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Self-Dual Plannar Lattice Ising Ferromagnet Within Generalized Statistics

by

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Abstract

Within generalized Boltzmann-Gibbs equilibrium statistics, we calculate the phase diagram and the correlation length critical exponent ν for the Ising ferromagnet in a self-dual hierarchical lattice which mimics the square lattice.

Key-words: Generalized statistics; Generalized entropy; Ising model; Critical Phenomena.

I INTRODUCTION

A generalized entropy S_q has been recently introduced [1] which enables the generalization of Boltzmann-Gibbs equilibrium distribution. It is given (in units of a conventional positive constant k) by

$$S_q = \frac{1 - \sum_{i=1}^{W} p_i^q}{q-1} \qquad (q \in \Re) \tag{1}$$

where $\{p_i\}$ are the occurrence probabilities of the W microstates of the system. We verify that $\lim_{q\to 1} S_q = -\sum_{i=1}^{W} p_i \ln p_i \equiv S^S$, hence the well known Shannon entropy [2] is recovered as a particular case. S_q is *concave but not extensive* [1,3-5] (more precisely speaking, S_q is concave for $q > 0$ and convex for $q < 0$). Generalized entropies which are *extensive but not concave* are also available, such as the Renyi entropy [6]

$$S_q^R = \frac{\ln \sum_{i=1}^{W} p_i^q}{1-q} \qquad (q \in \Re) \tag{2}$$

as well as [7]

$$S_q^C = -\frac{\sum_{i=1}^{W} p_i^q \ln p_i}{\sum_{i=1}^{W} p_i^q} \qquad (q \in \Re) \tag{3}$$

They also satisfy $\lim_{q\to 1} S_q^R = \lim_{q\to 1} S_q^C = S^S$ (we recall that S^S is *simultaneously concave and extensive.*

In the present paper we follow the line associated with expression (1). Indeed, concavity is an extremely basic requirement since it guarantees the thermodynamic stability of the system. Extensivity is a mathematically convenient property which has proved to be correct for an enormous variety of systems; however, it might be untrue for a certain class of essentially nonlinear systems. Furthermore, expression (1) has enabled various (nontrivial though mathematically simple and natural) generalizations of important properties such as

(i) The canonical equilibrium distribution now becomes [1, 3], for $q < 1$,

$$p_i = \begin{cases} \frac{[1-\beta(1-q)\varepsilon_i]^{\frac{1}{1-q}}}{Z_q} & \text{if } 1-\beta(1-q)\varepsilon_i > 0 \\ 0, \text{ otherwise} & \end{cases} \tag{4}$$

and, for $q > 1$,

$$p_i = \begin{cases} [1-\beta(1-q)\varepsilon_i]^{\frac{1}{1-q}} & \text{if } \; 1-\beta(1-q)\varepsilon_1 > 0 \\ \delta_{i1}/g_1 \quad \text{otherwise} & \end{cases} \tag{4'}$$

with the generalized partition function consistently given by

$$Z_q = \sum_{i=1}^{W} [1-\beta(1-q)\varepsilon_i]^{\frac{1}{1-q}} \tag{5}$$

where $\beta \equiv 1/kT > 0$ and $\{\varepsilon_i\}$ is the spectrum (set of given real numbers), ε_1 and g_1 being respectively the energy and degeneracy of the ground state, and δ_{i1} being equal to unity if $\varepsilon_i = \varepsilon_1$ and equal to zero otherwise.

(ii) The Thermodynamics associated with (4) and (4') is invariant under Legendre transformations; in particular, it was proved [3] that

$$\frac{1}{T} = \frac{\partial S_q}{\partial U_q} \tag{6}$$

with

$$U_q \equiv \sum_{i=1}^{W} p_i^q \varepsilon_i \tag{7}$$

and, also, that

$$U_q = -\frac{\partial}{\partial \beta} \frac{Z_q^{1-q} - 1}{1-q} \tag{8}$$

and

$$F_q \equiv U_q - TS_q = -\frac{1}{\beta} \frac{Z_q^{1-q} - 1}{1-q} \tag{9}$$

(iii) Shannon additivity now becomes [3]

$$\begin{aligned} S_q(p_1, p_2, \cdots, p_W) &= S_q(p_A, p_B) \\ &+ p_A^q S_q(p_1/p_A, p_2/p_A, \cdots, p_{W_A}/p_A) \\ &+ p_B^q S_q(p_{W_A+1}/p_B, \cdots, p_W/p_B) \end{aligned} \tag{10}$$

with $p_A \equiv \sum_{i=1}^{W_A} p_i$ and $p_B \equiv \sum_{i=W_A+1}^{W} p_i$ (hence $p_A + p_B = 1$).

