ISSN 0029 - 3865

CBPF-NF-051/88

ON THE DETECTION OF MATTER VORTICITY AND SPACETIME TORSION

by

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ABSTRACT

We discuss the detection of spacetime torsion and matter vorticity by using the splitting of the energy spectrum of fermions coupled gravitationally to torsion and matter vorticity. We show that matter vorticity splits the spectral lines of fermions in the same manner as torsion. These effects are additive and result from the existence of the *same* constant of motion for the fermions in both cases. The two effects can be more precisely distinguished by a further test involving Klein-Gordon particles.

Key-words: Fermions in Einstein-Cartan theory; Detection of torsion and vorticity; Torsion versus matter vorticity; Gravitational splitting of fermion spectra; Gödel-type spacetimes.

It seems well established that spacetime torsion can in principle be detected by means of a body endowed with a net intrinsic spin. According to Stoeger[1] such detection could otherwise be achieved by the spectral line splitting of a fermion, by the classical paths of massive particles with spin or by the precession of the spin vector of a large polarized body. These tests concern to general theories of gravitation with torsion. Here we deal with fermions in the context of Hehl's gravitational theory[2]. Our purpose is to show that the matter vorticity can also split the spectral lines in the *same* manner as torsion, and a further test involving the spectra of Klein-Gordon particles is discussed to distinguish the two effects, allowing to determine matter vorticity and torsion separately.

We assume that the torsion is generated by the spin $S^{i}{}_{jk}$ of a Weyssenhoff-Raabe fluid[3],

$$S^{i}{}_{jk} = u^{i}S_{jk} \quad , \quad u^{i}S_{ij} = 0 \quad , \tag{1}$$

where u^{i} is the fluid four-velocity and $S_{ij} = -S_{ji}$ is the spin density. It turns out that the spin tensor and the torsion $\tau^{i}{}_{jk}$ are related by the field equation[2]

$$\tau^{i}{}_{jk} = \frac{1}{2}(\Gamma^{i}{}_{jk} - \Gamma^{i}{}_{jk}) = ku^{i}S_{jk} \quad , \tag{2}$$

k being the gravitational constant. From the metricity postulate ($g_{ij;k} = 0$) equation (2) can be solved for the connection $\Gamma^{i}_{\ jk}$, which is found to be

$$\Gamma^{i}_{\ jk} = \{^{\ i}_{jk}\} - k\ (u^{i}S_{jk} + u_{j}S^{i}_{\ k} - u_{k}S^{i}_{\ j}) \tag{3}$$

For an observer comoving with the fluid we choose the four-velocity

$$u^{i} = \delta^{i}_{o} \tag{4}$$

and assume the fluid spin vector to be constant and directed along the x^3-axis, that is,

$$S_{12} = S = \text{const} \quad , \quad S_{ij} = 0 \text{ (other indices)} \tag{5}$$

Now let us consider a spacetime endowed with a Minkowski metric (expressed in Cartesian coordinates) and a connection given by (3), (4) and (5). In this background Dirac's equation[4] for a spinorial test field ψ with mass M becomes[5]

$$i\,\frac{\partial\psi}{\partial t} = (\gamma^{5}\ \vec{\Sigma}\cdot\vec{\hat{p}} + M\gamma^{0} - \frac{k}{2}\,S\,\Sigma^{3})\,\psi = \hat{H}\psi \tag{6}$$

Here

$$\vec{\Sigma}\cdot\vec{\hat{p}} = \Sigma^{1}\hat{p}_{1} + \Sigma^{2}\hat{p}_{2} + \Sigma^{3}\hat{p}_{3} \tag{7a}$$

with the spin matrices and momentum operators given by

$$\sum^{i} = \gamma^{5}\gamma^{0}\gamma^{i} \quad , \quad \hat{p}_{i} = -i \frac{\partial}{\partial x^{i}} \tag{7b}$$

