

# Multiflavor Correlation Functions in Non-Abelian Gauge Theories at Finite Fermion Density in Two Dimensions<sup>‡</sup>

by

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## ABSTRACT

We compute vacuum expectation values of products of fermion bilinears for two-dimensional Quantum Chromodynamics at finite flavored fermion densities. We introduce the chemical potential as an external charge distribution within the path-integral approach and carefully analyse the contribution of different topological sectors to fermion correlators. We show the existence of chiral condensates exhibiting an oscillatory inhomogeneous behavior as a function of a chemical potential matrix. This result is exact and goes in the same direction as the behavior found in  $QCD_4$  within the large  $N$  approximation.

**Key-words:**  $QCD_2$ ; Topological sectors; Fermionic correlators; Chemical potential.

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# 1 Introduction

In order to understand the structure of the  $QCD$  vacuum [1] one should analyse possible mechanisms for chiral symmetry breaking and the formation of fermion condensates. The existence of such correlators can be understood as the result of condensation of pairs of particles and holes and it can have interesting implications in particle physics and cosmology. For example, a color nonsinglet condensate may be related to superfluidity and color superconductivity of cold quark matter at high fermion densities [2]. In this respect the results of Deryagin, Grigoriev and Rubakov [3] are of particular importance. Analysing the large  $N_c$  limit of  $QCD$  these authors have shown that the order parameter for chiral symmetry, the quark condensate  $\langle \bar{\psi}\psi \rangle$ , is at high quark densities inhomogeneous and anisotropic so that, regarding the order parameter, the ground state of quark matter has the structure of a standing wave.

Two-dimensional models like the Schwinger model and  $QCD_2$  provide a natural laboratory to test these phenomena since, although simplified, the basic aspects (chiral symmetry features, non-trivial topological sectors, etc) are still present and exact calculations can be in many cases performed.

An analysis of two-dimensional  $QED$  at finite density was originally presented in [4]-[5]. More recently, studies on this theory [6]-[7] showed that inhomogeneous chiral condensates do exist as a result of the contribution of non-trivial topological sectors.

Extending our work on  $QED_2$  [7] we analyse in the present paper vacuum expectation values of products of local bilinears  $\bar{\psi}(x)\psi(x)$ , at finite density for two-dimensional Quantum Chromodynamics with flavor. Using a path-integral approach which is very appropriate to handle non-Abelian gauge theories, we show that the multipoint chiral condensates exhibit an oscillatory inhomogeneous behavior depending on a chemical potential matrix. Our results are exact and, remarkably, go in the same direction as those revealed in four dimensions using the  $1/N_c$  approximation to  $QCD$  [3].

To study the effect of finite fermion density in  $QCD_2$  a chemical potential may be introduced. Within the path-integral approach this amounts to consider a classical background charge distribution in addition to that produced by topologically non-trivial gauge configurations. Concerning this last point, it is well-known that in two space-time dimensions the role of instantons is played by vortices. In the Abelian case, these vortices are identified with the Nielsen-Olesen solutions of the spontaneously broken Abelian Higgs model [8]. Also in the non-Abelian case, regular solutions with topological charge exist when symmetry breaking is appropriately achieved via Higgs fields [9]-[10]. In both cases the associated fermion zero modes have been found [11]-[13].

Properties of the vortex solutions and the corresponding Dirac equation zero-modes are summarized in section 2. We then describe in sections 3 and 4 how topological effects can be taken into account within the path-integral formulation leading to a compact form for the partition function in the presence of a chemical potential. Our approach, following ref. [14], starts by decomposing a given gauge field belonging to the  $n^{th}$  topological sector in the form

$$A_\mu(x) = A_\mu^{(n)} + A_\mu^{ext} + a_\mu \quad (1)$$

Here  $A_\mu^{(n)}$  is a (classical) fixed gauge field configuration belonging to the  $n^{th}$  class,  $A_\mu^{ext}$  is the background charge field taking account of the chemical potential, and  $a_\mu$  is the

path-integral variable which represents quantum fluctuations. Both  $A_\mu^{ext}$  and  $a_\mu$  belong to the trivial topological sector and can be then decoupled by a chiral rotation with the sole evaluation of a Fujikawa jacobian [15]. This last calculation can be easily performed since it is to be done in the trivial topological sector.

The complete calculation leading to the minimal non-trivial correlation functions of fermion bilinears is first presented for multiflavour  $QED_2$  (Section 3) and then extended to multiflavour  $QCD_2$  (Section 4). In both cases the oscillatory behavior of correlators as a function of the chemical potential is computed, the result showing a striking resemblance with the  $QCD_4$  answer obtained within the large  $N_c$  approximation [3]. We summarize our results and conclusions in section 5.

## 2 Zero Modes

Topological gauge field configurations and the corresponding zero-modes of the Dirac equation play a central role in calculations involving fermion composites. We summarize in this section the main properties of vortices, the relevant topological objects in the model we shall consider, both for the Abelian and non-Abelian cases. We also present the corresponding Dirac operator zero-modes.

### 2.1 The Abelian case

In two-dimensional Euclidean space-time, topologically non-trivial gauge field configurations are available since Nielsen and Olesen [8] presented their static  $z$ -independent vortex. In the  $U(1)$  case the topological charge for such a configuration, working in an arbitrary compact surface (like a sphere or a torus) is defined as

$$\frac{1}{4\pi} \int d^2x \epsilon_{\mu\nu} F_{\mu\nu}^{(n)} = n \in \mathbb{Z} \quad (2)$$

A representative gauge field configuration carrying topological charge  $n$  can be written as

$$A_\mu^{(n)} = n \epsilon_{\mu\nu} \frac{x_\nu}{|x|} A(|x|) \quad (3)$$

with  $A(|x|)$  a function which can be calculated numerically (an exact solution exists under certain conditions on coupling constants, [9]). The adequate boundary conditions are

$$A(0) = 0 \quad , \quad \lim_{|x| \rightarrow \infty} A(|x|) = -1 \quad (4)$$

There are  $|n|$  zero-modes associated with the Dirac operator in the background of an  $A_\mu^{(n)}$  configuration in a suitable compactified space-time [14]. (For the non-compact case see [11]). For  $n > 0$  ( $n < 0$ ) they correspond to right-handed (left-handed) solutions  $\eta_R$  ( $\eta_L$ ) which in terms of light-cone variables  $z = x_0 + ix_1$  and  $\bar{z} = x_0 - ix_1$  can be written in the form

$$\eta_R^m = \begin{pmatrix} z^m h(z, \bar{z}) \\ 0 \end{pmatrix} \quad (5)$$

$$\eta_L^m = \begin{pmatrix} 0 \\ \bar{z}^{-m} h^{-1}(z, \bar{z}) \end{pmatrix} \quad (6)$$

where  $m = 0, 1, \dots, |n| - 1$ ,

$$h(z, \bar{z}) = \exp[\phi^{(n)}(|z|)] \quad (7)$$

and

$$\frac{d}{d|z|} \phi^{(n)}(|z|) = nA(|z|). \quad (8)$$

## 2.2 The non-Abelian case

As in the Abelian case, two-dimensional gauge field configurations  $A_\mu^{(n)}$  carrying a topological charge  $n \in Z_N$  can be found for the  $SU(N)$  case. As explained in ref.[16] the relevant homotopy group is in this case  $Z_N$  and not  $Z$  as in the  $U(1)$  case.

