J^P)(\rho_\kappa)} \right) , \tag{2.8}$$

where $(2J+1)$ stands for all the multiplicity of each spin contribution.

Given the definition for $V_\kappa(\rho)$ as above and the inverse propagator coefficients, $a_\kappa(J^P)$, stemming from the linear quadratic action (2.4), we are able to compute the β-functions (2.2) and (2.3) and analyse the scale-dependence of the parameters that characterize the effective action. The general functions are

$$\gamma(\kappa) = \frac{-1}{32\pi^2 \kappa^2 \lambda_\kappa} \int dx\, x \mathcal{R}_\gamma(P_\kappa, \rho_\kappa)\, \kappa \frac{\partial P_\kappa}{\partial \kappa} , \tag{2.9}$$

$$\beta(\kappa) = \frac{1}{32\pi^2} \int dx\, x \mathcal{R}_\beta(P_\kappa, \rho_\kappa)\, \kappa \frac{\partial P_\kappa}{\partial \kappa} ; \tag{2.10}$$

where $x = \mid q^2 \mid$ and $\mathcal{R}$ are rational functions of P_κ and ρ_κ. Notice that as a by-product of our appropriate infrared cutoff ΔS_κ introduced at a scale κ, due to the behavior of $P_\kappa(x)$ and its k-derivative, the momentum integration in the flow equations (2.9) and (2.10) are ultraviolet and infrared finite. $\kappa \frac{\partial P_\kappa}{\partial \kappa}$ receives an effective contribution at $x \approx \kappa^2$ and large momentum fluctuations $q^2 \gg \kappa^2$ are exponentially suppressed, while the κ-scale acts like a mass term in the inverse propagators $P_\kappa(q^2)$ in the infrared limit $q^2 \ll \kappa^2$. Thus, only a finite number of degrees of freedom effectively contributes to (2.9) and (2.10), as an exact evolution equation should be.

III Scalar-QED.

Let us now compute the flow equations, described in the previous section, for the case of Scalar-QED. The truncated effective action we shall work with is given by

$$\begin{aligned}\Gamma(A_\mu,\varphi^*,\varphi) &= \int d^4x \; \Big[\, V(\rho) + Z_\rho (D_\mu\varphi)^*(D^\mu\varphi) + \\ &+ \frac{1}{4} Z_F F_{\mu\nu}F^{\mu\nu} + \frac{1}{2\alpha}(\partial_\mu A^\mu)^2 \Big] \; ,\end{aligned} \tag{3.1}$$

where $D_\mu \equiv \partial_\mu + i e_o A_\mu$ and $\rho \equiv \varphi^*\varphi$. Z_ρ and Z_F are fixed wave-function renormalization constants [3] and e_o is a bare coupling constant.

Following the steps described in the previous section, the associated average effective potential is

$$V_\kappa(\rho) = \frac{1}{2}\int \frac{d^4q}{(2\pi)^4}\left[3\ln\left(\frac{a_\kappa(1^-)(\rho)}{a_\kappa(1^-)(\rho_\kappa)}\right) + \ln\left(\frac{\det_\kappa a_\kappa(0^+)(\rho)}{\det_\kappa a_\kappa(0^+)(\rho_\kappa)}\right)\right] . \tag{3.2}$$

Here, we have split the fields A_μ, φ^* and φ in classical backgrounds, that will be identified as vacuum expectation values, plus fluctuations as below:

$$\begin{aligned}\varphi^* &= \varphi^*_{cl.} + \delta\varphi^* \, , \\ \varphi &= \varphi_{cl.} + \delta\varphi \, , \\ A_\mu &= \delta A_\mu \; ; \; A^{cl.}_\mu \equiv 0 \, .\end{aligned} \tag{3.3}$$

The inverse average propagator coefficients $a_\kappa(J^P)(\rho)$ of eq. (3.2) are given by

$$a_\kappa(1^-) = \sqrt{3}\left(P_\kappa(q^2) + 2e_o^2\,\rho\right) \; , \tag{3.4}$$

$$a_\kappa(0^+) = \begin{pmatrix} \frac{1}{\alpha}P_\kappa(q^2) + 2e_o^2\rho & e_o\varphi^* \mid P_\kappa^{\frac{1}{2}}(q^2) \mid & -e_o\varphi \mid P_\kappa^{\frac{1}{2}}(q^2) \mid \\ e_o\varphi^* \mid P_\kappa^{\frac{1}{2}}(q^2) \mid & (\varphi^*)^2 V''(\rho) & P_\kappa(q^2) + V'(\rho) + \rho V''(\rho) \\ -e_o\varphi \mid P_\kappa^{\frac{1}{2}}(q^2) \mid & P_\kappa(q^2) + V'(\rho) + \rho V''(\rho) & (\varphi)^2 V''(\rho) \end{pmatrix} . \tag{3.5}$$

[3] In this work, we do not compute the running of the wave function renormalization constants Z_F and Z_ρ. It was shown in ref. [7] that for the pure scalar field case, in the ultraviolet and infrared limits, the anomalous dimension in four dimensions can be neglected; $\eta \ll 1$. We set here $Z_\rho = Z_F = 1$ because we don't expect it to give significant different results in these limits.

They can be easily read off from the linearized quadratic action in coordinate space

$$
\begin{aligned}
S^{(2)}(A_\mu;\varphi^*,\varphi) &= \int d^4x \,\Big[\delta A_\mu \Big((-\partial^2)\Theta^{\mu\nu} + \frac{1}{\alpha}(-\partial^2)\,\omega^{\mu\nu} + 2e_o^2\,\rho_{cl.}\,\delta^{\mu\nu}\Big)\,\delta A_\nu + \\
&+ 2\,\delta\varphi^* \,\big((-D^2) + V'(\rho)\,|_{\varphi_{cl.}} + \rho_{cl.}\,V''(\rho)\,|_{\varphi_{cl.}}\,\big)\,\delta\varphi + \\
&+ V''(\rho)\,|_{\varphi_{cl.}}\,\big((\delta\varphi^*)^2\varphi_{cl.}^2 + (\varphi_{cl.}^*)^2(\delta\varphi)^2\big) + \\
&+ 4e_o^2\,A^\mu_{cl.}\,\delta A_\mu\,\big(\varphi_{cl.}\,\delta\varphi^* + \varphi^*_{cl.}\,\delta\varphi\,\big) + \\
&- 2\,i\,e_o\,\delta A_\mu\,\big(\,\delta\varphi^*\partial^\mu\varphi\,|_{\varphi_{cl.}} - \varphi_{cl.}\partial^\mu\delta\varphi^* + \varphi^*_{cl.}\partial^\mu\delta\varphi - \delta\varphi\,\partial^\mu\varphi^*\,|_{\varphi_{cl.}}\,\big)\Big]\;;
\end{aligned}
\tag{3.6}
$$

after it is Fourier transformed to momentum space, the different spin-parity contributions[4] are identified, the field configurations (3.3) are chosen and q^2 is replaced by $P_\kappa(q^2)$.