The operator $\hat{H}$ defined by the second equality in (6) is the Hamiltonian of the fermionic system. It explicitly includes a term of interaction of the spin $\vec{\sum}$ with the torsion. As the energy and the momenta are constants of motion we can perform the separation of variables

$$\psi = \overset{0}{\psi} e^{-i(\vec{p}.\vec{x} + \varepsilon t)} \tag{8}$$

where ε and $\vec{p}$ are the eingenvalues of the operators of energy and momenta, and $\vec{p}.\vec{x} = p_1 x + p_2 y + p_3 z$; $\overset{0}{\psi}$ is a constant four-spinor. For zero mass γ^5 is also a constant of motion and the number of independent components of $\overset{0}{\psi}$ is reduced to only two. In the general case $(M \neq 0)$ a similar role is played by the operator

$$\hat{C} = -i\gamma^{5}\frac{\partial}{\partial z} + M\gamma^{3}\gamma^{5} \quad , \quad [\hat{C},\hat{H}] = 0 \tag{9}$$

Choosing ψ to be simultaneous eigenstate of $\hat{C}$ and $\hat{H}$, we obtain from (8) and (9)

$$\hat{C}\psi = -e\sqrt{M^{2}+(p_{3})^{2}}\,\psi \quad , \quad e = \pm 1 \qquad (M \neq 0) \tag{10a}$$

$$\hat{C}\psi = -p_{3}L\psi \qquad (M = 0) \;. \tag{10b}$$

$L = \pm 1$ is the eigenvalue of the helicity operator γ^5, and is related to e by

$$e = Lp_3/|p_3| \tag{11}$$

in the limit $M \to 0$. Now the substitution of (8) into (10a) yields

$$\overset{0}{\psi} = \begin{pmatrix} \overset{0}{\psi}_1 \\ \overset{0}{\psi}_2 \\ \gamma_+ \overset{0}{\psi}_1 \\ \gamma_- \overset{0}{\psi}_2 \end{pmatrix} \tag{12}$$

where $\gamma_\pm = \left(\pm M + e\sqrt{M^2 + (p_3)^2}\right)/p_3$. Replacing (8) and (12) into the equation

$$\varepsilon\psi = \{\gamma^5(\textstyle\sum^1 \hat{p}_1 + \sum^2 \hat{p}_2) + \sum^3 (\hat{C} - \frac{kS}{2})\}\psi \quad , \tag{13}$$

(which is equivalent to (6)) we obtain a system of algebraic equations for the two independent spinor components $\overset{0}{\psi}_1$ and $\overset{0}{\psi}_2$. The compatibility condition for this systems gives

$$\varepsilon^2 = (p_1)^2 + (p_2)^2 + \left[e\sqrt{M^2 + (p_3)^2} + \frac{kS}{2}\right]^2 \tag{14}$$

This is all we need. If $S = 0$ this formula coincides with the familiar one for flat-space. If however $S \neq 0$ we have for a given value of momenta a splitting of the energy level into two levels, one for each value of the quantum number e. In this manner the effect mentioned by Stoeger manifests itself due to the existence of the double valued quantum number e.

Now, the second part of our argument: the splitting of energies can also be generated by matter vorticity (this effect is present even in Riemannian spacetimes, as found in Refs. 6 and 7). In the context of Hehl's theory and for technical simplicity let us consider Gödel-type metrics[8],[9] which are the simplest known cosmological solutions with matter vorticity. These spacetimes have as a particular case the spacetimes previously considered with Minkowski metric and torsion. In cylindrical coordinates (t,r,ϕ,z) they can be expressed as

$$ds^2 = (dt + Hd\phi)^2 - dr^2 - D^2 d\phi^2 - dz^2 \quad , \tag{15}$$

$$H = \frac{\Omega}{\ell^2}\sinh^2(\ell r) \quad , \quad D = \frac{\sinh(2\ell r)}{2\ell}$$