Calling  $\varphi$  the angle characterizing the direction at infinity, a mapping  $g_n(\varphi) \in SU(N)$  belonging to the  $n^{th}$  homotopy class ( $n = 0, 1, \dots, N-1$ ) satisfies, when one turns around a close contour,

$$g_n(2\pi) = \exp\left(\frac{2\pi i n}{N}\right) g_n(0) \quad (9)$$

Such a behavior can be achieved just by taking  $g_n$  in the Cartan subgroup of the gauge group. For example, in the  $SU(2)$  case one can take

$$g_n(\varphi) = \exp\left[\frac{i}{2} \sigma^3 \Omega_n(\varphi)\right] \quad (10)$$

with

$$\Omega_n(2\pi) - \Omega_n(0) = 2\pi(2k + n) \quad (11)$$

Here  $n = 0, 1$  labels the topological charge and  $k \in Z$  is a second integer which connects the topological charge with the vortex magnetic flux (Only for abelian vortices both quantities coincide).

We can then write a gauge field configuration belonging to the  $n^{th}$  topological sector in the form

$$A_\mu^{(n)} = iA(|x|) g_n^{-1} \partial_\mu g_n \quad (12)$$

with the boundary conditions

$$A(0) = 0 \quad , \quad \lim_{|x| \rightarrow \infty} A(|x|) = -1 \quad (13)$$

These and more general vortex configurations have been thoroughly studied in [10]-[16].

Concerning zero-modes of the Dirac operator in the background of non-Abelian vortices, they have been analysed in refs.[12]-[13]. The outcome is that for topological charge  $n > 0$  ( $n < 0$ ) there are  $Nn$  ( $N|n|$ ) square-integrable zero modes  $\eta_L$  ( $\eta_R$ ) analogous to those arising in the Abelian case. Indeed, one has

$$\eta_R^{(m,i)j} = \begin{pmatrix} z^m h_{ij}(z, \bar{z}) \\ 0 \end{pmatrix} \quad (14)$$

$$\eta_L^{(m,i)j} = \begin{pmatrix} 0 \\ \bar{z}^{-m} h_{ij}^{-1}(z, \bar{z}) \end{pmatrix} \quad (15)$$

with

$$h(z, \bar{z}) = \exp[\phi^{(n)}(|z|)M] \quad (16)$$

and

$$M = \frac{1}{N} \text{diag}(1, 1, \dots, 1 - N) \quad (17)$$

Here  $i, j = 1, 2, \dots, N$  and  $m = 0, 1, \dots, |n| - 1$ . The pair  $(m, i)$  labels the  $N|n|$  different zero modes while  $j$  corresponds to a color index. Due to the ansatz discussed in refs.[10]-[16] for the non-Abelian vortex, the function  $\phi^{(n)}(|z|)$  appearing in eq.(16) coincides with that arising in eqs.(7)-(8) for the abelian vortex.

As it happens in the abelian case, the partition function of two dimensional Quantum Chromodynamics only picks a contribution from the trivial sector because  $\det(\not{D}[A^{(n)}]) = 0$  for  $n \neq 0$  (see eq.(68) below). In contrast, various correlation functions become non-trivial precisely for  $n \neq 0$  thanks to the “absortion” of zero-mode contributions when Grassman integration is performed.

It is our aim to see how these non-trivial correlators are modified when a fermion finite density constraint is introduced, comparing the results with those of the unconstrained (zero chemical potential) case. As explained in the introduction, we are motivated by the results of Deryagin, Grigoriev and Rubakov [3] in four dimensional  $QCD$ . They were able to show, in the large  $N_c$  and high fermion density limits, the existence of oscillatory condensates (the frequency given by the chemical potential) which are spatially inhomogeneous. For  $QED_2$  the same oscillatory behavior was found approximately in [6] and confirmed analytically in [7], by examining an arbitrary number of fermion bilinears for which the exact  $\mu$ -dependence of fermionic correlators was computed. In order to improve our understanding of the large  $N_c$  results found in  $QCD_4$ , we shall extend in what follows our two-dimensional approach to the non-Abelian case but before, we shall consider the case of flavored  $QED_2$  as a clarifying step towards multiflavor  $QCD_2$ .

### 3 Multiflavour $QED_2$

We developed in ref.[7] a path-integral method to compute fermion composites for Abelian gauge theories including chemical potential effects. In this section we briefly describe our approach while extending our treatment so as to include flavour. We then leave for section 4 the analysis of the non-Abelian multiflavour  $QCD_2$  model at finite density.

#### (i) Handling the chemical potential in the Abelian case

We start from the Lagrangian

$$L = -\frac{1}{4e^2} F_{\mu\nu} F_{\mu\nu} + \bar{\psi}(i\not{\partial} + \not{A} - i\mathcal{M}\gamma_0)\psi \quad (18)$$

where  $\psi$  is the fermion field isospinor. A chemical potential term has been included by considering the diagonal matrix  $\mathcal{M}$  defined as

$$\mathcal{M} = \text{diag}(\mu_1 \dots \mu_{N_f}) \quad (19)$$

where  $N_f$  is the total number of flavors and  $\mu_k$  are Lagrange multipliers carrying a flavour index, so that each  $k$ -fermion number is independently conserved. The corresponding partition function is defined as

$$Z[\mu_1 \dots \mu_{N_f}] = \int \mathcal{D}\bar{\psi} \mathcal{D}\psi \mathcal{D}A_\mu \exp(- \int d^2x L). \quad (20)$$

Since our interest is the computation of fermionic correlators, we have to carefully treat non-trivial topological configurations of the gauge fields which have been seen to be crucial in the obtention of non-vanishing condensates, see refs.[19]-[20]. Then, following the approach of refs.[14]-[20], we decompose gauge field configurations belonging to the  $n^{th}$  topological sector in the form

$$A_\mu(x) = A_\mu^{(n)}(x) + a_\mu(x) \quad (21)$$

where  $A_\mu^{(n)}$  is a fixed classical configuration carrying all the topological charge  $n$ , and  $a_\mu$ , the path integral variable, accounts for the quantum “fluctuations” and belongs to the trivial sector  $n = 0$ .

As it is well-known [18], the chemical potential term can be represented by a vector field  $A_\mu^{ext}$  describing an *external* charge density acting on the quantum system. Indeed, taking  $A_\mu^{ext}$  as  $i$  times the chemical potential matrix (see eqs.(19) and (22)) it corresponds to a uniform charge background for each fermionic flavor. As explained in [7], it is convenient to first consider a finite length ( $2L$ ) distribution and then take the  $L \rightarrow \infty$  limit. In this way translation symmetry breaking associated to the chemical potential becomes apparent and simultaneously, ambiguities in the definition of the finite density theory are avoided (see ref.[4] for a discussion on this issue). Therefore, we define

$$A_\nu^{ext} = -i\mathcal{M} \delta_{\nu 0}, \quad (22)$$

so that the Dirac operator

$$i\cancel{\partial} + \cancel{A} - i\mathcal{M}\gamma_0 \quad (23)$$

can be compactly written as

$$i\cancel{\partial} + \cancel{A}' \quad (24)$$

with

$$A'_\mu = A_\mu + A_\mu^{ext} \quad (25)$$

We shall now proceed to a decoupling of fermions from the chemical potential and the  $a_\mu$  fluctuations following the steps described in [7] for the case of only one flavor. In that case, we wrote

