A simple remark on three dimensional gauge theories

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Abstract

Classical three dimensional Yang-Mills is seen to be related to the topological Chern-Simons term through a nonlinear but fully local and covariant gauge field redefinition. A classical recursive cohomological argument is provided.

1 Introduction

Since many years topological three dimensional massive Yang-Mills theory [1] is a continuous source of investigation and has led to a large amount of interesting applications in different areas of theoretical physics.

As it is well known, the action of the model is characterized by two parameters (g, m), a gauge coupling and a mass term respectively, and can be written as the sum of a Yang-Mills and of a Chern-Simons term, *i.e.*

$$\mathcal{S}_{YM}(A) \ + \ \mathcal{S}_{CS}(A) \ , \tag{1.1}$$

where, adopting the same parametrization of refs.[2, 3],

$$\mathcal{S}_{YM}(A) = \frac{1}{4m} tr \int d^3x F_{\mu\nu} F^{\mu\nu} \ , \tag{1.2}$$

and

$$\mathcal{S}_{CS}(A) = \frac{1}{2} tr \int d^3x \varepsilon^{\mu\nu\rho} \left(A_\mu \partial_\nu A_\rho + \frac{2}{3} g A_\mu A_\nu A_\rho \right) \ , \tag{1.3}$$

with.

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + g[A_\mu, A_\nu] \ . \tag{1.4}$$

Although being only power counting superrenormalizable, topological massive Yang-Mills (1.1) turns out to be ultraviolet finite to all orders of perturbation theory. This remarkable feature was first detected by explicit one loop computations [4] and later on has been put on firm basis and extended to all orders by [2] with a careful study of the behaviour of higher loops three dimensional Feynman integrals. More recently, this result has been proven to hold [5] by similar arguments for the $N = 1$ supersymmetric version of (1.1).

It is also worthwhile to mention that a partial proof of the ultraviolet finiteness of the $N = 2$ version of (1.1) in the Wess-Zumino gauge has been achieved in [6] with a purely algebraic cohomological analysis. The interesting result obtained here is that the possible invariant counterterms turn out to be related to only one of the two parameters (g, m).

Let us come now to the main purpose of this work. Our aim here is to report on a very elementary classical geometrical aspect which we shall hope to be useful for a better understanding of the model. We shall be concerned, in particular, with the observation that the topological massive Yang-Mills action (1.1) can be actually traced back to a pure Chern-Simons action through a nonlinear redefinition of the gauge field, namely

$$\mathcal{S}_{YM}(A) \ + \ \mathcal{S}_{CS}(A) = \mathcal{S}_{CS}(\widehat{A}) \ , \tag{1.5}$$

with

$$\widehat{A}_\mu = A_\mu + \sum_{n=1}^{\infty} \frac{1}{m^n} \vartheta^n_\mu(D, F) \tag{1.6}$$

where the coefficients $\vartheta^n_\mu(D,F)$ turn out to be *local* and *covariant*, meaning that they are built up only with the field strength F and the covariant derivative ($D_\mu = \partial_\mu + g[A_\mu,\ ]$). The above formulas represent the essence of the present letter. Their meaning is that, at the *classical level*, topological massive three dimensional Yang-Mills can be seen as being related, through a nonlinear but *local* and *covariant* field redefinition, to the topological Chern-Simons term.

However, before going any further, some necessary remarks are in order. We underline that the equation (1.5) has to be understood here in pure classical terms. Although the quantum aspects are out of the aim of this work, let us observe here that at the level of the quantized theory the nonlinear field redefinition (1.6) could seem to allow for a transfer of the properties of topological massive Yang-Mills theory from the initial action (1.1) to the gauge fixing and the Faddeev-Popov terms. However, as one can easily infer from the presence of the expansion parameter $1/m$ in the eq.(1.6), the use of the redefined gauge field $\widehat{A}_\mu$ will introduce in these terms an infinite number of power counting nonrenormalizable interactions which would render the quantum analysis more involved. In other words, as far as the quantum aspects are concerned, the use of a manifest power counting renormalizable gauge coordinate system and of the usual action (1.1) as the starting points are more suitable, as proven by [1, 4, 2, 5, 6].

