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The Physics of the Sagnac-Mashoon Effects

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Abstract

We carry out a complete examination of the effects of rotation on the physics of spin-1/2 particles. The origin of these effects is connected to the fact that the frame of the rotating experiment is Fermi-Walker transported, and is related to the inertia rest frame (with respect to which the apparatus rotates with angular velocity $\vec{\omega}$) by a Lorentz boost with instantaneous velocity $\vec{\omega} \wedge \vec{R}$ plus a time dependent spatial rotation. Two distinct sets of effects are obtained. The first depends on the spin-rotation interaction, and consists of the Mashoon effect and a mass split effect due to the Lorentz boost mentioned above. In a neutron interferometer the latter effect produces a phase shift smaller than the Mashoon phase shift by a factor of order $0(v^2/c^2)$, where v is the velocity of the particles in the beams of the interferometer. The detection of the Mashoon effect is crucial to decide if free spin-1/2 particles actually behave a gyroscopes. The second set appears already in the eikonal approximation and is due to the dragging of the particles by the rotating apparatus, boosting the four momentum of the particles; it results in the Sagnac effect. We also discuss a criterion to fix the frame of the experiment among all mathematically admissible frames in the Brill-Wheeler formulation, and show that frames used in the literature to derive the Sagnac-Mashoon effect cannot be physically realized.

Key-words: Spin-Rotation Coupling; Sagnac Effect; Mashoon Effect.

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1 Introduction

In the realm of Special Relativity, the laws of Physics are basically formulated on inertial frames. However in many situations the description of a physical system may require the use of non-inertial frames, as is the case of experiments in laboratories rotating with the earth. The rotation of the apparatus may have a measurable effect on experiments, like the Sagnac effect originally predicted and measured for light [1], and more recently measured for neutrons in experiments of neutron interferometry [2]. Concerning the Sagnac effect the established fact is that the rotation of the experimental apparatus relative to the inertia rest frame [3] produces a phase-shift in the interference pattern proportional to $\vec{\omega} \cdot \vec{A}$, where $\vec{\omega}$ is the angular velocity of the apparatus and $\vec{A}$ is the oriented area enclosed by the interfering beams (this is correct to the first order of ω/v, where v is the velocity of the neutron or photon). A few years ago Mashoon [4] suggested the existence of a new effect for spin-1/2 particles – the spin-rotation coupling described by the interaction term $-\vec{\omega} \cdot \vec{S}$, where $\vec{S}$ is the spin angular momentum operator of the particle, adding a small contribution to the Sagnac effect in neutron interferometry experiments.

With the exception of the paper by Dresden and Yang [5] (where a derivation of the Sagnac effect is made) all derivations of the Sagnac-Mashoon effect are incorrect or incomplete. The origin of this problem lies in the use of non-physical frames to describe the motion of the experimental apparatus. We also show that two sets of effects are to be distinguished in experiments with the rotating apparatus. The first is a typical wave effect, related to the fact that the frame of the rotating experiment is non-inertial while the second set is due to the dragging of the particles of the experiment by the rotating apparatus. The origin of the effects in both cases is due to *active* Lorentz transformations realized on the system, taking it from rest to a state of uniform rotation about an axis.

In the following we adopt a procedure analogous to Dresden and Yang [5] in their derivation of the Sagnac effect, *staying throughout in the inertia rest frame.* For simplicity we consider an experimental apparatus of interferometry with cylindrical symmetry. The interfering beams have the semi-circular paths as depicted in Fig. 1. The beam is split at S, the semi circular trajectories being produced by a large number of mirrors/cristals placed along the path C. Between two successive mirrors/cristals the particles of the beams are assumed to be free.

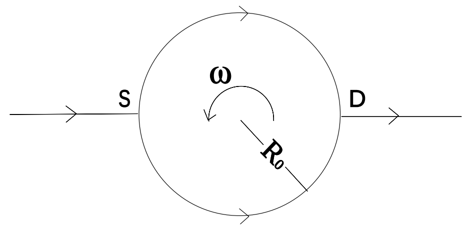

Fig. 1

Fig. 1: The rotating interferometry apparatus. D is the point of recombination of the beams. The incident beam is polarized parallel to the rotation axis.