$$a_\mu = -\epsilon_{\mu\nu} \partial_\nu \phi + \partial_\mu \eta \quad (26)$$

and made a chiral rotation to decouple both the  $\phi - \eta$  fields together with the chemical potential. In order to include  $N_f$  flavors in the analysis, one has to replace  $(\phi, \eta) \rightarrow (\phi, \eta)\mathbf{1}_f$  and  $\mu \rightarrow \mathcal{M}$  as we shall see below. Then, we can straightforwardly apply what we have learnt for one flavor [7] in the multiflavor case. The change of variables accounting

for the decoupling of fermions from the  $a_\mu$  field together with the chemical potential is given by

$$\psi = \exp[\gamma_5 (\phi(x)\mathbf{1}_f + i\mathcal{M}x_1) + i\eta(x)\mathbf{1}_f] \chi \quad (27)$$

$$\bar{\psi} = \bar{\chi} \exp[\gamma_5 (\phi(x)\mathbf{1}_f + i\mathcal{M}x_1) - i\eta(x)\mathbf{1}_f]$$

$$\not{d} = (-i\not{\partial}U) U^{-1} \quad (28)$$

where

$$U = \exp[\gamma_5 (\phi\mathbf{1}_f + i\mathcal{M}x_1) + i\eta\mathbf{1}_f] \quad (29)$$

For notation compactness we have included in  $\not{d}$  the external field  $A_\mu^{ext}$  describing the chemical potential term. From here on we choose the Lorentz gauge to work in (which in our notation corresponds to  $\eta = 0$ ).

After transformation (27) the resulting Dirac operator takes the form

$$i\not{D} = i\not{\partial} + \not{A}^{(n)} + \not{d} \rightarrow i\not{\partial} + \not{A}^{(n)}. \quad (30)$$

The jacobian associated with the chiral rotation of the fermion variables can be easily seen to be [7]

$$J = \exp\left(\frac{\mathbf{1}_f}{2\pi} \int d^2x (\phi + i\mathcal{M}x_1) \square(\phi + 2\phi^{(n)})\right) \quad (31)$$

where  $\phi^{(n)}$  is defined by

$$A_\mu^{(n)} = -\epsilon_{\mu\nu} \partial_\nu \phi^{(n)} \mathbf{1}_f$$

Together with eq.(27) we consider the change in the gauge-field variables  $a_\mu$  so that

$$\mathcal{D}a_\mu = \Delta_{FP} \delta(\eta) \mathcal{D}\phi \mathcal{D}\eta \quad (32)$$

with  $\Delta_{FP} = \det \square$

As thoroughly analysed by Actor [18],  $A_\mu^{ext}$  does not correspond to a pure gauge. Were it not so, the introduction of a chemical potential would not have physical consequences and this would be the case in any space-time dimensions. In fact, one cannot gauge away  $L_{chem}$  by means of a bounded gauge transformation. As explained in [7], the chiral rotation which decouples the chemical potential, although unbounded can be properly handled by putting the system in a spatial box, then introducing adequate counterterms and finally taking the infinite volume limit.

After the decoupling, the partition function, can be written in the form

$$Z = \mathcal{N} \sum_n \int \mathcal{D}\bar{\chi} \mathcal{D}\chi \mathcal{D}\phi \exp(-S_{eff}^{(n)}) \quad (33)$$

where  $S_{eff}^{(n)}$  is the effective action in the  $n^{th}$  topological sector,

$$S_{eff}^{(n)} = \int d^2x \bar{\chi} (i\not{\partial} + \not{A}^{(n)}) \chi - \frac{\mathbf{1}_f}{2e^2} \int d^2x ((\square\phi)^2 + \epsilon_{\mu\nu} F_{\mu\nu}^{(n)} \square\phi)$$

$$- \frac{\mathbf{1}_f}{4e^2} \int d^2x (F_{\mu\nu}^{(n)})^2 - \frac{\mathbf{1}_f}{2\pi} \int d^2x (\phi + i\mathcal{M}x_1) \square(\phi + 2\phi^{(n)}) + S_c \quad (34)$$

The usual divergency associated to the electromagnetic energy carried by fermions has to be eliminated by an appropriate counterterm  $S_c$  [4]. In our approach the divergency manifests through the term  $i\mathcal{M}x_1\Box\phi^{(n)}$  in eq.(34). This counterterm is the Lagrangian counterpart of the one usually employed in the Hamiltonian approach to handle this problem [4]. In the canonical formulation of QFT this is equivalent to a redefinition of creation and annihilation operators which amounts to a shift in the scale used to measure excitations. Note that with the choice of the counterterm discussed above, the effective action written in terms of the decoupled fermions does not depend on the chemical potentials  $\mu_k$ . This does not mean that this term has no physical consequences. In fact,  $\mathcal{M}$  reappears when computing correlation functions of fermion fields, once  $\bar{\psi}$  and  $\psi$  are written in terms of the decoupled fields  $\bar{\chi}$  and  $\chi$  through eq.(46). We shall see in the following sections how fermionic correlators are changed, exhibiting oscillatory inhomogeneities in the spatial axes which depend on  $\mathcal{M}$ . The fact that zero modes make certain v.e.v.'s not to vanish, leads to a highly non-trivial dependence on the chemical potentials.

## (ii) The Correlation Functions

The introduction of a flavor index implies additional degrees of freedom which result in  $N_f$  independent fermionic field variables. Consequently, the growing number of Grassman (numeric) differentials calls for additional Fourier coefficients in the integrand.

It is well known that each coefficient is related to the quantum numbers of the chosen basis, which is normally builded up from the eigenfunctions of the Dirac operator. As we have for one flavor, one has that  $n$  of these eigenfunctions are zero-modes, implying a vanishing fermionic exponential. Hence, in order to make Grassman integrals non-trivial, one has to insert several bilinears depending on the number of zero modes. When the path-integral measure contains  $N_f$  independent fermionic fields instead of one, the number of composite insertions is multiplied by  $N_f$  in order to saturate Grassman integration algebra, with some selection rules which will become apparent below.

For the sake of brevity let us readily give the result for general correlation functions of  $p$  points with arbitrary right and left insertions

$$C(w_1, w_2, \dots) = \langle \prod_{k=1}^{N_f} \prod_{i=1}^{r_k} s_+^k(w^i) \prod_{j=1'}^{s_k} s_-^k(w^j) \rangle \quad (35)$$

where

$$s_{\pm}^k(w^i) \equiv \bar{\psi}_{\pm}^k(w^i) \psi_{\pm}^k(w^i) , \quad (36)$$

$$p = \sum_{k=1}^{N_f} p_k$$

and

$$r_k + s_k = p_k$$

is the total number of insertions in the flavor sector  $k$ .