We call the readers attention to the fact that we have not constrained any longitudinal mode in order to obtain manifest gauge invariant results. Terms containing $(\delta\varphi)^2$, $(\delta\varphi^*)^2$ and longitudinal vector fields do propagate in our approach due to the presence of an infrared cutoff. In this way, the gauge invariance could be achieved as the limit $\kappa \to 0$ of the modified Ward-identities as in refs. [16].

After taking derivatives of the coefficients (3.4) and (3.5), with respect to κ and ρ as in eqs. (2.2) and (2.3), we substitute into the flow equations the 'classical' parameters and the 'classical' potential V, from (3.1), by their running counterparts. In connection to what was called a 'classical' action, as an effective action at a lower scale, the procedure can be iterated at each scale κ. This updating is called the renormalization group improvement, or, sometimes, fine-tunning, and the flow of $V'_\kappa(\rho)$ and $V''_\kappa(\rho)$ with κ shall be the flow equations we are searching for.

Analytic solutions of eqs. (2.9) and (2.10) for this model in closed form are not possible to be found. However, we are able to analyse asymptotic limits and obtain analytic solutions for κ^2 very large or very small compared to ρ_κ .

In the ultraviolet limit, where $q^2 \gg \rho_\kappa$, the γ- and β-functions describing the running of the v.e.v. of ρ_κ and the running of λ_κ are dominated by powers of $P_\kappa(x)$, which is of order κ^2, since the factor $\kappa\frac{\partial P_\kappa}{\partial\kappa}$, in eqs. (2.9) and (2.10), suppressing exponentially large momentum fluctuations in comparison to κ and as a power for lower q, receives its effective contribution at $q \approx \kappa$, where it is peaked. So, in this limit we neglect ρ_κ with respect to κ^2. The leading contributions are

$$
\gamma(\kappa) = \frac{-1}{32\,\pi^2\,\kappa^2\,\lambda_\kappa}\int dx\,x\,\kappa\,\frac{\partial\,P_\kappa(x)}{\partial\,\kappa}\,\big[-6e_o^2 - 4\alpha e_o^2 - 4\lambda_\kappa\big]\;P_\kappa^{-2}(x)\,, \tag{3.7}
$$

$$
\kappa\frac{\partial\,\rho_\kappa}{\partial\kappa} = \frac{\kappa^2}{16\pi^2\lambda_\kappa}\,\big[3e_o^2 + 2\alpha e_o^2 + 2\lambda_\kappa\big]\;I_{-2}(0) \tag{3.8}
$$

[4] See eq. (2.4) and Appendix.

and

$$\beta(\kappa) = \frac{1}{32\,\pi^2}\int dx\,x\,\kappa\,\frac{\partial\,P_\kappa(x)}{\partial\,\kappa}\left[24e_o^4+32\alpha^2e_o^4+8\alpha e_o^2\lambda_\kappa+20\lambda_\kappa^2\right]\,P_\kappa^{-3}(x)\,, \quad (3.9)$$

$$\kappa\frac{\partial\,\lambda_\kappa}{\partial\kappa} = \frac{1}{8\pi^2}\left[6e_o^4+8\alpha^2e_o^4+2\alpha e_o^2\lambda_\kappa+5\lambda_\kappa^2\right]\,I_{-3}(0)\,; \quad (3.10)$$

where the integrals $I_n(w)$ are defined by

$$\kappa^{2(n+3)}\,I_n(w) = \int\, dx\,x\quad(\,P_\kappa(x)+w\,)^n\,\kappa\frac{\partial P_\kappa(x)}{\partial\kappa}\,, \quad (3.11)$$

$$I_{-2}(0) = \kappa^{-2}\int\, dx\,x\quad P_\kappa^{-2}(x)\,\kappa\frac{\partial P_\kappa(x)}{\partial\kappa} = \frac{2}{(2a)^{\frac{1}{b}}}\Gamma(1+\frac{1}{b}) \quad (3.12)$$

and

$$I_{-3}(0) = \int dx\,x\,P_\kappa^{-3}(x)\,\kappa\frac{\partial P_\kappa(x)}{\partial\kappa} = 1\,. \quad (3.13)$$

From this we see that, independently of the constants a and b, ρ_κ runs quadratically in the ultraviolet regime, as dimensional arguments suggest perturbatively, and λ_κ scales logarithmically with sublogarithmic corrections at high energies.

There is an apparent incompatibility between the running of ρ_κ and the approximation $\kappa^2 \gg \rho_\kappa$. Looking at eqs. (3.7) - (3.10), we see that ρ_κ runs in fact quadratically, which makes the approximation questionable, but λ_κ runs logarithmically with some sublogarithmic corrections. Thus, the net result is that ρ_κ grows slower than κ^2 what justifies the approximation in this case.