We introduce inertial cylindrical coordinates (T, R, ϕ, Z). The apparatus rotates about the Z-axis with angular velocity ω as observed from the inertia rest frame. The coordinates of each cristal/mirror of the apparatus are given by $(T, R, \phi + \omega T, Z)$ with corresponding four velocity

$$u_{(0)} = \gamma(\partial_T + \omega\partial_\phi) \tag{1}$$

where $\gamma = (1 - \omega^2 R^2)^{-1/2}$. To complete the description of the apparatus we introduce three spacelike vector fields, as prescribed by Irvine [6],

$$\begin{aligned} u_{(1)} &= \partial_R \\ u_{(2)} &= \gamma(\omega R\partial_T + \frac{1}{R}\,\partial_\phi) \\ u_{(3)} &= \partial_Z \end{aligned} \tag{2}$$

connected to the inertia rest frame by the Lorentz boost

$$L_A^{\ B} = \begin{pmatrix} \gamma & 0 & \gamma\omega R & 0 \\ 0 & 1 & 0 & 0 \\ \gamma\omega R & 0 & \gamma & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \quad A, B = 0, \cdots, 3 \tag{3}$$

A general prescription for describing the dynamics of spin-1/2 particles in non-inertial frames is given by the Brill-Wheeler formulation [7]. The equation of motion for the spin-1/2 Dirac particles in the rotating frame (1)-(2) is then given by [7, 8]

$$\left[i\gamma(\partial_T + \omega\partial_\phi) + i\gamma^0\gamma^1\left(\partial_R + \tfrac{1}{2R}\right) + i\gamma^0\gamma^2\left(\gamma\omega R\partial_T + \tfrac{\gamma}{R}\,\partial_\phi\right) + \right.$$
$$\left. - \,\Sigma^3\left(\hat{C} + \tfrac{\omega}{2}\,(\gamma)^2\right)\right]\psi = 0 \tag{4}$$

where γ is the relativistic factor of the Lorentz boost (3) and

$$\hat{C} = M\gamma^3\gamma^5 - i\gamma^5\,\frac{\partial}{\partial Z} \tag{5}$$

is a constant of motion (M is the mass of the particle). From (4) we see that $\hat{C}$ corresponds to a trivial symmetry of the inertial equation ($\omega = 0$) involving the longitudinal motion along a given direction, the Z axis. If we choose ψ_{inert} to be simultaneously eigenstate of $\hat{\pi}_3 = -i\partial_Z$ and $\hat{C}$ we obtain

$$\hat{C}\;\psi_{inert} = e\sqrt{M^2 + \pi_3^2}\;\psi_{inert}\;, e = \pm 1\;. \tag{6}$$

$\hat{C}$ ceases to be trivial when the system rotates because the associated degeneracy is raised by the interaction of the spin of the Dirac particle with rotation; the symmetry is preserved but the degeneracy is raised by a split in the energy spectrum due to the split term $\hat{C} + (\gamma)^2\;\frac{\omega}{2}$ appearing in (4). For planar motion ($\pi_3 = 0$), as is the case of the experiment of Fig. 1, and low rotation approximation, this actually corresponds to a mass split term $M \to M + e\omega/2$. Equation (4) also results from the inertial equation $(-i\gamma^A D_A + M)\psi_{inert} = 0$ by the substitution

$$\psi_{inert} = U\psi \tag{7}$$
$$U = e^{\gamma^0\gamma^2\beta(R)/2} = \cosh\beta/2 + \gamma^0\gamma^2 sh\beta/2$$

where $D_A = (\partial_T, \partial_R + \frac{1}{2R}\;,\;\frac{1}{R}\,\partial_\phi\;,\;\partial_z)$ and $tgh\beta = \omega R$. The matrix U is a spin-1/2 representation of the Lorentz tranformation (3).