After the abelian decoupling, eq.(35) results in

$$C(w_1, w_2, \dots) = \frac{1}{Z} \sum_{n=1}^{\infty} \int \mathcal{D}\phi \exp\left[\frac{N_f}{2\pi} \int d^2x \phi \Box(\phi + \phi^{(n)})\right] \times$$



$$\begin{aligned}
& \exp\left[-\frac{1}{e^2} \int d^2x (\phi + \phi^{(n)}) \square \square (\phi + \phi^{(n)})\right] \times \\
& \exp\left[2 \sum_{k=1}^{N_f} \left(\sum_{i=1}^{r_k} \phi(w^i) - \sum_{j=1'}^{s_k} \phi(w^j)\right)\right] \times \\
& \prod_{k=1}^{N_f} \exp\left[2i\mu_k \left(\sum_{i=1}^{r_k} w_1^i - \sum_{j=1'}^{s_k} w_1^j\right)\right] \int \mathcal{D}\bar{\chi}^k \mathcal{D}\chi^k \prod_{i=1}^{r_k} \bar{\chi}_+^k(w^i) \chi_+^k(w^i) \times \\
& \prod_{j=1'}^{s_k} \bar{\chi}_-^k(w^j) \chi_-^k(w^j) \exp\left[-\int d^2x \bar{\chi}^k (i\not{\partial} + A^{(n)}) \chi^k\right]
\end{aligned} \tag{37}$$

where  $w_1^i$  is the space component of  $w^i$ . We see from eq.(37) that the chemical potential contribution is, as expected, completely factorized. Concerning the bosonic integral, it can be written as

$$\begin{aligned}
B &= \exp\left[N_f/2\pi \int d^2x \phi^{(n)} \square \phi^{(n)}\right] \exp\left[-2 \sum_{k=1}^{N_f} \left(\sum_{i=1}^{r_k} \phi^{(n)}(w^i) - \sum_{j=1'}^{s_k} \phi^{(n)}(w^j)\right)\right] \times \\
& \exp\left[-2 \sum_{k,k'=1}^{N_f} \sum_{i=1}^{p_k} \sum_{j=1}^{p_{k'}} e_i e_j O^{-1}(w^i, w^j)\right]
\end{aligned} \tag{38}$$

with

$$O^{-1}(w^i, w^j) = K_0(m|w^i - w^j|) + \ln(c|w^i - w^j|); \quad m = e\sqrt{N_f/\pi}$$

The fermionic path-integral determines the topological sectors contributing to equation (35). More precisely, once the correlator to be computed has been chosen, Grassman integration leads to a non-zero answer only when the number of right insertions minus the number of left insertions is the same in every flavor sector. It means that  $r_k - s_k = t \quad \forall k$ , where  $t$  is the only topological flux number surviving the leading sumatory in eq.(37). (Notice that mixed flavor indices in the elementary bilinear are avoided, i.e. we are not including flavor-violating vertices, in accordance with  $QED_4$  interactions). It is important to stress that each term explicitly including the classical configuration of the flux sector cancels out. Consequently, classical configurations only appear through by means of their global (topological) properties, namely, through the difference in the number of right and left handed bilinears [14].

To conclude, we give the final result for the general correlator defined in eq.(35) making use of the explicit form of abelian zero modes

$$\begin{aligned}
& \left\langle \prod_{k=1}^{N_f} \prod_{i=1}^{r_k} s_+^k(w^i) \prod_{j=1'}^{s_k} s_-^k(w^j) \right\rangle = \left(-\frac{me^\gamma}{4\pi}\right)^p \\
& \exp\left[2i \sum_{k=1}^{N_f} \mu_k \left(\sum_{i=1}^{r_k} w_1^i - \sum_{j=1'}^{s_k} w_1^j\right)\right] \prod_{k>k'=1}^{N_f} \exp\left[-4 \sum_{i=1}^{p_k} \sum_{j=1}^{p_{k'}} e_i e_j \ln(c|w^i - w^j|)\right] \\
& \exp\left[-\sum_{k,k'=1}^{N_f} \sum_{i=1}^{p_k} \sum_{j=1}^{p_{k'}} e_i e_j K_0(m|w^i - w^j|)\right]
\end{aligned} \tag{39}$$

(see Ref.[7] and [24] for details).

In order to clearly see the meaning of this expression, let us show the result for the simplest non-trivial flavored correlation functions including mixed right and left handed insertions

$$\begin{aligned} \sum_n \langle \bar{\psi}^1 \psi^1(x) \bar{\psi}^1 \psi^1(y) \bar{\psi}^1 \psi^1(z) \bar{\psi}^2 \psi^2(w) \rangle_n = \\ 2 \cos[\mu_1(z_1 - x_1 - y_1) - \mu_2 w_1] \langle s_+^1(x) s_+^1(y) s_-^1(z) s_+^2(w) \rangle_1 + \\ 2 \cos[\mu_1(y_1 - x_1 - z_1) - \mu_2 w_1] \langle s_+^1(x) s_-^1(y) s_+^1(z) s_+^2(w) \rangle_1 + \\ 2 \cos[\mu_1(x_1 - z_1 - y_1) - \mu_2 w_1] \langle s_-^1(x) s_+^1(y) s_+^1(z) s_+^2(w) \rangle_1, \end{aligned} \quad (40)$$

$$\begin{aligned} \sum_n \langle \bar{\psi}^1 \psi^1(x) \bar{\psi}^1 \psi^1(y) \bar{\psi}^2 \psi^2(z) \bar{\psi}^2 \psi^2(w) \rangle_n = \\ 2 \cos[\mu_1(x_1 - y_1) - \mu_2(z_1 - w_1)] \langle s_+^1(x) s_-^1(y) s_-^2(z) s_+^2(w) \rangle_0 + \\ 2 \cos[\mu_1(x_1 - y_1) + \mu_2(z_1 - w_1)] \langle s_+^1(x) s_-^1(y) s_+^2(z) s_-^2(w) \rangle_0 + \\ 2 \cos[\mu_1(x_1 + y_1) + \mu_2(z_1 + w_1)] \langle s_+^1(x) s_-^1(y) s_-^2(z) s_+^2(w) \rangle_2 \end{aligned} \quad (41)$$

These expressions make apparent: (i) How the topological structure of the theory exhibits itself through the existence of non-trivial vacuum expectation values of fermionic bilinears. (Notice that those on the right hand side are the only surviving terms of the whole sumatory). (ii) In the multiflavor case, the path-integrals are non-zero only when the number of right insertions minus the number of left insertions are identical in every flavor sector. (iii) The sum over spatial coordinates dramatically exhibits the translation symmetry breaking discussed above. (iv) The fixing of various fermion densities implies a somehow reacher spatial inhomogeneity of the results with respect to the one flavor case that we have analyzed in [7], in the sense that now the “angles” depend on various chemical potentials. (v) Another difference with respect to the one flavor case, concerns the trivial cancellation of logarithms coming from bosonic and fermionic integration respectively. Now, this cancellation occuring for one flavor, does not take place anymore, see eq.(39).

## 4 Multiflavour $QCD_2$

In the present section we consider two dimensional  $SU(N_c)$  Yang-Mills gauge fields coupled to massless Dirac fermions in the fundamental representation. Due to the non-Abelian character of the gauge symmetry, gluons are charged fields that preserve color flux at each vertex. Since a colored quark density is not a quantity to be kept constant, no chemical potential related to color should be considered but only that associated with the global symmetry that yields fermion number conservation. Hence, we first include one chemical potential term and then consider a different lagrange multiplier for each fermionic flavor.

Let us stress that once the topological effects arising from vortices are taken into account and the chemical potential behavior of fermion correlators is identified, we do not pursue calculations in the bosonic sector (neither we consider the inclusion of Higgs scalars, necessary at the classical level for the existence of regular vortex solutions). As we shall see, the boson contribution to the fermion condensate just factorizes and all the chemical potential effects can be controlled by calculations just performed within the fermionic sector.