When the source of spacetime curvature is a Weyssenhoff-Raabe fluid with spin vector along the vorticity field, we have in the realm of Hehl's theory that[9]

$$\Omega = \Omega_0 - kS \tag{16}$$

where Ω_0 is the magnitude of the vorticity and S is the spin density of the fluid (cf. Eqs. (1) and (5)). In the tetrad basis

$$e_o^{(o)} = e_1^{(1)} = e_3^{(3)} = 1 \ , \ e_2^{(o)} = H \ , \ e_2^{(2)} = D$$

Dirac's equation now reads

$$i\,\frac{\partial\psi}{\partial t} = (\gamma^5\vec{\Sigma}\cdot\vec{\pi} + M\gamma^0 + \frac{\Omega - kS}{2}\,\Sigma^3)\psi$$

$$\pi_1 = -i\,\frac{\partial}{\partial r} - \frac{i}{D}\frac{dD}{dr} \ , \ \pi_2 = \frac{iH}{D}\frac{\partial}{\partial t} - \frac{i}{D}\frac{\partial}{\partial\phi} \ , \ \pi_3 = -i\,\frac{\partial}{\partial z} \tag{17}$$

The operator (9) is again a constant of motion with respect to the Hamiltonian defined by (17). We then choose a simultaneous eigenstates of (9) and the Hamiltonian, and in the momenta modes defined by the Killing vectors[10],[11] $\frac{\partial}{\partial z}$ and $\frac{\partial}{\partial\phi}$ the eigenvalues of (9) are given by (10). The solutions of (17) which are regular at r = 0 have the expression

$$\psi = A \begin{pmatrix} \begin{pmatrix} 1 \\ \gamma_+ \end{pmatrix} \ell\gamma_-(\varepsilon - K)\,\alpha_+(x) \\ \\ \begin{pmatrix} 1 \\ \gamma_- \end{pmatrix} \ell(2m+1)\,\alpha_-(x) \end{pmatrix} e^{-i(\varepsilon t + m\phi + p_3 z)} \tag{18}$$

for $m \geq 1/2$, and

$$\psi = A \begin{pmatrix} \begin{pmatrix} 1 \\ \gamma_+ \end{pmatrix} \ell\gamma_-(1-2m)\,\beta_+(x) \\ \\ \begin{pmatrix} 1 \\ \gamma_- \end{pmatrix} i(\varepsilon + K)\,\beta_-(x) \end{pmatrix} e^{-i(\varepsilon t + m\phi + p_3 z)} \tag{19}$$

for $m \le -1/2$. Here ε, p_3 and the half-integer m are eigenvalues of the Hamiltonian and of the momenta operators $i\frac{\partial}{\partial z}$ and $i\frac{\partial}{\partial\phi}$, respectively. We have also denoted

$$A = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad \gamma_\pm = \frac{\pm M + \sqrt{M^2 + (p_3)^2}}{p_3},$$

$$K = e\sqrt{M^2 + (p_3)^2} - \frac{\Omega - kS}{2}, \quad x = \cosh(2\ell r),$$

and

$$\alpha_\pm = (x^2-1)^{\frac{2m\pm 1}{4}} (1+x)^{\frac{\Omega\varepsilon}{2\ell^2}} F(a,b,m+1 \pm \tfrac{1}{2}; \tfrac{1-x}{2}), \tag{19}$$

$$\beta_\pm = (x^2-1)^{-\frac{2m\pm 1}{4}} (1+x)^{-\frac{\Omega\varepsilon}{2\ell^2} \pm \frac{1}{2}} F(1-a,1-b,1-m \mp \tfrac{1}{2}; \tfrac{1-x}{2}), \tag{20}$$

where $F(a,b,c;y)$ is the hypergeometric function[12], with the parameters

$$a = m + \frac{1}{2} + \frac{\Omega\varepsilon}{2\ell^2} + \frac{n}{2}, \quad b = m + \frac{1}{2} + \frac{\Omega\varepsilon}{2\ell^2} - \frac{n}{2}$$
$$n = \left(\frac{\Omega^2 - \ell^2}{\ell^4}\varepsilon^2 + \frac{K^2}{\ell^2}\right)^{1/2} \tag{21}$$