In the remaining of the paper we adopt the low rotation approximation $\omega^2 R^2 << 1$ so that in (3) and (4) we may take $\gamma \simeq 1$. We choose ψ in the form

$$\psi = \chi_0(R, Z)e^{-iET + ip_\phi\phi} \tag{8}$$

where E and p_ϕ are respectively eigenvalues of the energy and angular momentum operators $i\partial_T$ and $-i\partial_\phi$. Equation (4) reduces then to

$$(E-\omega p_\phi)\psi = -i\gamma^5\left[\Sigma^1\left(\partial_R+\frac{1}{2R}\right)+\Sigma^2\left(-iE\omega R+i\,\frac{p_\phi}{R}\right)+\Sigma^3\left(\hat{C}+\frac{\omega}{2}\right)\right]\psi \tag{9}$$

Now by two successive Foldy-Wouthuysen transformations [9], [10] the Hamiltonian equation (9) and the constant of motion (5) are diagonalized to

$$(E-\omega p_\phi)\psi'' = \left[-\partial_R^2-\frac{1}{R}\,\partial_R+\frac{(p_\phi-\Sigma^3/2)^2}{R^2}-2E\omega p_\phi-E\omega\Sigma^3+\left(\hat{C}''+\frac{\omega}{2}\right)^2\right]^{1/2}\gamma^0\psi'' \tag{10}$$

$$\hat{C}'' = \sqrt{M^2+\pi_3^2}\,\gamma^0\Sigma^3 \tag{11}$$

with $\hat{C}''\psi''=\sqrt{M^2+\pi_3^2}\,e\psi''$. If we are now restricted to positive energy states $\gamma^0\psi_e''=\psi_e''$, namely, $\psi_e''=\frac{1}{2}\begin{pmatrix}1+e\\1-e\\0\\0\end{pmatrix}f_e$ (as in the non-relativistic limit), we can interpret the constant of motion $\hat{C}$ (or the quantum number e) as proportional to the projection of the spin of the particle along the rotation axis [11].

Squaring (10) we obtain

$$\begin{aligned}(E-\omega p_\phi)^2 f_e \;&=\; \left[-\partial_R^2-\frac{1}{R}\,\partial_R+\left(\frac{(p_\phi-e/2)}{R}\right)^2-2E\omega p_\phi-eE\omega+\right.\\ &+\;\left.\left(e\sqrt{M^2+\pi_3^2}+\frac{\omega}{2}\right)^2\right]f_e\end{aligned} \tag{12}$$

The solutions of (12) which are regular at the origin $R=0$, are Bessel functions of the first kind [13]

$$f_e = J_{p_\phi-e/2}(P(\omega)R) \tag{13}$$

where

$$P(\omega)=\sqrt{\left(E+\frac{e\omega}{2}\right)^2+\omega^2\left(p_\phi^2-\frac{1}{4}\right)-\left(\sqrt{M^2+\pi_3^2}+\frac{e\omega}{2}\right)^2} \tag{14}$$

We must now impose boundary conditions on the solutions (13) connected to the interferometry apparatus described in Fig. 1. Let us consider the inertial case ($\omega=0$) which corresponds to the experiment at rest. The presence of cristals/mirrors at R_0, producing the semi-circular beams, imply that boundary conditions must be imposed at R_0. Although we have not a model for the cristals/mirrors, the boundary conditions imposed by them will actually correspond to fixing the momentum of the beams at R_0. In fact from (13) we have

$$P(\omega = 0)R_0 = j'_{p_\phi - e/2} \tag{15}$$

where $j'_{p_\phi - e/2}$ is a number depending on $p_\phi - e/2$, associated to the boundary conditions at R_0. On solving (15) we obtain for the inertial energy

$$E = \sqrt{\frac{\left(j'_{p_\phi - e/2}\right)^2}{R_0^2} + M^2 + \pi_3^2} \tag{16}$$

leading us to interpret $P_0 = (j'_{p_\phi - e/2}/R_0)$ as the azimuthal momentum of the particles in the beams. For instance in the eikonal approximation [15] for the classical radius R_0 we have $j'_{p_\phi - e/2} \sim p_\phi - e/2$ which corresponds to impose that $|\psi''|^2$ has its first maximum approximately at R_0. We remark that in the circular beams of the experiment the particels have a fixed polarization e, and therefore a fixed energy. The spin polarization dependence of (16) is an effect analogous to the spin-orbit interaction in atoms. The azimuthal momentum P_0 does not coincide with the eikonal momentum $(p_\phi - e/2)/R_0$ due to the contribution of the spin-orbit type interaction to the kinetic energy of the planar motion. For the non-inertial case ($\omega \neq 0$) – noting from (7) and (8) that the energy eigenvalue E remains equal to the inertial energy – we must have at R_0,

$$P(\omega)R_0 = j'_{p_\phi - e/2} + \delta j'_{p_\phi - e/2}$$

corresponding to a momentum in the beams ($\pi_3 = 0$)