### (i) Handling the Chemical Potential in $QCD_2$

We start from the massless  $QCD_2$  (Euclidean) Lagrangian

$$L = \bar{\psi}^q (i\partial_\mu \gamma_\mu \delta^{qq'} + A_{\mu,a} t_a^{qq'} \gamma_\mu - i\mu \gamma_0 \delta^{qq'}) \psi^{q'} + \frac{1}{4g^2} F_{\mu\nu}^a F_{\mu\nu}^a. \quad (42)$$

where we have included a chemical potential term in the form

$$L_{chem} = -i\mu \psi^\dagger \psi \quad (43)$$

in order to take care of the fermion density constraint. Here  $a = 1 \dots N_c^2 - 1$ , and  $q = 1 \dots N_c$ . The partition function reads

$$Z[\mu] = \int \mathcal{D}\bar{\psi} \mathcal{D}\psi \mathcal{D}A_\mu \exp[-\int d^2x \exp L]. \quad (44)$$

Again, one can decouple the chemical potential by performing an appropriate chiral rotation for the fermion variables. Indeed, under the transformation

$$\begin{aligned} \psi &= \exp(i\mu \gamma_5 x_1) \chi \\ \bar{\psi} &= \bar{\chi} \exp(i\mu \gamma_5 x_1) \end{aligned} \quad (45)$$

the fermion Lagrangian becomes

$$L = \bar{\psi} \not{D}[A, \mu] \psi \rightarrow \bar{\chi} \not{D}[A] \chi \quad (46)$$

so that the chemical potential completely disappears from the fermion Lagrangian. As we have seen, chiral transformations may generate a Fujikawa jacobian which has to be computed using some regularization procedure. For example, using the heat-kernel regularization one introduces a resolution of the identity of the form

$$1 = \lim_{M \rightarrow \infty} \exp(-\not{D}(\alpha)^2/M^2). \quad (47)$$

where  $D_\mu(\alpha)$  ( $\alpha \in (0, 1)$ ) is an interpolating Dirac operator such that  $D_\mu(\alpha = 0) = D_\mu[A, \mu]$  and  $D_\mu(\alpha = 1) = D_\mu[A]$ .

After some standard calculation [21] one ends with a Jacobian of the form

$$J = \exp \left( i\epsilon_{\mu\nu}/4\pi \int_0^1 d^2x \, d\alpha \, \text{tr}^c [\mu x_1 F_{\mu\nu}(\alpha)] \right) \quad (48)$$

where  $\text{tr}^c$  is the trace with respect to color indices and

$$F_{\mu\nu}(\alpha) = F_{\mu\nu}^a(\alpha) t^a, \quad a = 1, 2, \dots, N_c^2 - 1 \quad (49)$$

Now, the color trace in eq.(48) vanishes and then the chiral Jacobian is in fact trivial,

$$J = 1 \quad (50)$$

We can then write the partition function (44) after the fermion rotation defined in eq.(27) in the form

$$Z[\mu] = \int \mathcal{D}A_\mu \mathcal{D}\bar{\chi} \mathcal{D}\chi \exp(-\int d^2x L) \quad (51)$$

As we have seen in the Abelian case, although  $\mu$  is absent from the r.h.s. of eq.(51) one should not conclude that physics is independent of the chemical potential. For correlation functions of composite operators which are not chiral invariant, the chemical potential will reappear when rotating the fermion variables in the fermionic bilinears. As in the abelian case, this happens when computing v.e.v.'s of products  $\bar{\psi}(x)\psi(x)$

## (ii) Correlation functions in $QCD_2$ with chemical potential

Our main interest is the computation of fermionic correlators containing products of local bilinears  $\bar{\psi}\psi(x)$  for which non-trivial topological gauge field configurations, and the associated Dirac operator zero-modes, will be crucial to the obtention of non-vanishing results as explained in refs.[7],[14]-[20].

As in section 3, we start by writing a gauge field belonging to the  $n^{th}$  topological sector, in the form

$$A_\mu^a(x) = A_\mu^{a(n)}(x) + a_\mu^a(x) \quad (52)$$

where  $A_\mu^{a(n)}$  is a fixed classical configuration (as described in section 2.2) carrying all the topological charge  $n$ , and  $a_\mu^a$ , will be the actual integration variable which belongs to the trivial sector  $n = 0$ . Then, we decouple the  $a_\mu$  field from the fermions through an appropriate rotation (the calculation of the Fujikawa Jacobian being standard since the decoupling corresponds to the topologically trivial sector). Now, it will be convenient to choose the background so that

$$A_+^{a(n)} = 0 \quad (53)$$

In this way, the Dirac operator takes the form<sup>1</sup>

$$\mathcal{D}[A^{(n)} + a] = \begin{pmatrix} 0 & \partial_+ + a_+ \\ \partial_- + A_-^{(n)} + a_- & 0 \end{pmatrix} \quad (54)$$

and we are left with the determinant of this operator once fermions are integrated out

$$Z[\mu] = \sum_n \int \mathcal{D}a_\mu \exp\left[\frac{1}{4g^2} F_{\mu\nu}^2[A^{(n)} + a]\right] \det \mathcal{D}[A^{(n)} + a]. \quad (55)$$

As before, we have introduced a sum over different topological sectors. Now, we shall factor out the determinant in the classical background so as to control the zero mode problem. Let us start by introducing group valued fields to represent  $A^{(n)}$  and  $a_\mu$

$$a_+ = iu^{-1}\partial_+u \quad (56)$$

$$a_- = id(v\partial_-v^{-1})d^{-1} \quad (57)$$

$$A_-^{(n)} = id\partial_-d^{-1}. \quad (58)$$

Consider first the light-cone like gauge choice [22]

$$A_- = A_-^{(n)} \quad (59)$$

implying

$$v = I. \quad (60)$$

In this gauge the Dirac operator (54) reads

$$\mathcal{D}[A^{(n)} + a]|_{lc} = \begin{pmatrix} 0 & \partial_+ + iu^{-1}\partial_+u \\ \partial_- + A_-^{(n)} & 0 \end{pmatrix} \quad (61)$$

---

<sup>1</sup>We are using  $\gamma_0 = \sigma_1$  and  $\gamma_1 = -\sigma_2$ .

where subscript  $lc$  means that we have used the gauge condition (59). One can easily see (for example by rotating the  $+$  sector with  $u^{-1}$  while leaving the  $-$  sector unchanged) that

$$\det \mathcal{D}[A^{(n)} + a]|_{lc} = \mathcal{N} \det \mathcal{D}[A^{(n)}] \times \exp(W[u, A^{(n)}]). \quad (62)$$

Here  $W[u, A^{(n)}]$  is the gauged Wess-Zumino-Witten action which in this case takes the form

$$W[u, A^{(n)}] = W[u] + \frac{1}{4\pi} tr_c \int d^2x (u^{-1} \partial_+ u) (d \partial_- d^{-1}) \quad (63)$$

and  $W[u]$  is the Wess-Zumino-Witten action

$$W[u] = \frac{1}{2\pi} tr_c \int d^2x \partial_\mu u^{-1} \partial_\mu u + \frac{e^{ijk}}{4\pi} tr_c \int_B d^3y (u^{-1} \partial_i u) (u^{-1} \partial_j u) (u^{-1} \partial_k u). \quad (64)$$

Note that in writing the fermion determinant in the form (62), the zero-mode problem has been circumscribed to the classical background fermion determinant.