In the opposite limit, $\kappa^2 \ll \rho_\kappa$, the infrared one, the flow equations are dominated by powers of ρ_κ greater than powers of $P_\kappa(x) \approx \kappa^2$. The leading contributions are those with higher powers of ρ_κ. So, the γ-function is given by

$$\gamma(\kappa) = \frac{-1}{32\,\pi^2\,\kappa^2\,\lambda_\kappa}\int dx\,x\,\kappa\,\frac{\partial\,P_\kappa(x)}{\partial\,\kappa}\left[\frac{-3}{2\rho_\kappa^2e_o^2}-\frac{\lambda_\kappa}{2P_\kappa^2(x)}\right]\,, \quad (3.14)$$

$$\kappa\frac{\partial\,\rho_\kappa}{\partial\kappa} = \frac{1}{32\pi^2\lambda_\kappa}\left[\frac{3}{2\rho_\kappa^2e_o^2}\,I_0(0)\,\kappa^6+\frac{\lambda_\kappa}{2}\,I_{-2}(0)\,\kappa^2\right]\,. \quad (3.15)$$

Solving the above p.d.e. for $\lambda_0 \neq 0$ and $\rho_0 \neq 0$, we get that the first term on the r.h.s. of eq. (3.15) becomes the dominant one which damps the scaling of ρ_κ by powers of $\frac{\kappa^2}{\rho_\kappa}$ and stops its running for $\kappa \to 0$. For small enough κ, ρ_κ runs as

$$\rho_\kappa = \rho_0 \left[1 + \frac{1}{128\pi^2} \frac{I_0(0)}{\lambda_0 e_o^2} \frac{\kappa^6}{\rho_0^3} \right] , \tag{3.16}$$

where

$$I_0(0) = \kappa^{-6} \int dx \; x \; \kappa \frac{\partial P_\kappa(x)}{\partial \kappa} \tag{3.17}$$

is a constant.

The β-function in this limit is given by

$$\beta(\kappa) = \frac{1}{32\pi^2} \int dx \; x \; \kappa \frac{\partial P_\kappa(x)}{\partial \kappa} \left[\frac{3}{\rho_\kappa^3 e_o^2} + \frac{\lambda_\kappa^2}{2 P_\kappa^3(x)} \right] , \tag{3.18}$$

$$\kappa \frac{\partial \lambda_\kappa}{\partial \kappa} = \frac{1}{32\pi^2} \left[\frac{3}{\rho_\kappa^3 e_o^2} \kappa^6 I_0(0) + \frac{\lambda_\kappa^2}{2} I_{-3}(0) \right] . \tag{3.19}$$

The first term in the r.h.s. contributes with the same behavior seen in the pure scalar case [7], although it comes from a purely vector contribution (spin-1) : the running of λ_κ is damped by powers of $\frac{\kappa^2}{\rho_\kappa}$, going to zero in the limit $\kappa \to 0$. The second term contributes logarithmically to the scaling of λ_κ; showing the same net behavior found in the ultraviolet regime. Thus, when $\kappa \to 0$, the damped term decouples but the theory still correlates at long distances.

IV Turning on the Gravity interaction.

We are now going to use the techniques shown in the sections before to compute and analyse the β-function of the Scalar-QED model when gravitational effects are taken into account; i.e., we are going to correct the β-functions of the last section when the energy scale is close to, but below, the Planck's scale. To do that, we take our effective action as

$$\begin{aligned} \Gamma(A_\mu, \varphi^*, \varphi; h_{\mu\nu}) &= \int d^4x \sqrt{g} \; [\; V(\rho) + g^{\mu\nu} (D_\mu \varphi)^* (D_\nu \varphi) + \\ &+ \frac{1}{4} g^{\mu\nu} g^{\kappa\lambda} F_{\mu\kappa} F_{\nu\lambda} + \frac{1}{2\alpha} g^{\mu\nu} g^{\kappa\lambda} (\partial_\mu A_\nu)(\partial_\kappa A_\lambda) + \mathcal{K} R \Big] , \end{aligned} \tag{4.1}$$

where, as before, $D_\mu \equiv \partial_\mu + i e_o A_\mu$ and $\rho \equiv \varphi^* \varphi$. $\mathcal{K} = \frac{1}{16\pi G}$, $g = det \, g_{\mu\nu}$, $R = g^{\mu\nu} R_{\gamma\mu}{}^{\gamma}{}_{\nu}$ is the scalar curvature and $R_{\mu\nu}{}^{\rho}{}_{\sigma} = \partial_\mu \Gamma_\nu{}^\rho{}_\sigma - \partial_\nu \Gamma_\mu{}^\rho{}_\sigma + \Gamma_\mu{}^\rho{}_\gamma \Gamma_\nu{}^\gamma{}_\sigma - \Gamma_\nu{}^\rho{}_\gamma \Gamma_\mu{}^\gamma{}_\sigma$ is the curvature tensor, where the connection is defined as usual; $\Gamma_\nu{}^\rho{}_\sigma = \frac{1}{2} g^{\rho\tau} (\partial_\nu g_{\tau\sigma} + \partial_\sigma g_{\nu\tau} - \partial_\tau g_{\sigma\nu})$.

Hereafter, we shall recognize the metric field $g_{\mu\nu}$ as a fluctuation around the Euclidean (flat) space geometry,

$$g_{\mu\nu}(x) = \delta_{\mu\nu} + \frac{1}{\sqrt{\mathcal{K}}}\, h_{\mu\nu}(x)\,, \tag{4.2}$$

and take $h_{\mu\nu}(x)$ as the field variable of the gravitational interaction. Also, we shall expand the fields A_μ, φ^* and φ as in (3.3).