Nevertheless, in our opinion the formulas (1.5) , (1.6) could give a simple pure geometric set up in order to improve our knowledge about three dimensional gauge theories. This is our motivation for the present letter.

The paper is organized as follows. Sect.2 is devoted to the computation of the coefficients $\vartheta^n_\mu(D,F)$ up to the fourth order in the $1/m$ expansion. In Sect.3 we present a simple classical cohomological argument which supports the formulas (1.5) , (1.6). Sect.4 deals with the $N=1$ superspace generalization. Finally, we conclude with a few remarks concerning possible further applications.

2 Some computations

In order to have a more precise idea of the coefficients $\vartheta^n_\mu(D,F)$ let us give here the explicit value of some of them. Their computation is really straightforward, one only needs to insert the eq.(1.6) into the eq.(1.5) and identify the terms with the same power in $1/m$. For instance, the first four coefficients are found to be

$$
\begin{aligned}
\vartheta^1_\mu &= \frac{1}{4}\varepsilon_{\mu\sigma\tau}F^{\sigma\tau} , \\
\vartheta^2_\mu &= \frac{1}{8}D^\sigma F_{\sigma\mu} , \\
\vartheta^3_\mu &= -\frac{1}{16}\varepsilon_{\mu\sigma\tau}D^\sigma D_\rho F^{\rho\tau} + \frac{g}{48}\varepsilon_{\mu\sigma\tau}\left[F^{\sigma\rho}, F_\rho{}^\tau\right] , \\
\vartheta^4_\mu &= -\frac{5}{128}D^2 D^\rho F_{\rho\mu} + \frac{5}{128}D^\nu D_\mu D^\lambda F_{\lambda\nu} \\
&\quad -\frac{7}{192}g\left[D^\rho F_{\rho\tau}, F_\mu{}^\tau\right] - \frac{1}{48}g\left[D_\nu F_{\mu\lambda}, F^{\lambda\nu}\right] .
\end{aligned}
\tag{2.7}
$$

Observe that, as already remarked, all the coefficients of eq.(2.7) are covariant, depending only on $F_{\mu\nu}$ and its covariant derivatives. Although the higher order coefficients can be easily obtained, let us now focus on a cohomological argument which will justify the formulas (1.5), (1.6).

3 A cohomological argument

In order to provide a cohomological argument for the eqs.(1.5), (1.6) we shall make use of the BRST antifield formulation. The construction of the corresponding classical Slavnov-Taylor (or master equation) identity does not present any difficulty and is easily carried out [2]. Only two antifields[1] (A^*_μ, c^*) are needed, corresponding respectively to the gauge connection A_μ and to the Faddeev-Popov ghost c. Let us observe now that within the BRST framework the reabsorption of the pure Yang-Mills term $\int FF$ through a gauge field redefintion, as it is implied by the eqs.(1.5), (1.6), lies in the possibility of (re)expressing $\int FF$ in the form of an exact BRST cocycle. That this is indeed the case follows from a simple inspection of the BRST transformation of the antifield A^*_μ, *i.e.*

$$sA^*_\mu = \frac{1}{2}\varepsilon_{\mu\nu\rho}F^{\nu\rho} + \frac{1}{m}D^\nu F_{\mu\nu} + \left\{c, A^*_\mu\right\} \ , \tag{3.8}$$

s denoting the BRST differential.