$$P(\omega) = \frac{j'_{p_\phi - e/2} + \delta j'_{p_\phi - e/2}}{R_0} \simeq P_0\sqrt{1 + \frac{e\omega}{E + M}}$$

where $P_0 = P(\omega = 0) = \sqrt{E^2 - M^2}$. This gives a momentum variation

$$\delta P_{mass-split} \cong \frac{e\omega}{2(E + M)}\, P_0 \simeq \frac{e\omega}{4}\, v_\phi \tag{17}$$

which depends basically on the mass-split term discussed above, where v_ϕ is the velocity of the particle along the circular beams [20].

We now derive the Mashoon effect, a wave effect of the same nature of (17). On completing the description of the apparatus we introduced in (2) a triad which is Frenet-Serret transported [6] but *not* Fermi Walker transported. The need for a Fermi-Walker transported triad comes from our assumption that free spin-1/2 particles behave as gyroscopes and may be used to define the inertia rest frame of our problem. The frame of the apparatus (Fermi-Walker transported) is obtained from the frame (1)-(2) by the additional time-dependent rotation $R^A{}_B = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\alpha & \sin\alpha & 0 \\ 0 & -\sin\alpha & \cos\alpha & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$, with $\alpha = \gamma\omega T$. In other words, we go from the inertia rest frame to the frame of the apparatus by the composed transformation $R^A{}_C L_A{}^B$, with $L_A{}^B$ given in (3). To avoid cumbersome calculations of the effects of this additional rotation, we instead extend the procedure used in obtaining (17) by noting the following.

A solution $\tilde{\psi}$ of the Dirac equation in the Fermi-Walker frame constructed above and a solution ψ of equation (4) are related by $\tilde{\psi} = S^{-1}\psi$, where $S = e^{i\gamma\Sigma^3 \frac{\omega T}{2}}$ satisfies $R^A{}_B\gamma^B = S^{-1}\gamma^A S$. Now we note that under the Foldy-Wouthuysen transformations leading from (9) to (10) the operator Σ^3 is transformed into

$$\tilde{\Sigma} = \frac{(-\vec{\gamma}\cdot\vec{\pi} + M)}{\sqrt{-(\vec{\gamma}\cdot\vec{\pi})^2 + M^2}}\ \Sigma^3$$

where $\vec{\gamma}\cdot\vec{\pi} = \gamma^1\pi_1 + \gamma^2\pi_2 = -i(\gamma^1 D_R + \gamma^2 D_\phi)$. A straightforward calculation gives that the Foldy-Wouthuysen transformed of $\tilde{\psi}$, which we denote by $\tilde{\psi}''$, is

$$\tilde{\psi}'' = e^{-i\frac{e\omega}{2}T}\psi'' + ie\left(1 - \frac{M}{E}\right)\sin\frac{\omega T}{2}\psi'' - \frac{eP(\omega)}{2E}\begin{pmatrix} 0 \\ 0 \\ 1-e \\ 1+e \end{pmatrix}\frac{J_{p_\phi - e/2-1}(PR)}{N(p_\phi - e/2)}\, e^{(-iET + ip_\phi\phi)}\sin\frac{\omega T}{2}$$

where $N(p_\phi - e/2)$ is a normalization factor for ψ''. In the limit $E \sim M$, we have $\tilde{E} \equiv \langle\tilde{\psi}''/i\partial_T\tilde{\psi}''\rangle = E + \frac{e\omega}{2}$, implying that $\tilde{\psi}''$ behaves approximately as ψ'' with $E \to E + e\omega/2$. An estimate of higher corrections to $\tilde{E}$ may be obtained by calculating $\tilde{E} = \langle\tilde{\psi}''/i\partial_T\ \tilde{\psi}''\rangle$, averaging in time, and considering that we may approximate $\frac{N^2(p_\phi - e/2)}{N^2(p_\phi - e/2-1)} \simeq 1$ for large quantum numbers $p_\phi - e/2$. We obtain

$$\tilde{E} = E + \frac{e\omega}{2} + O(v_\phi^4)$$

Hence in the non-relativistic limit $\tilde{\psi}''$ will describe particles in the beams with azimuthal momentum given by equation (14) with $E \to E + e\omega/2$, namely