(iv) Boltzmann H-theorem has been proved for arbitrary q (within detailed balance hypothesis, by Mariz [8], and, under much less restrictive hypothesis, by Ramshaw [9]); the connection of S_q with irreversibility has been further analysed by Ramshaw [10].

(v) Both Ehrenfest theorem and Jaynes Information Theory duality relations remain true for arbitrary q [11]; the same happens with von Neumann equation [12].

(vi) Both Langevin and Fokker-Planck equations have been consistently generalized for all values of q [13].

(vii) Standard additivity and the Kullback-Leibler cross entropy are respectively generalized as follows [14]:

$$S_q(\{p_i^A p_j^B\}) = S_q(\{p_i^A\}) + S_q(\{p_j^B\}) + (1-q) S_q(\{p_i^A\}) S_q(\{p_j^B\}) \tag{11}$$

and

$$I_q(f,g) \equiv \int dx\, f(x) \frac{[f(x)/g(x)]^{q-1}-1}{q-1} \begin{array}{ll} \geq & 0 \ \text{if } q>0 \\ = & 0 \ \text{if } q=0 \\ \leq & 0 \ \text{if } q<0 \end{array} \tag{12}$$

where the equality holds if and only if the arbitrary distributions $f(x)$ and $g(x)$ are equal.
(viii) Bogolyubov inequality (hence the Variational Method) is generalized as follows [15]:

$$\begin{aligned} F &\leq \frac{Fo}{H} + \left(1-\frac{1}{H}\right)\frac{1}{\beta(1-q)} \quad \text{if } q<2 \\ &= \frac{Fo}{H} - \left(1-\frac{1}{H}\right)\frac{1}{\beta} \quad \text{if } q=2 \\ &\geq \frac{Fo}{H} + \left(1-\frac{1}{H}\right)\frac{1}{\beta(1-q)} \quad \text{if } q>2 \end{aligned} \tag{13}$$

where

$$H \equiv < \frac{1-\beta(1-q)\mathcal{H}_0}{1-\beta(1-q)\mathcal{H}} >_0 \tag{14}$$

$\mathcal{H}$ and $\mathcal{H}_0$ being respectively the real and trial Hamiltonians and $< \cdots >_0$ denoting the canonical thermal average associated with $\mathcal{H}_0$ (F and Fo are respectively the exact and trial generalized free energies).
(ix) The fluctuation-dissipation theorem has been generalized [16] for all values of q; the same holds for Callen's identity [17].
(x) Fermi-Dirac and Bose-Einstein statistics have been generalized [18] for all values of q.

A physical application is already available for this generalized statistics. Indeed, the use of standard ($q = 1$) Statistical Mechanics to discuss collisionless stellar dynamics [19] (within the polytropic model as studied by Chandrasekhar and others) leads to a physically undesirable result, namely that the galaxy is infinitely massive (untrue). Plastino and Plastino recently showed [20] that this unphysical consequence is removed if q sufficiently differs from unity; in fact, this situation nicely illustrates Balian's point [21] that nonextensive entropy might be relevant for the discussion of astrophysical matter stability. A second physical application has been recently found [22] for d-dimensional Levy flights, q being directly determined by the fractal dimension associated with these flights. Finally, it might be not useless mentionning that this nonextensive physics exhibits intriguing analogies with very recent quantum group developments [23, 24] (q-deformations, q-oscillators, Eqs. (4.1) and (4.34) of [25]).