The last term in eq.(3.8) states the simple fact that under a rigid gauge transformation the antifield A^*_μ transforms according to the adjoint representation of the gauge group, and can be neglected when the BRST differential s acts on the space of the gauge invariant quantities, as for instance $\int FF$. Contracting[2] now both sides of eq.(3.8) with $\varepsilon_{\mu\nu\rho}$, the eq.(3.8) can be cast in the following more convenient form

$$F_{\mu\nu} = s(\varepsilon_{\mu\nu\rho}A^{*\rho}) - \frac{1}{m}\varepsilon_{\mu\nu\rho}D_\lambda F^{\rho\lambda} \ - \{c, \varepsilon_{\mu\nu\rho}A^{*\rho}\} \ . \tag{3.9}$$

It becomes now apparent that the above formula[3] allows us to replace in any gauge invariant quantity the field strength $F_{\mu\nu}$ by a pure BRST variation with, in addition, a term of the order $1/m$ containing a covariant derivative D_μ, *i.e.*

$$\begin{aligned} tr\int d^3x F_{\mu\nu}F^{\mu\nu} \ &= \ tr\int d^3x \ \left(s(A^{*\mu}\varepsilon_{\mu\nu\rho}F^{\nu\rho}) \ - \frac{1}{m}F^{\mu\nu}\varepsilon_{\mu\nu\rho}D_\lambda F^{\rho\lambda}\right) \\ &= \ tr\int d^3x \ s\left(A^{*\mu}\varepsilon_{\mu\nu\rho}F^{\nu\rho} \ + \frac{2}{m}A^{*\mu}D^\sigma F_{\sigma\mu}\right) \ + \ O(\frac{1}{m^2}) \\ &= \ \ldots\ldots\ldots\ldots \ . \end{aligned} \tag{3.10}$$

The expression (3.9) yields then a recursive procedure since $F_{\mu\nu}$ appears on both sides. At each step of the iteration a new factor $(\varepsilon D/m)$ will appear in the right hand side of

[1] As it is well known [7] the introduction of the antifields allows to implement in cohomology the classical equations of motion.

[2] We use here the euclidean normalization $\varepsilon^{\mu\nu\rho}\varepsilon_{\mu\sigma\tau} = (\delta^\nu_\sigma\delta^\rho_\tau - \delta^\rho_\sigma\delta^\nu_\tau)$.

[3] A similar cohomological argument has been already used by [6] (see for instance eqs.(6.31)) in the algebraic analysis of the $N = 2$ version of the topological massive Yang-Mills.

eq.(3.10). We will end up therefore with an infinite power series whose generic element of the order n is characterized by the presence of a factor of the kind $(\varepsilon D/m)^n$. The eq.(3.9) implies thus that the Yang-Mills term $\int FF$ can always be rewritten as a pure BRST variation of an infinite series in the expansion parameter $1/m$. It is this property which allows us to reabsorb $\int FF$ into the Chern-Simons action by means of a nonlinear gauge field redefinition. Moreover, all coefficients will be *local* and *covariant*[4].

Let us conclude this section with the following remark. If we had started with the pure Yang-Mills term as initial action, it would be impossible to reach such a kind of conclusion. In fact the left hand side of the expression (3.9) would be vanishing, due to the absence of the Chern-Simons term. The formula (3.9) would become thus useless. However, as soon as the Chern-Simons is switched on, the Yang-Mills term can be seen as being generated by pure Chern-Simons by means of a local and covariant gauge field redefinition.

4 Supersymmetric generalization

It is very easy to generalize the previous set up to the supersymmetric version of topological massive Yang-Mills (1.1). Considering for instance the case of $N = 1$ in superspace we have, following [5],

$$\mathcal{S}_{CS}(\Gamma) = -\frac{1}{2}tr\int dV\left(\Gamma^\alpha D^\beta D_\alpha\Gamma_\beta + \frac{g}{3}\Gamma^\alpha\left[\Gamma^\beta, D_{(\beta}\Gamma_{\alpha)}\right] + \frac{g^2}{6}\Gamma^\alpha\left[\Gamma^\beta,\{\Gamma_\alpha,\Gamma_\beta\}\right]\right)\ , \quad (4.11)$$

and

$$\mathcal{S}_{YM}(\Gamma) = \frac{1}{m}tr\int dV\ W^\alpha W_\alpha\ , \qquad dV = d^3x\ d^2\theta\ , \quad (4.12)$$