The linearized quadratic action is then written as

$$\begin{aligned}
S^{(2)}(A_\mu;\varphi^*,\varphi;h_{\mu\nu}) \;=\; & \int d^4x \Big[\delta A_\mu \Big(\delta^{\mu\nu}\,\delta^{\kappa\lambda}(-\partial^2\Theta^{\mu\nu}) + \frac{1}{\alpha}\delta^{\mu\nu}\,\delta^{\kappa\lambda}(-\partial^2\,\omega_{\nu\kappa}) + \\
+ \; & 2e_o^2\,\rho_{cl.}\,\delta^{\mu\lambda}\Big)\,\delta A_\lambda \,+ \\
+ \; & 2\,\delta\varphi^*\,\big(-\delta^{\mu\nu}\,\partial_\mu\,\partial_\nu + V'(\rho)\,|_{\rho_{cl.}} + \rho_{cl.}\,V''(\rho)\,|_{\rho_{cl.}}\big)\,\delta\varphi \,+ \\
+ \; & V''(\rho)\,|_{\rho_{cl.}}\,\big((\delta\varphi^*)^2\varphi_{cl.}^2 + (\varphi_{cl.}^*)^2(\delta\varphi)^2\big) \,+ \\
- \; & i\,2\,e_o\,\delta A_\mu\,\delta^{\mu\nu}\,\big(\varphi_{cl.}^*\,\partial_\nu\,\delta\varphi - \varphi_{cl.}\,\partial_\nu\,\delta\varphi^*\big) \,+ \\
+ \; & \frac{1}{\sqrt{\mathcal{K}}}\,h^{\mu\nu}\,\left(\frac{1}{2}\,\delta_{\mu\nu}\,V'(\rho)\,|_{\rho_{cl.}}\;\varphi_{cl.}^*\right)\delta\varphi \,+ \\
+ \; & \frac{1}{\sqrt{\mathcal{K}}}\,\delta\varphi^*\,\left(\frac{1}{2}\,\delta_{\mu\nu}\,V'(\rho)\,|_{\rho_{cl.}}\;\varphi_{cl.}\right)h^{\mu\nu} \,+ \\
+ \; & \frac{1}{\mathcal{K}}\,h^{\mu\nu}\,\left(\frac{-1}{4}\,\delta_{\mu\kappa}\delta_{\nu\lambda}\,V(\rho_{cl.}) + \frac{1}{8}\,\delta_{\mu\nu}\delta_{\kappa\lambda}\,V(\rho_{cl.})\right)h^{\kappa\lambda} \,+ \\
+ \; & \frac{1}{2}\,h_{\mu\nu}\,\Big(\delta^{\mu\nu}\,\partial^\rho\,\partial^\sigma - \frac{1}{2}\,\delta^{\mu\nu}\delta^{\rho\sigma}\,\partial^2 - \delta^{\nu\sigma}\,\partial^\mu\,\partial^\rho \,+ \\
+ \; & \frac{1}{2}\,\delta^{\mu\rho}\,\delta^{\nu\sigma}\,\partial^2\Big)\,h_{\rho\sigma} + \frac{1}{2\,\xi}\;\partial_\mu\,h^{\mu\nu}\,\partial^\sigma\,h_{\sigma\nu}\;\Big]\,,
\end{aligned} \tag{4.3}$$

where we have fixed the gauge symmetry

$$\delta h_{\mu\nu}(x) = \partial_\mu\zeta_\nu(x) + \partial_\nu\zeta_\mu(x)\,, \tag{4.4}$$

with $\zeta_\nu(x)$ being an infinitesimal coordinate transformation, by adding the gravitational gauge-fixing sector

$$S_{_{G.F.}} = \frac{1}{2\,\xi}\int d^4x\,\partial_\mu\,h^{\mu\nu}\,\partial^\sigma\,h_{\sigma\nu}\,. \tag{4.5}$$

The Faddeev-Popov ghosts decouple from $h_{\mu\nu}$ in this gauge and shall not be considered in what follows.

The linearized quadratic action (4.3) can be rewritten in compact form spanning all its elements on the corresponding spin-projection operators as in eq. (2.4). The coefficient matrices $a^{AB}(J^P)$ are

$$
\begin{aligned}
a^{hh}(2^+) &= \frac{-1}{4}q^2 - \frac{1}{4\mathcal{K}}\,V(\rho)\,, && (4.6a)\\
a^{hh}(1^-) &= \frac{1}{4\,\xi}q^2 - \frac{1}{4\mathcal{K}}\,V(\rho)\,, && (4.6b)\\
a^{hh}_{s}(0^+) &= \frac{1}{2}q^2 + \frac{1}{8\mathcal{K}}\,V(\rho)\,, && (4.6c)\\
a^{hh}_{w}(0^+) &= \frac{1}{2\,\xi}q^2 - \frac{1}{8\mathcal{K}}\,V(\rho)\,, && (4.6d)\\
a^{hh}_{sw}(0^+) &= \frac{\sqrt{3}}{8\mathcal{K}}\,V(\rho) \;=\; a^{hh}_{ws}(0^+)\,, && (4.6e)\\
a^{h\varphi}_{s1}(0^+) &= \frac{\sqrt{3}}{4\sqrt{\mathcal{K}}}\,V'(\rho)\,\varphi^* \;=\; a^{\varphi h}_{1s}(0^+)\,, && (4.6f)\\
a^{h\varphi^*}_{s1}(0^+) &= \frac{\sqrt{3}}{4\sqrt{\mathcal{K}}}\,V'(\rho)\,\varphi \;=\; a^{\varphi^* h}_{1s}(0^+)\,, && (4.6g)\\
a^{h\varphi}_{w1}(0^+) &= \frac{1}{4\sqrt{\mathcal{K}}}\,V'(\rho)\,\varphi^* \;=\; a^{\varphi h}_{1w}(0^+)\,, && (4.6h)\\
a^{h\varphi^*}_{w1}(0^+) &= \frac{1}{4\sqrt{\mathcal{K}}}\,V'(\rho)\,\varphi \;=\; a^{\varphi^* h}_{1w}(0^+)\,, && (4.6i)\\
a^{AA}_{s}(1^-) &= \sqrt{3}\,(q^2 + 2\,e_o^2\,\rho)\,, && (4.6j)\\
a^{AA}_{w}(0^+) &= \frac{1}{\alpha}\,q^2 + 2\,e_o^2\,\rho\,, && (4.6k)\\
a^{A\varphi}_{w1}(0^+) &= e_o\,\varphi^*\;\mid q\mid = \; a^{\varphi A}_{1w}(0^+)\,, && (4.6l)\\
a^{A\varphi^*}_{w1}(0^+) &= -e_o\,\varphi\;\mid q\mid = \; a^{\varphi^* A}_{1w}(0^+)\,, && (4.6m)\\
a^{\varphi\varphi}_{11}(0^+) &= V''(\rho)(\varphi^*)^2\,, && (4.6n)\\
a^{\varphi^*\varphi}_{11}(0^+) &= q^2 + V'(\rho) + \rho V''(\rho) \;=\; a^{\varphi\varphi^*}_{11}(0^+)\,, && (4.6o)\\
a^{\varphi^*\varphi^*}_{11}(0^+) &= V''(\rho)\varphi^2\,. && (4.6p)
\end{aligned}
$$