$$P(\omega) \simeq \sqrt{(E + e\omega)^2 - \left(M + \frac{e\omega}{2}\right)^2} \simeq P_0\left(1 + \frac{e\omega}{2}\,\frac{E}{E^2 - M^2} + \frac{e\omega}{2(E+M)}\right) \qquad (18)$$

where E is the inertial ($\omega = 0$) energy. The third term in (18) is exactly (17), having its origin from the pure Lorentz boost (3). The second term appears as a direct consequence of the time-dependent rotation $R^A{}_B$, and gives the Mashoon shift

$$\delta P_{Mashoon} = \frac{e\omega}{2}\,\frac{E}{P_0} \simeq \frac{e\omega}{2v_\phi}\left(1 + \frac{v_\phi^2}{2}\right) \tag{19}$$

We note in (19) a $O(v_\phi^2)$ correction to the Mashoon effect predicted in Ref. [4] Although of distinct origin, it is equal to the mass-split term (17).

The origin of the Mashoon effect is thence the additional time dependent rotation necessary to produce a Fermi-Walker frame. Therefore the existence of the Mashoon effect is crucial to decide if free spin-1/2 particles actually behave as gyroscopes when rotational motion is involved. The phase shift resulting from (17) and (19) will be calculated at the end of the paper.

This exhausts the wave effects due to the motion of the frame of the experimental apparatus.

The second set of effects is due to the fact that the rotating apparatus drags the particles in the beams, producing a boost in their energy-momentum. It results in the Sagnac effect. The dragging of the particles by the apparatus is not described in the Brill-Wheeler formulation [14] but we may incorporate this effect in the formalism by the following procedure. Let us rewrite equation (4) in the form

$$(-i\gamma^A D_A + M)U\psi = 0 \tag{20}$$

where the matrix U defined in (7) is a solution of

$$L_B^{\ A}\gamma^B = U^{-1}\gamma^A U \tag{21}$$

and gives a spin-1/2 representation of the Lorentz transformation (3). The solution U of (21) is defined up to an operator V satisfying $[U, V] = 0 = [V, \gamma^A]$, and we use V to describe the boost due to the apparatus. If in (20) we substitute U by $W = VU$, we obtain

$$\{-i(U^{-1}\gamma^A U)V^{-1}D_A V - iU^{-1}\gamma^A(D_A U) + M\}\psi = 0 \tag{22}$$

To describe the drag of the particle by the apparatus we define V by

$$V^{-1}D_A V = L_A^{\ B} D_B \tag{23}$$

and the wave function of the experiment is now

$$\psi = V^{-1}U^{-1}\psi_{inert} \tag{24}$$

In the eikonal approximation for the circular ray R_0 [15](cf. experiment of Fig. 1) equation (23) has the solution

$$V = e^{\beta(R_0)L(R_0)} \simeq 1 + \omega R_0 L(R_0) \tag{25}$$

where $L(R) = \left(\frac{T}{R}\,\partial_\phi + R\phi\partial_T\right)$. The wave function (24) is an inertial solution with boosted four-momentum, and

$$W^{-1} = V^{-1}U^{-1} = e^{-\beta(R_0)(\gamma^0\gamma^2/2+L(R_0))}$$

is the complete boost operator of the rotating experiment. In fact, under (25) the coordinates X^A transform as $\tilde{X}^A = V^{-1}X^A V$, and we have

$$\begin{aligned}\tilde{T} &= T - \omega R_0^2\phi \\ \tilde{\phi} &= \phi - \omega T\end{aligned}$$

implying that $V^{-1}e^{-i(ET-p_\phi\phi)}V = e^{-i(\tilde{E}T-\tilde{p}_\phi\phi)}$ where

$$\begin{aligned}\tilde{E} &= E + \omega p_\phi \\ \tilde{p}_\phi &= p_\phi + \omega R_0^2 E\end{aligned} \tag{26}$$

The eikonal wave function of the experiment has thus the form (cf. (8))

$$\psi = e^{-\tilde{E}T+i\cdot\tilde{p}_\phi\phi}U^{-1}(R_0)\chi(R_0,Z)$$

where $U^{-1}\chi(R_0)$ is the boosted inertial spinor.

It is important to note here that the dragging of the particles by the apparatus is associated to a boost. No drag effect arises associated to the time dependent rotation ${R^A}_B$.