One can easily extend the result (62) to an arbitrary gauge, in terms of the group-valued fields  $u$  and  $v$  defined by eqs.(56)-(57), by repeated use of the Polyakov-Wiegmann identity [23]

$$W[pq] = W[p] + W[q] + \frac{1}{4\pi} tr_c \int d^2x (p^{-1} \partial_+ p) (q \partial_- q^{-1}) \quad (65)$$

The answer is

$$\det \mathcal{D}[A^{(n)} + a] = \mathcal{N} \det \mathcal{D}[A^{(n)}] \times \exp(S_{eff}[u, v; A^{(n)}]) \quad (66)$$

$$\begin{aligned} S_{eff}[u, v; A^{(n)}] &= W[u, A^{(n)}] + W[v] + \frac{1}{4\pi} tr_c \int d^2x (u^{-1} \partial_+ u) d(v \partial_- v^{-1}) d^{-1} \\ &\quad + \frac{1}{4\pi} tr_c \int d^2x (d^{-1} \partial_+ d) (v \partial_- v^{-1}). \end{aligned} \quad (67)$$

Once one has the determinant in the form (66), one can work with any gauge fixing condition. The gauge choice (59) is in principle not safe since the corresponding Faddeev-Popov determinant is  $\Delta = \det D_-^{adj}[A^{(n)}]$  implying the possibility of new zero-modes. A more appropriate choice would be for example  $A_+ = 0$ , having a trivial FP determinant. In any case one ends with a partition function showing the following structure

$$\begin{aligned} Z &= \sum_n \det(\mathcal{D}[A^{(n)}]) \int \mathcal{D}a_\mu \Delta \delta(F[a]) \\ &\quad \exp \left( -S_{eff}[A^{(n)}, a_\mu] - \frac{1}{4g^2} \int d^2x F_{\mu\nu}^2[A^{(n)}, a_\mu] \right) \end{aligned} \quad (68)$$

Concerning the divergency associated to the external charge distribution, we have learnt from the Abelian case that one has to carefully handle this term in order to define excitations with respect to the external background. In section 3 we have seen that it came from the interaction of  $A^{ext}$  with  $F_{\mu\nu}^{(n)}$ , appearing in the fermionic jacobian. Performing a similar calculation in the present case we would find the non-Abelian analogue of this term with  $tr^c$  acting on it. As we have mentioned above, this color trace operation implies

the vanishing of the corresponding divergency so that no counterterm might be added in  $QCD_2$ , meaning that the relevant vacuum is properly defined.

As we have seen, the Lagrangian for  $QCD_2$  at finite density can be written in terms of  $\mu$ -rotated fields which hide the chemical potential from the partition function. This result however, does not exhausts the physics of the theory in the sense that correlation functions do depend on  $\mu$ . Actually, it will be shown that the chemical potential dependence appears as a factor multiplying the result for correlators of the unconstrained theory. For this reason, we shall first describe the computation of vacuum expectation values of fermion bilinears in the  $\mu = 0$  case and then consider how this result is modified at finite fermion density. Hence, we proceed with the analysis of v.e.v's of products of bilinears like  $\bar{\chi}\chi$ . Let us start by noting that with the choice (53) for the classical field configuration, the Dirac equation takes the form

$$\not{D}[A^{(n)} + a] \begin{pmatrix} \chi_+ \\ \chi_- \end{pmatrix} = \begin{pmatrix} 0 & u^{-1}i\partial_+ \\ dv d^{-1}D_-[A^{(n)}] & 0 \end{pmatrix} \begin{pmatrix} \zeta_+ \\ \zeta_- \end{pmatrix} \quad (69)$$

where  $\zeta$  is defined as

$$\begin{aligned} \chi_+ &= dv d^{-1} \zeta_+ \\ \chi_- &= u^{-1} \zeta_- \end{aligned} \quad (70)$$

so that the Lagrangian in the  $n^{th}$  flux sector can be written as

$$\begin{aligned} L &= \bar{\chi} \not{D}[a + A^{(n)}]\chi = \zeta_-^* i\partial_+ \zeta_- + \zeta_+^* \not{D}_-[A^{(n)}]\zeta_+ \\ &\equiv \bar{\zeta} \widetilde{D}[A^{(n)}]\zeta. \end{aligned} \quad (71)$$

In terms of these new fields, the bilinears  $\bar{\chi}\chi$  take the form

$$\bar{\chi}\chi = \zeta_-^* u dv d^{-1} \zeta_+ + \zeta_+^* dv^{-1} d^{-1} u^{-1} \zeta_- . \quad (72)$$

We observe that the jacobian associated to (70) is nothing else but the effective action defined in the previous section by eq.(67). Hence, an explicit expression for the non-abelian correlators reads

$$\begin{aligned} \langle \bar{\chi}\chi(x^1) \dots \bar{\chi}\chi(x^l) \rangle &= \sum_n \int \mathcal{D}a_\mu \Delta \delta(F[a_\mu]) \exp[-S_{eff}(A^{(n)}, a)] \\ &\int \mathcal{D}\bar{\zeta} \mathcal{D}\zeta \exp(\bar{\zeta} \begin{pmatrix} 0 & i\partial_+ \\ D_-[A^{(n)}] & 0 \end{pmatrix} \zeta) \\ &B^{q_1 p_1}(x^1) \dots B^{q_l p_l}(x^l) \zeta_-^{*q_1} \zeta_+^{p_1}(x^1) \dots \zeta_-^{*q_l} \zeta_+^{p_l}(x^l) + B^{q_1 p_1}(x^1) \dots \\ &B^{-1 q_1 p_1}(x^l) \zeta_-^{*q_1} \zeta_+^{p_1}(x^1) \dots \zeta_+^{*q_l} \zeta_-^{p_l}(x^l) + B^{q_1 p_1}(x^1) \dots \\ &B^{-1 q_{l-1} p_{l-1}}(x^{l-1}) B^{-1 q_l p_l}(x^l) \zeta_-^{*q_1} \zeta_+^{p_1}(x^1) \dots \zeta_+^{*q_{l-1}} \zeta_-^{p_{l-1}}(x^{l-1}) \zeta_+^{*q_l} \zeta_-^{p_l}(x^l) \\ &+ \dots \end{aligned} \quad (73)$$

where the group-valued field  $B$  is given by

$$B = u dv d^{-1}.$$

For brevity we have written the gauge field measure in terms of the original fields  $a_\mu$  although for actual calculations in the bosonic sector one has to work using  $u$  and  $v$  variables and proceed to a definite gauge fixing. That is, the measure should be written according to

$$Da_\mu \rightarrow DuDvJ_B(u, v, d)$$

and then the gauge condition and Faddeev-Popov determinant should be included (For example, in the light-cone gauge  $a_+ = 0$ ,  $u = 1$  and the FP determinant is trivial). Finally, notice that we have obtained a general and completely decoupled result, from which one sees that due to color degrees of freedom, the simple product that one finds in the Abelian case becomes here an involved summatory.