Some one-body systems have already been discussed within the present generalized statistics: the two-level system [1, 26], the harmonic oscillator [26], the free particle [5]. One many-body system, namely, the $d = 1$ Ising model, has been discussed as well [27]. However, the study of a system presenting a phase transition has never been undertaken for $q \neq 1$. This is the purpose of the present work, and we shall address the square-lattice Ising ferromagnet. To do this we shall discuss its criticality, more precisely, its phase diagram and its correlation length critical exponent ν. The theoretical framework will be that of a simple real space renormalization group (RG) which replaces the (self-dual) square lattice by the (self-dual) Wheatstone-bridge hierarchical one. This approach has already proved, for $q = 1$, its efficiency for both classical (Ising, bond percolation, Potts models; see [28] and references therein) and quantum (Heisenberg and Hubbard models; see [29] and references therein) systems. In Section II we present the formalism, in Section III the results, and we finally conclude in Section IV.

II MODEL AND RENORMALIZATION GROUP

We consider the dimensionless Ising Hamiltonian

$$-\beta\mathcal{H} = K \sum_{<i,j>} S_i S_j + K_0 \tag{15}$$

where $S_i = \pm 1, < i, j >$ runs over all pairs of first-neighbouring spins on a square lattice, $K \equiv \beta J$ and $K_0 = \beta J_0$ ($J > 0$ is the coupling constant and J_0 fixes the origin of the energy scale). In the framework of the generalized statistics, the probability of a configuration $\{S_i\}$ is given, for $q < 1$, (see Eq. (4)) by

$$p(\{S_i\}) = \begin{cases} \frac{[1-(1-q)\beta\mathcal{H}]^{\frac{1}{1-q}}}{Z_q} & \text{if } 1-(1-q)\beta\mathcal{H} > 0; \\ 0, \text{ otherwise} & \end{cases} \tag{16}$$

with

$$Z_q \equiv \sum_{\{S_i\}} [1-(1-q)\beta\mathcal{H}]^{\frac{1}{1-q}} \tag{17}$$

while, for $q > 1$, an analogous expression is obtained from Eq. (4')

In order to study the criticality of this model, we shall use the hierarchical lattice generated by the Wheatstone-bridge cluster indicated in Fig. 1. This cluster being self-dual (as the square lattice itself), the RG will provide, for $q = 1$, the exact critical temperature $kT_c/J = 2.269\cdots$ (see, for instance, [28]). More precisely, to the left side cluster of Fig. 1 we associate the Hamiltonian

$$-\beta\mathcal{H}_{1234} = K(S_1S_3 + S_1S_4 + S_3S_4 + S_2S_3 + S_2S_4) + K_0 \tag{18}$$

and to its right side cluster we associate

$$-\beta\mathcal{H}'_{12} = K'S_1S_2 + K'_0 \tag{19}$$

We then impose, for $S_1, S_2 = \pm 1$,

$$[1-(1-q)\beta\mathcal{H}'_{12}]^{\frac{1}{1-q}} = \sum_{S_3=\pm 1}\sum_{S_4=\pm 1} [1-(1-q)\beta\mathcal{H}_{1234}]^{\frac{1}{1-q}} \tag{20}$$

which completely determines the RG recursive relation in the (K, K_0) space, or, equivalently, in the (t, x) space with $t \equiv 1/K = kT/J$ and $x \equiv K_0/K = J_0/J$. Eq. (20) holds, for $q < 1$, as it stands for all $\{S_1, S_2, S_3, S_4\}$ configurations satisfying $[1-(1-q)\beta\mathcal{H}_{1234}] \geq 0$ and only for them (i.e., it excludes all the configurations for which the condition is not satisfied); this equation implies

$$Z'_{q(12)} = Z_{q(1234)} \tag{21}$$

where

$$Z'_{q(12)} \equiv \sum_{S_1,S_2} [1-(1-q)\beta\mathcal{H}'_{12}]^{\frac{1}{1-q}} \tag{22}$$

and

$$Z_{q(1234)} \equiv \sum_{S_1,S_2,S_3,S_4} [1-(1-q)\beta\mathcal{H}_{1234}]^{\frac{1}{1-q}} \tag{23}$$