In matrix form, we can write

$$
a(0^+) \;=\; \begin{pmatrix}
a^{hh}_{s}(0^+) & a^{hh}_{sw}(0^+) & 0 & a^{h\varphi}_{s1}(0^+) & a^{h\varphi^*}_{s1}(0^+)\\
a^{hh}_{ws}(0^+) & a^{hh}_{w}(0^+) & 0 & a^{h\varphi}_{w1}(0^+) & a^{h\varphi^*}_{w1}(0^+)\\
0 & 0 & a^{AA}_{w}(0^+) & a^{A\varphi}_{w1}(0^+) & a^{A\varphi^*}_{w1}(0^+)\\
a^{\varphi h}_{1s}(0^+) & a^{\varphi h}_{1w}(0^+) & a^{\varphi A}_{1w}(0^+) & a^{\varphi\varphi}_{11}(0^+) & a^{\varphi\varphi^*}_{11}(0^+)\\
a^{\varphi^* h}_{1s}(0^+) & a^{\varphi^* h}_{1w}(0^+) & a^{\varphi^* A}_{1w}(0^+) & a^{\varphi^*\varphi}_{11}(0^+) & a^{\varphi^*\varphi^*}_{11}(0^+)
\end{pmatrix}, \qquad (4.7)
$$

$$a(1^-) \;=\; \begin{pmatrix} a^{hh}(1^-) & 0 \\ 0 & a_s^{AA}(1^-) \end{pmatrix} \,, \tag{4.8}$$

$$a(2^+) \;=\; a^{hh}(2^+) \,. \tag{4.9}$$

To each element $a^{AB}(J^P)$, there is a spin-projector $P^{AB}(J^P)$. The complete set of projectors for this model is cast in the appendix. The lower labels s, w, and 1 stands for transversal, longitudinal and pure-scalar contributions.

Again, the associated average effective potential reads as

$$\begin{aligned} V_\kappa(\rho) \;&=\; \frac{1}{2}\int \frac{d^4q}{(2\pi)^4}\left[5\cdot \ln\left(\frac{a_\kappa(2^+)(\rho)}{a_\kappa(2^+)(\rho_\kappa)}\right) + 3\cdot \ln\left(\frac{\det_\kappa a_\kappa(1^-)(\rho)}{\det_\kappa a_\kappa(1^-)(\rho_\kappa)}\right) + \right. \\ &\left. + \;\ln\left(\frac{\det_\kappa a_\kappa(0^+)(\rho)}{\det_\kappa a_\kappa(0^+)(\rho_\kappa)}\right)\right] \,. \end{aligned} \tag{4.10}$$

Given the set of parametrizations (2.1) and the average potential above, the flow of ρ_κ and λ_κ are given by

$$\begin{aligned} \gamma(\kappa) \;&=\; \frac{-1}{32\,\pi^2\,\kappa^2\,\lambda_\kappa}\int dx\; x\;\kappa\,\frac{\partial\, P_\kappa(x)}{\partial\,\kappa}\left[5\,\cdot\,\frac{\partial}{\partial\, P_\kappa(x)}\,\frac{\partial}{\partial\,\rho}\;\ln\left(\frac{a_\kappa(2^+)(\rho)}{a_\kappa(2^+)(\rho_\kappa)}\right)_{\rho\,=\,\rho_\kappa} + \right. \\ &+\; 3\,\cdot\,\frac{\partial}{\partial\, P_\kappa(x)}\,\frac{\partial}{\partial\,\rho}\;\ln\left(\frac{\det_\kappa a_\kappa(1^-)(\rho)}{\det_\kappa a_\kappa(1^-)(\rho_\kappa)}\right)_{\rho\,=\,\rho_\kappa} + \\ &\left. +\;\frac{\partial}{\partial\, P_\kappa(x)}\,\frac{\partial}{\partial\,\rho}\;\ln\left(\frac{\det_\kappa a_\kappa(0^+)(\rho)}{\det_\kappa a_\kappa(0^+)(\rho_\kappa)}\right)_{\rho\,=\,\rho_\kappa}\right] \end{aligned} \tag{4.11}$$

and

$$\begin{aligned} \beta(\kappa) \;&=\; \frac{1}{32\,\pi^2}\int dx\; x\;\kappa\,\frac{\partial\, P_\kappa(x)}{\partial\,\kappa}\left[5\,\cdot\,\frac{\partial}{\partial\, P_\kappa(x)}\,\frac{\partial^2}{\partial\,\rho^2}\;\ln\left(\frac{a_\kappa(2^+)(\rho)}{a_\kappa(2^+)(\rho_\kappa)}\right)_{\rho\,=\,\rho_\kappa} + \right. \\ &+\; 3\,\cdot\,\frac{\partial}{\partial\, P_\kappa(x)}\,\frac{\partial^2}{\partial\,\rho^2}\;\ln\left(\frac{\det_\kappa a_\kappa(1^-)(\rho)}{\det_\kappa a_\kappa(1^-)(\rho_\kappa)}\right)_{\rho\,=\,\rho_\kappa} + \\ &\left. +\;\frac{\partial}{\partial\, P_\kappa(x)}\,\frac{\partial^2}{\partial\,\rho^2}\;\ln\left(\frac{\det_\kappa a_\kappa(0^+)(\rho)}{\det_\kappa a_\kappa(0^+)(\rho_\kappa)}\right)_{\rho\,=\,\rho_\kappa}\right] \,. \end{aligned} \tag{4.12}$$

In the ultraviolet limit, $\kappa^2 \gg \rho_\kappa$, where again $P_\kappa(x) \approx \kappa^2$, we collect the terms with highest power of $P_\kappa(x)$ and neglect ρ_κ with respect to κ^2. Thus, we have for the γ-function

$$\kappa \frac{\partial \rho_\kappa}{\partial \kappa} \;=\; \kappa^2\, \gamma(\kappa) \;=\; \frac{\kappa^2}{16\pi^2 \lambda_\kappa} \left[3e_o^2 + 2\alpha e_o^2 + 2\lambda_\kappa \right] I_{-2}(0)\;. \tag{4.13}$$

This result is the same as in the case without gravity coupling; ρ_κ runs quadratically.