We remark that the c-number p_ϕ is the eigenvalue of the operator $\vec{n}\cdot\vec{J}$, where $\vec{J}$ is the total angular momentum of the particle and $\vec{n}$ is the unitary vector of the axis for definition of ϕ, which coincides with axis of rotation. To see this we use that the inertial wave function in cylindrical coordinates and Cartesian coordinates are related by [16]

$$\psi_{inert\ Cartes} = P\psi_{inert\ cyl}$$

where $P = e^{-\frac{i\phi}{2}\,\vec{n}\cdot\vec{\Sigma}}$, and therefore we have

$$P(-i\partial_\phi)P^{-1} = (-i\partial_\phi + \Sigma^3/2) = \vec{n}\cdot(-i\vec{R}\wedge\vec{\nabla} + \frac{\vec{\Sigma}}{2})$$

and

$$\vec{n}\cdot\vec{J}\,\psi_{inert\ Cartes} = p_\phi\,\psi_{inert\ Cartes} \tag{27}$$

(the domain of p_ϕ are half-integer values).

Now for an observer at the inertia rest frame the phase shift due to the rotation of the interferometer is given by the variation of the difference of wave numbers along the two semi circular paths of the interfering beams *at a given instant of time*, namely

$$\delta f = \int_C \delta\vec{p}\cdot d\vec{x}$$

where the line integral is calculated along the closed path C of the beams (cf. Fig. 1). In the approximations considered we may assume that the momentum variations arising from (17), (19) and (26) are additive, and it results

$$\delta f = 2\pi\omega\left(\frac{(p_\phi - e/2)}{v_\phi} + \frac{e}{2v_\phi} + 2\frac{ev_\phi}{4}\right)R_0$$

where we have used that $v_\phi = \frac{dE}{d(p_\phi/R_0)} \simeq \frac{(p_\phi - e/2)}{R_0 E}$ in the eikonal approximation. From (27) we may finally express

$$\delta f = \delta_{Sagnac} + \delta_{Mashoon} + \delta_{mass-split} \tag{28}$$

where

$$\begin{aligned}
\delta_{Sagnac} &= \frac{2\pi {R_0}^2}{v_\phi}\,\frac{\langle\vec{\omega}\cdot\vec{L}\rangle}{R_0} \\
\delta_{Mashoon} &= \frac{2\pi R_0}{v_\phi}\left(1 + \frac{v_\phi^2}{2}\right)\langle\vec{\omega}\cdot\frac{\vec{\Sigma}}{2}\rangle \\
\delta_{mass-split} &= \pi\ v_\phi\ \langle\vec{\omega}\cdot\frac{\vec{\Sigma}}{2}\rangle R_0
\end{aligned}$$

The first effect appears already in the eikonal approximation and is due to the drag of the particles by the rotating apparatus, in agreement with the derivation by Dresden and Yang [5] of the Sagnac effect. The third effect is due to an effective mass split, being of $0(v_\phi^2)$ smaller than the predicted Mashoon shift.

To conclude our analysis we show that other frames used in the literature give a null Sagnac effect, although they are mathematically admissible and the Hamiltonian in these frames contains interaction terms analogous to the ones in equation (3). Let us adopt the frame introduced in Ref. [17] , which is frequently used to discuss the interaction of spin-1/2 particles with rotation. We restrict ourselves to the case of the experiment of Fig. 1. We denote Cartesian inertial coordinates by $x^\alpha = (T, X, Y, Z)$, and Cartesian rotating coordinates by $x'^\alpha = (T', X', Y', Z')$, related by

$$\begin{aligned} T' &= T \\ X' &= X\cos\omega T + Y\sin\omega T \\ Y' &= -X\sin\omega T + Y\cos\omega T \\ Z' &= Z \end{aligned} \tag{29}$$

It is straightforward to show that a solution of Dirac equation in Hehl-Ni's frame is related to the inertial solution by [18]

$$\psi_{HN}(x') = U\psi_{inert}(x) \tag{30}$$

where $U = e^{\frac{i}{2}\,T'\vec{\omega}\cdot\vec{\Sigma}}$. From (30) we can check that $\delta\vec{p}\,' = \vec{p}_{HN} - \vec{p}\,'_{inert} = 0$, or equivalent $\vec{p}_{HN} = R^{-1}p_{inert}$ where R is the 3 dim time dependent rotation matrix defined through (29). Thus no interference Sagnac effect would be measured by the apparatus [19]. In other words the Hehl-Ni frame cannot be physically realized as the frame of the experiment. A non-null Sagnac effect indicates that the experiment is rotating with respect to the inertia rest frame, defining a rotating frame connected to the inertia rest frame by an active (instantaneous) Lorentz boost.