hence (using (20))

$$p'(S_1S_2) = \sum_{S_3,S_4} p(S_1,S_2,S_3,S_4) \tag{24}$$

Eq. (20) straightforwardly leads, for $q < 1$, to

$$t' = \frac{2(1-q)t}{(f_1+2f_2+f_4)^{1-q} - 2^{1-q}(f_2+f_3)^{1-q}} \tag{25.a}$$

$$x' = \frac{(f_1+2f_2+f_4)^{1-q} + 2^{1-q}(f_2+f_3)^{1-q} - 2t}{(f_1+2f_2+f_4)^{1-q} - 2^{1-q}(f_2+f_3)^{1-q}} \tag{25.b}$$

where

$$f_i = \begin{cases} g_i^{\frac{1}{1-q}} & \text{if } g_i > 0 \\ 0 & \text{otherwise} \end{cases} \qquad (i = 1,2,3,4) \tag{26}$$

with

$$\begin{aligned} g_1 &\equiv t + (1-q)(x-3) \\ g_2 &\equiv t + (1-q)(x-1) \\ g_3 &\equiv t + (1-q)(x+1) \\ g_4 &\equiv t + (1-q)(x+5) \end{aligned} \tag{27}$$

Analogous expressions are obtained for $q > 1$.

Eqs. (25-27), together with the corresponding ones for $q > 1$, determine the criticality of the system for arbitrary q. Although *exact* for a single cluster, the present RG procedure is only *approximative* for the entire hierarchical lattice. This is due to the fact that generically $p(\mathcal{H}_1\mathcal{H}_2) \neq p(\mathcal{H}_1)p(\mathcal{H}_2)$, $\mathcal{H}_1$ and $\mathcal{H}_2$ being the Hamiltonians associated with two different clusters of the lattice. However, it is easy to see that it becomes asymptotically exact in the limit $(q-1)\beta \to 0$, i.e. for all temperatures if $q=1$, or for all values of q if the temperature is high enough. This situation is very analogous to what occurs for $q=1$ quantum systems (see [29] and references therein).

III RESULTS

It is convenient to discuss (for $t \geq 0$ and $-\infty < x < \infty$) separately the $q=1$, $q<1$ and $q>1$ cases.

III.1 - q=1 Criticality

The $q \to 1$ recurrence relations (25) become (see,for instance, [28])

$$t' = \frac{2}{\ln\{2[e^{-2/t}\cosh(1/t) + e^{2/t}\cosh(3/t)]\} - \ln\{4\cosh(1/t)\}} \tag{28.a}$$

$$x' = \frac{2x/t + \ln\{2[e^{-2/t}\cosh(1/t) + e^{2/t}\cosh(3/t)]\} + \ln\{4\cosh(1/t)\}}{\ln\{2[e^{-2/t}\cosh(1/t) + e^{2/t}\cosh(3/t)]\} - \ln\{4\cosh(1/t)\}} \tag{28.b}$$

where we notice that t' depends only on t (for $q \neq 1$, t' depends on both t and x). The RG flow diagram includes a $t=0$ invariant subspace and is indicated in Fig. 2. Two trivial (fully stable) fixed points exist, namely $(t,x)=(0,3)$ and (∞,∞), respectively characterizing the *ferromagnetic* (Fe) and *paramagnetic* (P) phases. The two corresponding attractive basins are separated by the critical line $t = 2.269\cdots (\forall x)$, thus recovering the square lattice exact result. The correlation length critical exponent is given by $\nu = \ln 2/\ln(dt'/dt|_{2.269\ldots}) \simeq 1.149$ ($b=2$ is the RG length rescaling). This result is exact for the Wheatstone-bridge hierarchical lattice, and is to be compared with the square lattice exact result ($\nu = 1$).