In the infrared limit, $\kappa^2 \ll \rho_\kappa$, we get

$$\kappa \frac{\partial \rho_\kappa}{\partial \kappa} \;=\; \kappa^2\, \gamma(\kappa) \;=\; \frac{1}{32\pi^2 \lambda_\kappa} \left[\frac{3}{2\rho_\kappa^2 e_o^2} I_0(0)\, \kappa^6 + \frac{\lambda_\kappa}{2} I_{-2}(0)\, \kappa^2 \right]. \tag{4.14}$$

The first term in the r.h.s. damps, when $\kappa \to 0$, the evolution of ρ_κ by powers of $\frac{\kappa^2}{\rho_\kappa}$. For $\rho_0 \neq 0$ and $\lambda_0 \neq 0$, we have

$$\rho_\kappa \;=\; \rho_0 \left[1 + \frac{1}{128\,\pi^2} \frac{I_0(0)}{\lambda_0\, e_o^2} \frac{\kappa^6}{\rho_0^3} \right]. \tag{4.15}$$

This exactly agrees with the case without gravity.

Now, the β-function, in the ultraviolet limit is given by

$$\begin{aligned} \beta(\kappa) \;=\; & \frac{1}{32\,\pi^2} \int dx\, x\, \kappa \frac{\partial P_\kappa(x)}{\partial \kappa} \left[5 \cdot \left(\frac{-\lambda_\kappa}{\mathcal{K}\, P_\kappa^2(x)} \right) + \right. \\ & + \; 3 \cdot \left(\frac{\lambda_\kappa\, \xi}{\mathcal{K}\, P_\kappa^2(x)} + \frac{8\, e_o^4}{P_\kappa^3(x)} \right) + \left(\frac{\lambda_\kappa\,(\xi-1)}{4\,\mathcal{K}\, P_\kappa^2(x)} + \frac{32\,\alpha^2\, e_o^4}{P_\kappa^3} + \right. \\ & + \left. \left. \frac{8\,\alpha\, e_o^2\, \lambda_\kappa}{P_\kappa^3} + \frac{20\,\lambda_\kappa^2}{P_\kappa^3} + \frac{3\,\alpha\, e_o^2\, \lambda_\kappa\, \rho_\kappa\,(\xi-1)}{\mathcal{K}\, P_\kappa^3} + \frac{5\,\lambda_\kappa^2\, \rho_\kappa\, \xi}{2\,\mathcal{K}\, P_\kappa^3} \right) \right], \end{aligned} \tag{4.16}$$

$$\begin{aligned} \kappa \frac{\partial \lambda_\kappa}{\partial \kappa} \;=\; & \frac{1}{32\,\pi^2} \left[\left(24\, e_o^4 + 32\,\alpha^2\, e_o^4 + 8\,\alpha\, e_o^2\, \lambda_\kappa + 20\,\lambda_\kappa^2 \right) I_{-3}(0) + \right. \\ & + \left(-5 + 3\,\xi + \frac{(\xi-1)}{4} \right) \frac{\lambda_\kappa\, \kappa^2}{\mathcal{K}} I_{-2}(0) \; + \\ & + \left. \left(\frac{3\,\alpha\, e_o^2\, \lambda_\kappa\, \rho_\kappa\,(\xi-1)}{\mathcal{K}} \right) I_{-3}(0) + \left(\frac{5\,\lambda_\kappa^2\, \rho_\kappa\, \xi}{2\,\mathcal{K}} \right) I_{-3}(0) \right]. \end{aligned} \tag{4.17}$$

As $\kappa^2 \gg \rho_\kappa$, the last two terms do not contribute in this limit. Considering $\kappa^2 \ll \mathcal{K}$, the terms proportional to $\frac{\kappa^2}{\mathcal{K}} I_{-2}(0)$ are not relevant, but when the energy scale goes near, but below, the Planck's mass, $\kappa^2 \approx \mathcal{K}$, the λ_κ coupling begins to run much faster than logarithmically and some care must be taken, as the validity of Einstein theory near this energy scale becomes questionable.

In the infrared limit, one gets for the β-function

$$\begin{aligned}\beta(\kappa) &= \frac{1}{32\,\pi^2} \int dx\, x\, \kappa \frac{\partial P_\kappa(x)}{\partial \kappa} \left[5 \cdot \left(\frac{-\lambda_\kappa}{\mathcal{K}\, P_\kappa^2(x)} \right) + \right. \\ &\left. + \; 3 \cdot \left(\frac{\lambda_\kappa\, \xi}{\mathcal{K}\, P_\kappa^2(x)} + \frac{1}{e_o^2\, \rho_\kappa^3} \right) + \left(\frac{\lambda_\kappa\,(\xi+1)}{2\,\mathcal{K}\, P_\kappa^2(x)} + \frac{\lambda_\kappa^2}{2\, P_\kappa^3} \right) \right] ,\end{aligned} \tag{4.18}$$

$$\kappa \frac{\partial \lambda_\kappa}{\partial \kappa} = \frac{1}{32\,\pi^2} \left[\frac{(-9+7\,\xi)}{2} \lambda_\kappa \frac{\kappa^2}{\mathcal{K}} I_{-2}(0) + \frac{3}{e_o^2} \frac{\kappa^6}{\rho_\kappa^3} I_0(0) + \frac{\lambda_\kappa^2}{2} I_{-3}(0) \right] . \tag{4.19}$$

The first term in the r.h.s. becomes relevant when the scale is close to the Planck's threshold where the Einstein's action is limited to. The other two terms are the same as those coming from the Scalar-QED model when gravity corrections are not considered.

V Concluding Remarks.