Now that we have an expression for correlators in the unconstrained case, let us include the chemical potential in our results. Recall that in this theory the partition function is (see eq.(51))

$$Z = \int \mathcal{D}A_\mu \mathcal{D}\bar{\chi} \mathcal{D}\chi \exp \left( - \int d^2x \bar{\chi} (i \not{\partial} + \not{A}) \chi + \frac{1}{4g^2} F_{\mu\nu} F_{\mu\nu} \right) \quad (74)$$

where  $\bar{\chi}, \chi$  represent the fermion fields after the chiral rotation (45) which eliminated the chemical potential from the Lagrangian. Since fermionic bilinears can be written as

$$\bar{\psi}\psi = \bar{\psi}_+\psi_+ + \bar{\psi}_-\psi_-,$$

one has

$$\langle \bar{\psi}\psi \rangle = \exp(2i\mu x_1) \langle \bar{\chi}_+\chi_+ \rangle + \exp(-2i\mu x_1) \langle \bar{\chi}_-\chi_- \rangle. \quad (75)$$

It can be easily seen that the same factorization occurs when flavor is introduced. The corresponding transformation is now

$$\begin{aligned} \psi &= \exp(i\mathcal{M}\mathbf{1}_c\gamma_5x_1)\chi \\ \bar{\psi} &= \bar{\chi} \exp(i\mathcal{M}\mathbf{1}_c\gamma_5x_1) \end{aligned} \quad (76)$$

and the bilinear v.e.v takes in this case the form

$$\langle \bar{\psi}\psi \rangle = \exp(2i\mathcal{M}\mathbf{1}_cx_1) \langle \bar{\chi}_+\chi_+ \rangle + \exp(-2i\mathcal{M}\mathbf{1}_cx_1) \langle \bar{\chi}_-\chi_- \rangle. \quad (77)$$

We shall then include from here on flavor degrees of freedom with the corresponding constraint on each fermion density. Since in this case one deals with  $N_f$  fermions coupled to the gauge field, we can use the fermionic jacobian we have computed for one flavor to the power  $N_f$  while the bosonic measure remains untouched. In the light-cone gauge it can be easily seen that the effective bosonic sector now involves  $N_c - 1$  massive scalars, their mass depending on flavor and color numbers by means of a factor  $(2N_c + N_f)^{1/2}$  with respect to the abelian counterpart (There is also the same number of unphysical massless particles [25]).

As we have previously explained, the Dirac operator has  $|n|N_c$  zero modes in the  $n^{th}$  topological sector, this implying that more fermion bilinears are needed in order to obtain a non-zero fermionic path-integral. Moreover, since the flavor index implies a factor  $N_f$

on the number of Grassman coefficients, the minimal non-zero product of fermion bilinears in the  $n^{th}$  sector requires  $|n|N_cN_f$  insertions.

Since the properties of the topological configurations are dictated by those of the torus of  $SU(N_c)$ , one can easily extend the results already obtained for  $QED_2$ . In particular, the chirality of the zero modes is dictated by the same index theorem found in the Abelian theory, this implying that in sector  $n > 0$  ( $n < 0$ ) every zero mode has positive (negative) chirality. In this way, the right (left) chiral projections of the minimal non-zero fermionic correlators can be easily computed. One gets

$$\begin{aligned} \langle \prod_k^{N_f} \prod_q^{N_c} \prod_i^{|n|} \bar{\psi}_+^{q,k} \psi_+^{q,k}(x_i^{q,k}) \rangle_n &= \frac{1}{Z^{(0)}} \int_{GF} \mathcal{D}u \mathcal{D}v J_B e^{-S_{Bef}^{(n)}(u,v,d)} \\ \prod_k^{N_f} \prod_q^{N_c} \prod_i^{|n|} \sum_{p_i, l_i}^{N_c} B_k'^{q, p_i l_i}(x_i^{q,k}) &\left( \int \mathcal{D}\bar{\zeta} \mathcal{D}\zeta e^{\int \bar{\zeta} \tilde{\mathcal{D}}[A^{(n)}] \zeta} \bar{\zeta}_+^{p_i} \zeta_+^{l_i}(x_i^q) \right)_k \end{aligned} \quad (78)$$

where

$$B_k'^{q, p_i l_i}(x) = \exp(2i\mu^k x_1) u^{p_i q}(x) (dv d^{-1})^{q l_i}(x), \quad (79)$$

$\bar{\zeta}_+ = \zeta_-^*$  and  $\tilde{D}[A^{(n)}]$  stands for the Dirac operator in the r.h.s of eq.(71). We have used the notation  $Z^{(0)}$  for the partition function since it is completely determined within the  $n = 0$  sector, see eq.(68). We have showed every color and flavor indices explicitly indicating sum and product operations. The  $GF$  label stands for the gauge fixing. The action  $S_{Bef}^{(n)}(u, v, d) = N_f S_{WZW}(u, v, d) + S_{Maxwell}(u, v, d)$  is given by the full gluon field  $A^{(n)}(d) + a(u, v)$ , and yields a high order Skyrme-type lagrangian [17].

Let us consider  $N_c = 2$  and  $N_f = 2$  in order to present the simplest illustration for the last expression. The minimal fermionic correlator then looks

$$\begin{aligned} \sum_n \langle \bar{\psi}_+^{1,1} \psi_+^{1,1}(x^1) \bar{\psi}_+^{1,2} \psi_+^{1,2}(x^2) \bar{\psi}_+^{2,1} \psi_+^{2,1}(y^1) \bar{\psi}_+^{2,2} \psi_+^{2,2}(y^2) \rangle_n &= \\ \frac{1}{Z^{(0)}} \sum_{p,q,r,s} \prod_{k=1}^2 \exp[2i\mu^k(x_1 + y_1)^k] \int_{GF} \mathcal{D}u \mathcal{D}v J_B e^{-S_{Bef}^{(1)}(u,v,d)} \times \\ B_k^{1,pq}(x^k) B_k^{2,rs}(y^k) \int \mathcal{D}\bar{\zeta}_k \mathcal{D}\zeta_k e^{\int \bar{\zeta}_k \tilde{D}[A^{(1)}] \zeta_k} \bar{\zeta}_+^{p,k} \zeta_+^{q,k}(x^k) \bar{\zeta}_+^{r,k} \zeta_+^{s,k}(y^k). \end{aligned} \quad (80)$$

The fermionic path-integral can be easily done, resulting in the product of eigenfunctions discussed in the sections above, as follows

$$\begin{aligned} \int \mathcal{D}\bar{\zeta}_k \mathcal{D}\zeta_k e^{\int \bar{\zeta}_k \tilde{D}[A^{(1)}] \zeta_k} \bar{\zeta}_+^{p,k} \zeta_+^{q,k}(x^k) \bar{\zeta}_+^{r,k} \zeta_+^{s,k}(y^k) &= \det \iota(\tilde{D}[A^{(1)}]) \times \\ \left( -\bar{\eta}_+^{(0,1)p,k} \eta_+^{(0,1)q,k}(x^k) \bar{\eta}_+^{(0,2)r,k} \eta_+^{(0,2)s,k}(y^k) + \bar{\eta}_+^{(0,1)p,k} \eta_+^{(0,2)q,k}(x^k) \right. \\ \bar{\eta}_+^{(0,2)r,k} \eta_+^{(0,1)s,k}(y^k) - \bar{\eta}_+^{(0,2)p,k} \eta_+^{(0,1)q,k}(x^k) \bar{\eta}_+^{(0,1)r,k} \eta_+^{(0,2)s,k}(y^k) \\ \left. + \bar{\eta}_+^{(0,2)p,k} \eta_+^{(0,2)q,k}(x^k) \bar{\eta}_+^{(0,1)r,k} \eta_+^{(0,1)s,k}(y^k) \right). \end{aligned} \quad (81)$$

Here  $\det \iota(\tilde{D}[A^{(1)}])$  is the determinat of the Dirac operator defined in eq.(71) omitting zero-modes and (e.g.)  $\eta_+^{(0,1)q,k}(x^k)$  is a non-Abelian zero-mode as defined in section 2, with an additional flavor index  $k$ . Concerning the bosonic sector, the presence of the  $F_{\mu\nu}^2$



(Maxwell) term crucially changes the effective dynamics with respect to that of a pure Wess-Zumino model. One then has to perform approximate calculations to compute the bosonic factor, for example, linearizing the  $U$  transformation, see [17]. In any case, once this task is achieved for the  $\mu = 0$  model, the modified (finite density) result can be obtained in an exact way.