III.2 - $q<1$ Criticality

We see, from Eqs. (27), that $g_4 > g_3 > g_2 > g_1$, consequently three different situations occur, namely

(i) $t \leq -(1-q)(x+5)$, hence $g_4 \leq 0$, hence $f_1 = f_2 = f_3 = f_4 = 0$, hence the normalization condition $\sum_{S_1,S_2} p'(S_1S_2) = 1$ cannot be satisfied. This region is physically unaccessible and we refer to it as *thermally forbidden* (Fo);

(ii) $-(1-q)(x+5) < t \leq -(1-q)(x+1)$, hence $g_4 > 0$ and $g_3 \leq 0$, hence $f_1 = f_2 = f_3 = 0$. Consequently, the only spin configuration whose probability is nonzero (hence, necessarily equals unity) is the ground state, i.e., all spins aligned to each other (and aligned to a vanishing external magnetic field which, rigorously speaking, ought to be included in the present theory). The existence of *finite* frozen (as well as forbidden) temperature intervals is a common phenomenon within $q \neq 1$ statistics (see [5] and references therein). However, in the present case, this is a finite size effect introduced by the RG approach; indeed, improved RG approximations based on larger clusters (e.g., clusters of higher order in the hierarchy generated by the Wheatstone-bridge cluster of Fig. 1) will drive the Fo and Fr regions towards $x = -\infty$. Consequently, such regions disappear in the limit of an infinite cluster, this is to say, for the full hierarchical lattice;

(iii) $t > -(1-q)(x+1)$, hence $g_4 > 0$ and $g_3 > 0$. Consequently different spin configurations can occur, thus characterizing a *thermally active* region. In fact, in the present model, two macroscopic phases can occur within this region, namely the ferromagnetic (Fe) and the paramagnetic (P) phases.

Let us now analyze the recurrence equations (25). We see that, as for the $q = 1$ case, $t = 0$ constitutes an invariant subspace of the (t, x) parameter space. In this case, $x'(x)$ is not defined for $x \leq -5$, equals unity for $-5 < x \leq -1$, and behaves as indicated in Fig. 3 for $x > -1$. A special value $q* \simeq 0.852$ exists such as for $q < q*$ only one attractive fixed point exists (namely $x = +\infty$), whereas for $q > q*$ two new fixed points appear (namely an attractive (trivial) one located at $x_1(q)$, and a repulsive (critical) one located at $x_c(q)$). When q increases from $q*$ to 1, $(x_1(q), x_c(q))$ continuous and monotonously varies from $(x*, x*)$ (with $x* \simeq 6.14$) to $(3, \infty)$, thus recovering, in the $q \to 1-0$ limit, the $q = 1$ result presented in Section III.1. Consistently, the $x_1(q)$ attractor is to be associated with the Fe phase, whereas the $x_c(q)$ repulsor corresponds to a Fe-P phase transition at vanishing temperature. The $x = \infty$ attractor corresponds to the P phase. We present in Fig. 4 the q-evolution of the critical point x_c. The corresponding critical exponent is given by

$$\nu(q) = \frac{\ln b}{\ln\left(\partial x'/\partial x\right)_{\substack{t=0 \\ x_c(q)}}} \qquad (b = 2) \tag{29}$$

and is represented in Fig. 5.

The frozen region $(-5 < x \leq -1)$ deserves a word. Indeed, the entire region is driven, under RG, towards $x = 1$; in other words, the flow in this region is a discontinuous one.

The presence, for $q \neq 1$ and $T = 0$, of a (continuous) phase transition can be understood by looking at the $T = 0$ limit of Eqs. (4) and (5), namely $p_i \to \frac{|\varepsilon_i|^{\frac{1}{1-q}}}{\sum_{i=1}^{W} |\varepsilon_i|^{\frac{1}{1-q}}}$ for all states i which satisfy $(q-1)\varepsilon_i > 0$, while the same limit (i.e., $T = 0$) for $q = 1$ yields, only for the ground state, a non vanishing probability.

In our case, for $x \to +\infty$ (i.e., $J_0 \to -\infty$, keeping J constant) all spin configurations $\{S_i\}$ will have an energy $\sim -J_0$, hence $p_i(\{S_i\}) \to \frac{1}{W}$. Such distribution corresponds to a *disordered* (i.e., P) phase. On the other hand, for $J_0 \sim J$, the ground state probability will be larger than the probabilities of all other configurations, thus allowing for the existence of an *ordered* (i.e, Fe) phase. The competition between these two tendencies yields a continuous (i.e., critical) phase transition when varying J_0/J.