References

[1] M.G. Sagnac, Compt. Rend. **152**, 310 (1911); **157**, 708 (1913); **157**, 1410 (1916); A. Silberstein, J. Op. Soc. **5**, 291 (1921); A.A. Michelson, H.G. Gale and F. Pearson, Ap. J. **61**, 140 (1925).

[2] S.A. Werner, J.L. Staudenmann and R. Collela, Phys. Rev. Lett. **42**, 1103 (1979); D.M. Greenberger and A.W. Overhauser, Rev. Mod. Phys. **51**, 43 (1979).

[3] The inertia rest frame may be determined by a Foulcault pendulum, or by gyroscopes. It is also referred to as the local compass of inertia.

[4] B. Mashoon, Phys. Rev. Lett. **61**, 2639 (1988).

[5] M. Dresden and C.N. Yang, Phys. Rev. **D20**, 1846 (1979).

[6] W.M. Irvine, Physica **30**, 1160 (1964). The orthonormal triad $u_{(i)}$ is obtained by Irvine under the prescription that they are Frenet-Serret transported along the observer world line $u_{(0)}$.

[7] D.R. Brill and J.A. Wheeler, Rev. Mod. Phys. **29**, 465 (1957).

[8] γ^A are the constant Dirac matrices; we use a representation such that $(\gamma^A)^+ = \gamma^0\gamma^A\gamma^0$, with $(\gamma^0)^2 = -(\gamma^k)^2 = 1,\ k = 1,2,3$. Also $\gamma^5 = i\gamma^0\gamma^1\gamma^2\gamma^3$ and $\Sigma^k = \gamma^5\gamma^0\gamma^k$. Throughout the paper we use units such that $\hbar = c = 1$.

[9] L.L. Foldy and S.A. Wouthuysen, Phys. Rev. **78**, 29 (1950).

[10] J.Tiomno, Physica **53**, 58 (1971).

[11] By a Cini-Toushek transformations we can show that, in the high momentum limit (or $M = 0$), the conserved quantum number e is the projection of the spin of the Dirac field along the direction of the momentum of the field (cf. ref. (12)).

[12] B.D.B. Figueiredo, I. Damião Soares and J. Tiomno, Class. Q. Grav. **9**, 1593 (1992).

[13] Up to a normalization factor $N_{p_\phi - e/2}$.

[14] The Brill-Wheeler formalism describes only the spinorial boost of the inertial spinor wave function, according to (7).

[15] In the eikonal approximation for the classical radius R_0, equation (2) assumes the form

$$\{\gamma^0\partial_T + \gamma^2\ \frac{\partial_\phi}{R_0} + \gamma^1\ \frac{1}{2R_0} + \gamma^3\partial_z + iM\}\psi = 0\ .$$

[16] We remark that P is the operator which transforms the Dirac inertial Hamiltonian in cylindrical coordinates into the Dirac inertial Hamiltonian in Cartesian coordinates.

[17] F.W. Hehl and W.T. Ni, Phys. Rev. **D42**, 2045 (1990).

[18] The operator U transforms the Dirac Hamiltonian in the inertial rest frame (in Cartesian rotating coordinates) into the Dirac Hamiltonian in the HN frame (in Cartesian rotating coordinates). The difference between HN tetrad and the inertial tetrad, in Cartesian rotating coordinates, is a spatial rotation depending on the angle $\omega T'$.

[19] We note that the Hehl-Ni frame is *not* Fermi-Walker transported. To transform it into a Fermi-Walker frame we need to realize a time dependent rotation, resulting in the inertial frame expressed in Cartesian rotating coordinates.

[20] The smallness of this term results from the fact that it is actually the difference between the mass-split term and the Foldy-Wouthuysen term $-i\vec{\omega}\cdot\vec{\Sigma}\partial_T$, which is a rotomagnetic moment interaction originating from the rotomagnetic potential $-iE\omega R$ produced by the Lorentz boost (cf. eq. (9)).