## 5 Summary

We have presented the correlation functions of fermion bilinears in multiflavour  $QED_2$  and  $QCD_2$  at finite fermion density, using a path-integral approach which is particularly appropriate to identify the contributions arising from different topological sectors. Analysing correlation functions for an arbitrary number of fermionic bilinears, we have been able to determine exactly its dependence with the chemical potentials associated to different flavor indices. As stressed in the introduction, our work was prompted by recent results by Deryagin, Grigoriev and Rubakov [3] showing that in the large  $N_c$  limit, condensates of  $QCD$  in four dimensions are inhomogeneous and anisotropic at high fermion density.

Two-dimensional models are a favorite laboratory to test phenomena which are expected to happen in  $QCD_4$ . In fact, an oscillatory inhomogeneous behavior in  $\langle \bar{\psi}\psi \rangle$  was found in the Schwinger model [6] using operator bosonization and then the analysis was completed by finding the exact behavior of fermion bilinear correlators in [7]. Here we have extended this analysis in order to include flavor and color degrees of freedom within a path-integral scheme which makes apparent how topological effects give rise to the non-triviality of correlators.

Remarkably, the oscillatory behavior related to the chemical potential that we have found with no approximation, coincides exactly with that described in [3] for  $QCD_4$  within the large  $N_c$  approximation (apart from the anisotropy that of course cannot be tested in one spatial dimension). In particular, the structure of the multipoint correlation functions, given by eqs.(39) and (78), shows a non-trivial dependence on spatial coordinates. This makes apparent that the ground state has, at finite density, an involved structure which is a superposition of standing waves with respect to the order parameter. Being our model two-dimensional, we were able to control the chemical potential matrix behavior in an *exact* way so that we can discard the possibility that the formation of the standing wave is a byproduct of some approximation. This should be considered when analysing the results of ref.[3] in  $d = 4$  dimensions, where one could argue that use of a ladder approximation as well as the fact of neglecting effects subleading in  $1/N_c$  play an important role in obtaining such a behavior.

Several interesting issues are open for further investigation using our approach. One can in particular study in a very simple way the behavior of condensates at finite temperature. The chiral anomaly is independent of temperature and plays a central role in the behavior of condensates through its connection with the index theorem. Therefore, one should expect that formulae like (39) or (78) are valid also for  $T > 0$ . Of course, v.e.v.'s at  $\mu = 0$  in the r.h.s. of this equation, should be replaced by those computed at finite temperature and hence the issue of zero-modes in a toroidal manifold should

be carefully examined (see e.g. [24]). At the light of recent results concerning  $QCD_2$  with adjoint fermions [26]-[28] it should be of interest to extend our calculation so as to consider adjoint multiplets of fermions.

Finally, it should be worthwhile to consider massive fermions and compute fermion correlation functions at finite density, via a perturbative expansion in the fermion mass following the approach of [29]. We hope to report on these problems in a future work.

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## References

- [1] see e.g. E. Shuryak, Rev. Mod. Phys. **65** (1993) 1.
- [2] D. Bailin and A. Love, Phys. Rep. **107** (1984) 325.
- [3] D. V. Deryagin, D.Yu. Grigoriev and V.A. Rubakov, Int. Jour. Mod. Phys. **A7** (1992) 659.
- [4] W. Fischler, J. Kogut and L. Susskind, Phys. Rev. **D19** (1979) 1188.
- [5] C. Aragão de Carvalho, Phys. Rev. **D21** (1980) 1100.
- [6] Y-Ch. Kao and Y-W. Lee, Phys. Rev. **D50** (1994) 1165.
- [7] H.R. Christiansen and F.A. Schaposnik, Phys. Rev. **D53** (1996) 3260.
- [8] H.B. Nielsen and P. Olesen, Nucl. Phys. **B61** (1973) 45.
- [9] H. de Vega and F.A. Schaposnik, Phys. Rev. **D14** (1976) 1100.
- [10] G. Lozano, M.V. Manías and F.A. Schaposnik, Phys. Rev. **D38** (1988) 601.
- [11] R. Jackiw and P. Rossi, Nucl. Phys. **B190** [**FS3**] (1981) 681.
- [12] H. de Vega, Phys. Rev. **D18** (1978) 2932.
- [13] L. Cugliandolo and G. Lozano, Phys. Rev. **D39** (1989) 3093.
- [14] K. Bardacki, L. Crescimano, Nucl. Phys. **B313** (1989) 269.
- [15] K. Fujikawa, Phys. Rev. Lett. **42** (1979) 95; Phys. Rev. **D21** (1980) 2824.
- [16] H. de Vega and F.A. Schaposnik, Phys. Rev. Lett. **56** (1986) 2564; Phys. Rev. **D34** (1986) 3206.
- [17] E. Fradkin, C.M. Naon and F.A. Schaposnik, Phys. Rev. **D36** (1987) 3809.
- [18] A. Actor, Phys. Lett. **B157** (1985) 53; Nucl. Phys. **B256** (1986) 689.
- [19] M.V. Manías, C.M. Naon and M. Trobo, Int. J. Mod. Phys. **A5** (1990) 2853; Phys. Rev. **D41** (1990) 3174.
- [20] D. Cabra, M.V. Manías, F.A. Schaposnik and M. Trobo, Phys. Rev. **D43** (1991) 3508.
- [21] R.G. Gamboa Saraví, M.A. Muschietti, F.A. Schaposnik and J.E. Solomin, Ann. Phys. (N.Y) **157** (1984) 360.
- [22] A.M. Polyakov in *Fields, Strings and Critical Phenomena* Proceedings of the XLIX Les Houches Summer School, eds. E. Brezin and J. Zinn-Justin, Elsevier, 1988.

- [23] A.M. Polyakov and P.B. Wiegmann, Phys. Lett. **131B** (1983) 121; Phys Lett. **141B** (1984) 223.
- [24] V. Steele, A. Subramanian and I. Zahed, Nucl. Phys. **B452** (1995) 545.
- [25] J. Lowenstein and J.A. Swieca, Ann. Phys. (NY) **68** 172 (1971).
- [26] A.V. Smilga, Phys. Rev. **D49** (1994) 6836.
- [27] D.J. Gross, I.R. Klebanov, A. Matytsin and A. Smilga, Nucl. Phys. **B461** (1996) 109.
- [28] A.V. Smilga, ITEP report, hep-th/9607007, 1996, unpublished.
- [29] C. M. Naon, Phys. Rev. **D31** (1985) 2035.