If we enlarge now our discussion to the entire (t, x) space, we can see that the RG flow associated with Eqs. (25) is attracted by the $t = 0$ axis, which controls, consequently, the criticality of the system. In Fig. 6(a) (Fig. 6(b)) we have represented the RG flow for a typical $q < q*$ $(q > q*)$ case. Therefore, for $q < q*$ we have three regions in the phase diagram, namely (in order of increasing x or increasing t) the Fo, the Fr and the P ones; the latter is attracted by the $(t, x) = (0, \infty)$ fully stable fixed point. For $q > q*$ we have four regions, namely (in order of increasing x or increasing t) the Fo, the Fr, the Fe and the P ones; the Fe (P) region is attracted by the fully stable $(0, x_1(q))((0, \infty))$ fixed point. The Fe-P critical line is attracted by the semistable $(0, x_c(q))$ fixed point, which determines the criticality of the entire line (in particular, along this line, $\nu(q)$ is given by Eq. (29)). In Fig. 7 we present the critical temperature t_c corresponding to $J_0 = 0$ $(x = 0)$.

III.3 - $q > 1$ Criticality

For arbitrary values of $q > 1$ three different situations occur, namely

(i) $t \leq (q-1)(x-3)$. This is a physically unaccessible region (Fo)

(ii) $(q-1)(x-3) < t \leq (q-1)(x+5)$. In this region the only spin configuration whose probability is nonzero is the ground state (ferromagnetic). Hence we have (as for $q < 1$), a frozen ferromagnetic region.

(iii) $t > (q-1)(x+5)$. This is a thermally active region, where different spin configurations can occur. Moreover, two macroscopic phases occur within this region for all $q > 1$, namely the Fe and the P phases.

The $t = 0$ recurrence equation $x'(x)$ presents an unstable fixed point $x_c(q)$ (in fact, it is fully unstable in the (t, x) space). As q increases from 1 to infinity, $x_c(q)$ continuously increases from $-\infty$ to -5; $x_c(q)$ is represented in Fig. 8. The points $x < x_c(q)$ exhibit a continuous flow towards an attractive fixed point at $x = -\infty$; for $x \to -\infty$, the probability associated with every spin configuration equals $1/W$ (like the $x \to +\infty$ limit in the $q > 1$ case); hence, the $x = -\infty$ fixed point is associated with a P phase; $x_c(q)$ plays the role of an unstable (critical) fixed point. The points $x_c(q) < x < -5$ are driven discontinuosly into the Fr region, which is a Fe region.

We enlarge now our analysis to the entire (t, x) space. Besides the Fr and Fo regions above mentioned, we found two active regions in the phase diagram, namely (for decreasing x or increasing t) the Fe region and then the P one. The P region is attracted by a fixed point located at $t = \infty$. The Fe region is characterized by a discontinuous flow into the Fr region. The $Fe - P$ critical line, which starts, for $t = 0$, at $x_c(q)$, is attracted by a fixed point located at $(+\infty, +\infty)$; the associated critical exponent $\nu(q)$ is represented in Fig. 5 and the critical temperature t_c corresponding to $J_0 = 0$ is shown in Fig. 7.

IV CONCLUSION

We have shown that, within the framework of the present generalized statistics, the criticality of the Ising ferromagnet in square lattice is modified in a continuous manner when we slightly depart from $q = 1$. Below a critical value $q* \simeq 0.852$ (i.e., $q < q*$) the system is paramagnetic for all nonvanishing temperatures for which it is not forbidden nor frozen. When q further increases, the reduced critical temperature $t_c \equiv kT_c/J$ for $J_0 = 0$ monotonously increases with increasing q; $t_c = 1$ for $q = q*$, $t_c = 2.269\cdots$ (exact answer for both square and hierarchical lattices) for $q = 1$, and $t_c \sim Aq$ for $q \to \infty$ with $A \simeq 5.5$ (this fact can be compared with the mean field square lattice result $t_c = 4q$, $\forall\ q \geq 0$). Consistently, the thermal critical exponent ν is defined only above q*, and monotonously decreases with increasing q; ν diverges for $q \to q * +0$, equals $1.149\cdots$ (exact answer for the hierarchical lattice [28] and 15% wrong for the square lattice) for $q = 1$, and vanishes for $q \to \infty$.