where Γ^α is the spinor gauge superfield and W_α is the superfield strength given by

$$W_\alpha = D^\beta D_\alpha\Gamma_\beta + g\left[\Gamma^\beta, D_\beta\Gamma_\alpha\right] + \frac{g^2}{3}\left[\Gamma^\beta,\{\Gamma_\alpha,\Gamma_\beta\}\right]\ , \quad (4.13)$$

with D_α being the ordinary superspace supersymmetric derivative (α,β are now spinor indices). Introducing the covariant supersymmetric gauge derivative

$$\nabla_\alpha = D_\alpha + g\left[\Gamma_\alpha,\ \right]\ , \quad (4.14)$$

for the first coefficients $\vartheta^n_\alpha(\nabla, W)$ of the supersymmetric version of the expansion (1.6) we get

$$\vartheta^1_\alpha(\nabla, W) \ = \ -W_\alpha\ , \quad (4.15)$$

[4]The covariance of the coefficients θ^n_μ in eq.(1.6) easily follows from the requirement of gauge invariance of $\mathcal{S}_{CS}(\widehat{A})$. As a consequence, the redefined field $\widehat{A}$ transforms as a connection under gauge transformations.

$$\vartheta^2_\alpha(\nabla, W) = -\frac{1}{2}\nabla^\beta\nabla_\alpha W_\beta \ ,$$
$$\vartheta^3_\alpha(\nabla, W) = -\frac{1}{2}\nabla^\beta\nabla_\alpha\nabla^\gamma\nabla_\beta W_\gamma + \frac{1}{3}g\left[W^\beta, \nabla_\alpha W_\beta\right] \ .$$

Again, they are all covariant. Of course, the previous cohomological argument applies to the present supersymmetric case as well.

5 Conclusion

Let us conclude with a few comments and remarks on possible further applications.

The first remark concerns the use of a nonlinear field redefinition. Although the present work deals with pure classical considerations, it is worthy to remind that nonlinear field redefinitions have already been used in several cases, being typically needed when dimensionless fields are present. This is the case, for instance, of the two dimensional nonlinar sigma model [8] and of the $N = 1$ superspace super Yang-Mills theories [9]. Let us underline that these nonlinear field redefinitions are, in analogy with our case, completely *local* and given explicitely by an infinite power series in the fields. Other kinds of nonlinear but *nonlocal* field redefinitions are also known. They are used in order to compensate the nonlocal divergences which arise when noncovariant gauges are employed. An example of such a kind of nonlinear nonlocal field redefinition is provided by [3] who analysed in fact the one loop renormalization of massive topological Yang-Mills theory in the light-cone gauge. It should be clear, however, that this nonlinear nonlocal field redefinition is completely different from that of eq.(1.6).

Although the higher order coefficients ϑ^n_μ of the series (1.6) can be computed straightforwardly, the possibility of obtaining a closed expression for the infinite expansion (1.6) is very tempting. We limit to observe here that the covariant character of the coefficients ϑ^n_μ naturally remind us the normal coordinate expansion frequently used in the nonlinear two dimensional sigma models [8].

Let us point out, finally, that the recursive cohomological argument of eq.(3.9) applies also in the case in which terms of higher order in F and its covariant derivatives ($FD^2F, ..., etc.$) are introduced in the classical initial action. As one can easily understand, the recursive cohomological argument relies essentially only on the presence of the Chern-Simons term. Of course, this follows from the fact that the variation of the Chern-Simons term yields exactly the field strength F. It is this basic property which is at the origin of the recursive cohomological argument (3.9). In other words, provided the Chern-Simons term is present in the game, the recursive argument still holds for whatever metric dependent gauge invariant Yang-Mills type term one likes to start with.

Possible quantum aspects related to the nonlinear field redefinition (1.6) as well as to a pure algebraic analysis of the ultraviolet finiteness of topological massive Yang-Mills are under investigation.

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