The first conclusion we can draw from the analysis of these flow equations is that the running of the v.e.v. of ρ_κ is not modified when the gravitational contribution is taken into account. Eqs. (4.13), (4.14) and (4.15) are exactly the same as eqs. (3.8), (3.15) and (3.16), respectively. We have promoted the metric field $g_{\mu\nu}$ to a quantum field without letting the Planck's constant to run; i.e., we took only the bilinear sector of Einstein's action and defined the effective action over flat background. In this way, the vacuum structure of the theory was not modified by gravity couplings; the equivalence principle applies.

For the β-functions, we can observe that eqs. (4.17) and (4.19) contain the same terms found in eqs. (3.10) and (3.19) plus corrections when the mass scale is increased. At $\mathcal{K} \gg \kappa^2 \gg \rho_\kappa$, the flow of λ_κ is given by the first term in r.h.s. of eq. (4.17). It is the same result found in eq. (3.10). For $\mathcal{K} \approx \kappa^2 \gg \rho_\kappa$, the scale is of order the Planck's mass and the $2^{nd.}$ term of (4.17) becomes relevant as a correction induced by gravity. So, the β-function reads

$$\begin{aligned}\kappa \frac{\partial \lambda_\kappa}{\partial \kappa} &= \frac{1}{32\,\pi^2} \left[\left(24\, e_o^4 + 32\,\alpha^2\, e_o^4 + 8\,\alpha\, e_o^2\, \lambda_\kappa + 20\,\lambda_\kappa^2 \right) I_{-3}(0) + \right. \\ &\left. + \left(-5 + 3\,\xi + \frac{(\xi-1)}{4} \right) \frac{\lambda_\kappa\, \kappa^2}{\mathcal{K}} I_{-2}(0) \right] .\end{aligned} \tag{5.1}$$

The $3^{rd.}$ and $4^{th.}$ terms of eq. (4.17) would become relevant, compared to the $2^{nd.}$ one, if $\mathcal{K} \approx \kappa^2 \approx \rho_\kappa$ or $\mathcal{K} \approx \rho_\kappa \gg \kappa^2$; but these possibilities would violate the basic assumption of a β-function in the ultraviolet limit $\kappa^2 \gg \rho_\kappa$. So, these last two terms can be neglected. Yet, the terms of correction by Einstein's gravity should be looked upon with some care, as the theory is questionable at this energy scale if one sticks to gravity as an effective field theory.

On the opposite range, $\rho_\kappa \gg \kappa^2$, the infrared one, the first term of eq. (4.19) appears as a correction by gravity and the last two are the same found in eq. (3.19). Comparing the behavior of the first two terms, one finds two different situations. If we set up the regime $\rho_\kappa^{3/2}\mathcal{K}^{-1/2} \ll \kappa^2 \ll \rho_\kappa$, only the $3^{rd.}$ term of eq. (4.19) contributes in the deep infrared limit. If $\kappa^2 \ll \rho_\kappa^{3/2}\mathcal{K}^{-1/2} \ll \rho_\kappa$, the $1^{st.}$ term corrects the case without gravity, but our threshold region is limited to $\mathcal{K}$ as $\kappa^2 \ll \rho_\kappa \ll \mathcal{K}$. So, effectively, only the $3^{rd.}$ term contributes to the running of λ_κ. Thus, the behavior of λ_κ in the infrared, with and without gravity, is kept the same:

$$\lambda_\kappa = \frac{\lambda_0}{1 - \frac{\lambda_0 I_{-3}(0)}{64\pi^2} \ln\left(\frac{\kappa}{\kappa_o}\right)} \,. \qquad (5.2)$$

The self-interacting coupling λ_κ scales slowly to zero in the deep infrared and the theory still correlates at long distances as the masses of the gauge particles are suppressed with ρ_κ.

We have computed the γ- and β-functions of the Scalar-QED corrected by Einstein's Gravity. The v.e.v. of ρ_κ was found to run quadratically at high energies and to suppress it's running at sufficient low energies. λ_κ scales faster than logarithmically in the ultraviolet in the presence of the gravitational coupling and logarithmically in the infrared due to the presence of massless particles in this limit.

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Appendix:

Below, we list all the spin-projection operators found in this work.

$$
\begin{aligned}
P^{hh}(2^+)_{\mu\nu,\kappa\lambda} &= \frac{1}{2}\left(\Theta_{\mu\rho}\Theta_{\nu\lambda}+\Theta_{\mu\lambda}\Theta_{\nu\kappa}\right)-\frac{1}{3}\Theta_{\mu\nu}\Theta_{\kappa\lambda}\ , \\
P^{hh}(1^-)_{\mu\nu,\kappa\lambda} &= \frac{1}{2}(\Theta_{\mu\kappa}\omega_{\nu\lambda}+\Theta_{\mu\lambda}\omega_{\nu\kappa}+\Theta_{\nu\kappa}\omega_{\mu\lambda}+\Theta_{\nu\lambda}\omega_{\mu\kappa})\ , \\
P^{hh}_{ss}(0^+)_{\mu\nu,\kappa\lambda} &= \frac{1}{3}\Theta_{\mu\nu}\Theta_{\kappa\lambda}\ , \\
P^{hh}_{ww}(0^+)_{\mu\nu,\kappa\lambda} &= \omega_{\mu\nu}\omega_{\kappa\lambda}\ , \\
P^{hh}_{sw}(0^+)_{\mu\nu,\kappa\lambda} &= \frac{1}{\sqrt{3}}\Theta_{\mu\nu}\omega_{\kappa\lambda}\ , \\
P^{hh}_{ws}(0^+)_{\mu\nu,\kappa\lambda} &= \frac{1}{\sqrt{3}}\omega_{\mu\nu}\Theta_{\kappa\lambda}\ , \\
P^{h\varphi}_{s1}(0^+)^{\mu\nu} &= \frac{1}{\sqrt{3}}\,\Theta^{\mu\nu}\ , \\
P^{\varphi h}_{1s}(0^+)^{\mu\nu} &= \frac{1}{\sqrt{3}}\,\Theta^{\mu\nu}\ , \\
P^{h\varphi^*}_{s1}(0^+)^{\mu\nu} &= \frac{1}{\sqrt{3}}\,\Theta^{\mu\nu}\ , \\
P^{\varphi^* h}_{1s}(0^+)^{\mu\nu} &= \frac{1}{\sqrt{3}}\,\Theta^{\mu\nu}\ , \\
P^{h\varphi}_{w1}(0^+)^{\mu\nu} &= \omega^{\mu\nu}\ , \\
P^{\varphi h}_{1w}(0^+)^{\mu\nu} &= \omega^{\mu\nu}\ , \\
P^{h\varphi^*}_{w1}(0^+)^{\mu\nu} &= \omega^{\mu\nu}\ , \\
P^{\varphi^* h}_{1w}(0^+)^{\mu\nu} &= \omega^{\mu\nu}\ , \\
P^{AA}_{s}(1^-)^{\mu\nu} &= \frac{1}{\sqrt{3}}\,\Theta^{\mu\nu}\ , \\
P^{AA}_{w}(0^+)^{\mu\nu} &= \omega^{\mu\nu}\ , \\
P^{A\varphi}_{w1}(0^+)^{\mu\,\cdot} &= \omega^{\mu\nu}\,\hat{\mathrm{q}}_\nu\ , \\
P^{\varphi A}_{1w}(0^+)^{\mu\,\cdot} &= \omega^{\mu\nu}\,\hat{\mathrm{q}}_\nu\ , \\
P^{A\varphi^*}_{w1}(0^+)^{\mu\,\cdot} &= \omega^{\mu\nu}\,\hat{\mathrm{q}}_\nu\ , \\
P^{\varphi^* A}_{1w}(0^+)^{\mu\,\cdot} &= \omega^{\mu\nu}\,\hat{\mathrm{q}}_\nu\ , \\
P^{\varphi\varphi}_{11}(0^+) &= 1\ , \\
P^{\varphi\varphi^*}_{11}(0^+) &= 1\ , \\
P^{\varphi^*\varphi}_{11}(0^+) &= 1\ , \\
P^{\varphi^*\varphi^*}_{11}(0^+) &= 1\ .
\end{aligned}
$$