Due to the character of test field we are ascribing to ψ, we must require that the solutions (18)-(21) be finite at any space-time point. If we restrict our attention to the class of hyperbolic metrics with $\Omega^2 \geq \ell^2 > 0$ we obtain[13] the discrete energy spectrum

$$|\varepsilon| = |\Omega|(2j+1) + \left((\Omega^2-\ell^2)(2j+1)^2 + \left[e\sqrt{M^2+(p_3)^2} - \frac{\Omega-kS}{2}\right]^2\right)^{1/2} \tag{22}$$

$$j = -1/2,\ 1/2,\ 3/2, \ldots$$

The term $(\Omega-kS)/2$ is again responsible for a splitting of each energy level into a doublet. In the Riemannian limit $(S=0)$ this splitting does remain but it disappears in a Riemann-Cartan spacetime with $\Omega = kS$. So the presence (or absence) of the splitting in general does not imply the presence (or absence) of torsion or vorticity. What matters is the simultaneous effect of torsion and vorticity. It is remarkable that even if no splitting is found in the fermion case, yet this is an evidence for the existence of spacetime torsion $S = \Omega/k$.

Finally we ask how to find out whether a given spacetime with matter vorticity presents torsion. In the context of Hehl's theory and for a Gödel-type spacetime this question could in principle be decided by the measurement of the energy levels of fermions, which is sufficient to determine from (22) the parameters Ω, ℓ^2 and consequently S. However, as this procedure involves adjustment to experimental values (and the effect of splitting being expected to be very small), we could improve the precision of the determination of Ω and ℓ if we performed an independent experiment involving scalar test particles. In fact we find that the solutions of the Klein-Gordon equation (for $\Omega^2 > \ell^2 > 0$) possess the discrete energy spectrum[14]

$$|\varepsilon| = |\Omega|(2j+1) + \left((\Omega^2-\ell^2)(2j+1)^2 + \ell^2 + M^2 + (p_3)^2\right)^{1/2} \qquad (23)$$

where j is now a non-negative integer. By measuring the separation of adjacent levels of energy we can determine (for a known value of p_3) the cosmological parameters Ω and ℓ^2. Returning to the fermion case we can adjust the value of S in (22) so as to reproduce the observed lines of those spectra and consequently we are in principle able to say whether the spacetime presents torsion or not.

In a forthcoming paper[14] we shall accomplish a complete analysis of scalar and spinorial solutions for all metrics comprised in (15) including the Som-Raychandhuri metric ($\ell^2 = 0$). In the present paper our interest was to call attention to the possible influence of matter vorticity on the splitting due to the torsion field, foreseen by Stoeger. It would also be interesting to study the influence of matter vorticity on the other tests proposed to detect torsion.

One of us (B.D.B.F.) wishes to acknowledge Conselho Nacional de Desenvolvimento Científico e Tecnológico (CNPq) and Fundação de Amparo à Pesquisa do Rio de Janeiro (FAPERJ) for financial support.

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13) The denomination *discrete spectrum* is used because of the half-integer parameter $j \geq -1/2$ appearing in the expression of the energy (22). For the other cases $\Omega^2 \leq \ell^2$ the spectrum presents also a continuum region-in a manner analogous to (14)-for higher values of the energy. Our conclusions however remain unaffected. Cf, B.D.B. Figueiredo, D.Sc. Thesis, CBPF, Rio de Janeiro (1987), and Ref. (14).

14) B.D.B. Figueiredo, I. Damião Soares and J. Tiomno, to be published.