The operators $\Theta^{\mu\nu}$ and $\omega^{\mu\nu}$ stand for the usual transverse and longitudinal projectors on the space of vectors,

$$\hat{\mathrm{q}}^{\mu} = \frac{\mathrm{q}^{\mu}}{\sqrt{\mathrm{q}^2}} \,, \quad \delta^{\mu\nu} = \Theta^{\mu\nu} + \omega^{\mu\nu} \;\; \text{and} \;\; \omega^{\mu\nu} = \hat{\mathrm{q}}^{\mu}\hat{\mathrm{q}}^{\nu} \,,$$

and are identified by the lower labels s and w, as in the Barnes and Rivers notation [14]. The lower index 1, label the unity contribution from the scalar fields to the spin operators.

References

[1] G. t'Hooft and M. Veltman, Ann. Inst. Henri Poicarè 20 (1974) 69; M Veltman, in *Methods in Field Theory* Proc. of the Les Houches Summer School, ed. by R. Balian and J. Zinn-Justin (North Holland, Amsterdam, 1976.

[2] M. Goroff and A. Sagnotti, *Nucl. Phys.* **B 266** (1986) 799.

[3] S. Deser and P. van Nieuwenhuizen, *Phys. Rev.* **D 10**, (1974) 401; K.S. Stelle, *Phys. Rev.* **D 16**, (1977) 953.

[4] I.L. Buchbinder, S.D. Odintsov and I.L. Shapiro, *Effective Action in Quantum Gravity*, (IOP, Bristol (UK), 1992).

[5] P. van Nieuwenhuizen, *Nucl. Phys.* **B 60** (1973) 478; E. Sezgin and P. van Nieuwenhuizen, *Phys. Rev.* **D 21**, (1980) 3269; E. Sezgin, *Phys. Rev.* **D 24**, (1981) 1677.

[6] John F. Donoghue, *Phys. Rev.* **D 50**, (1994) 3874.

[7] R. Floreanini and R. Percacci, *Phys. Rev.* **D 52** (1995) 896; L. Griguolo and R. Percacci, *Phys. Rev.* **D 52** (1995) 5787.

[8] K.G. Wilson and J.B. Kogut, *Phys. Rep.* **12 C** (1974) 75; K.G. Wilson, *Rev. Mod. Phys.* **47** (1975).

[9] J. Polchinski, *Nucl. Phys.* **B 231** (1984) 269.

[10] A. Ringwald and C. Wetterich, *Nucl. Phys.* **B 334** (1990) 506; C. Wetterich, *Nucl. Phys.* **B 352** (1991) 529; *Z. Phys* **C 57** (1993) 451; ibid **C60** (1993) 461; N. Tetradis and C. Wetterich, *Nucl. Phys.* **B 422** (1994) 541.

[11] C. Wetterich, *Phys. Lett.* **B 301**; (1993) 90.

[12] M. Reuter and C. Wetterich, *Nucl. Phys.* **B 408** (1993) 91; *Nucl. Phys.* **B 417** (1994) 181.

[13] S. Coleman and E. Weinberg, *Phys. Rev.* **D 7** (1973) 1888.

[14] K. J. Barnes, *Ph.D. Thesis* (1963) unpublished; R. J. Rivers, *Nuovo Cim.* **34** (1964) 387.

[15] C. Pinheiro and G. O. Pires, *Phys. Lett.* **B 301** (1993) 339; C. Pinheiro, G. O. Pires and N. Tomimura, *Il Nuovo Cim.* **111 B** (1996) 1023; Carlos Pinheiro, Gentil O. Pires and C. Sasaki, *Gen. Rel. Grav.* **29** (1997) 409.

[16] C. Becchi, "On the Construction of Renormalized Quantum Field Theory Using Renormalization Group Techniques", in *Elementary Particles, Field Theory and Statistical Mechanics*, eds. M. Bonini, G. Marchesini and E. Onofri, *Parma University* (1993); U. Ellwanger, *Phys. Lett.* **B 335** (1994) 364; M. Bonini, M. D'Attanasio and G. Marchesini, *Nucl. Phys.* **B 437** (1995) 163 and *Phys. Lett.* **B 346** (1995) 87; Ulrich Ellwanger, Manfred Hirch and Axel Weber, *Z. Phys.* **C 69** (1996) 687.