When T increases for $q \neq 1$, we might have a finite forbidden region (physically unaccessible), above which always is present a frozen region (which possibly shrinks and finally disappears in the limit of an infinite lattice), above which thermally active phases exist: paramagnetic if $0 < q < q*$; ferromagnetic and then paramagnetic if $q* \leq q$. In the $q \to 1$ limit, the forbidden and frozen regions shrink into the $T = 0$ point.

We have also performed the RG analysis for the $d = 1$ case. For $q < 1$ the system qualitatively behaves as the $d = 2$ case for $q < q*$, i.e., the entire active region is attracted by the fixed point $(t, x) = (0, +\infty)$. For $q > 1$ the entire active region for $t \neq 0$ is attracted by a fixed point located at $t = \infty$. In other words, the system presents no phase transition for any value of q, and, as expected, is paramagnetic for all temperatures outside from the Fr and Fo regions.

While the present RG results clearly seem quite reliable (for both square and hierarchical lattices) for $q \simeq 1$ and not too low temperatures, one can not exclude the possibility that some of them be spurious when we depart from this situation. Consequently, the study of this system through different techniques (e.g., Monte Carlo) would be very welcome.

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FIGURE CAPTIONS

Fig. 1: Renormalization group cell transformation. The two-rooted cluster on the left of the figure generates , through infinite iterations, an hierarchical lattice. (K_0, K) and (K_0', K') are the parameters of the Hamiltonians $\mathcal{H}$ and $\mathcal{H}'$ respectively.
Fig. 2: Schematic RG flow for $q = 1$.
Fig. 3: Recurrence equation $x' = x'(x)$ for $q < 1$, $t = 0$ and $x > -1$.
Fig. 4: Inverse of the critical point $x_c \, vs \, q$ at $t = 0$ for $q* < q < 1 (q* \approx 0.852)$.
Fig. 5: Correlation length critical exponent ν of the $Fe - P$ transitions $vs \, q$.
Fig. 6: Schematic RG flow for $q < 1$. (a) $q = 0.8 < q*$; (b) $q = 0.9 > q*$. Fig. 7: Dimensionless critical temperature $t_c \, vs \, q$ for $J_0 = 0$.
Fig. 8: Critical point $x_c \, vs \, q$ at $t = 0$ for $q > 1$ (this figure is the continuation of Fig. 4).

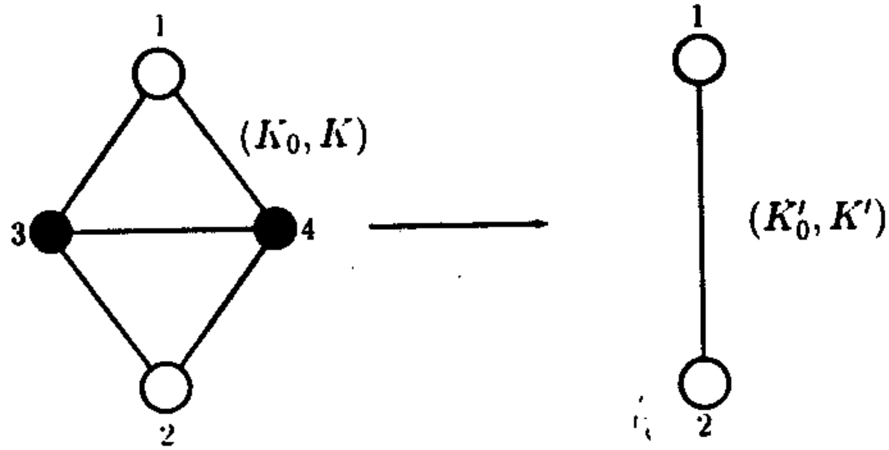

FIG. 1

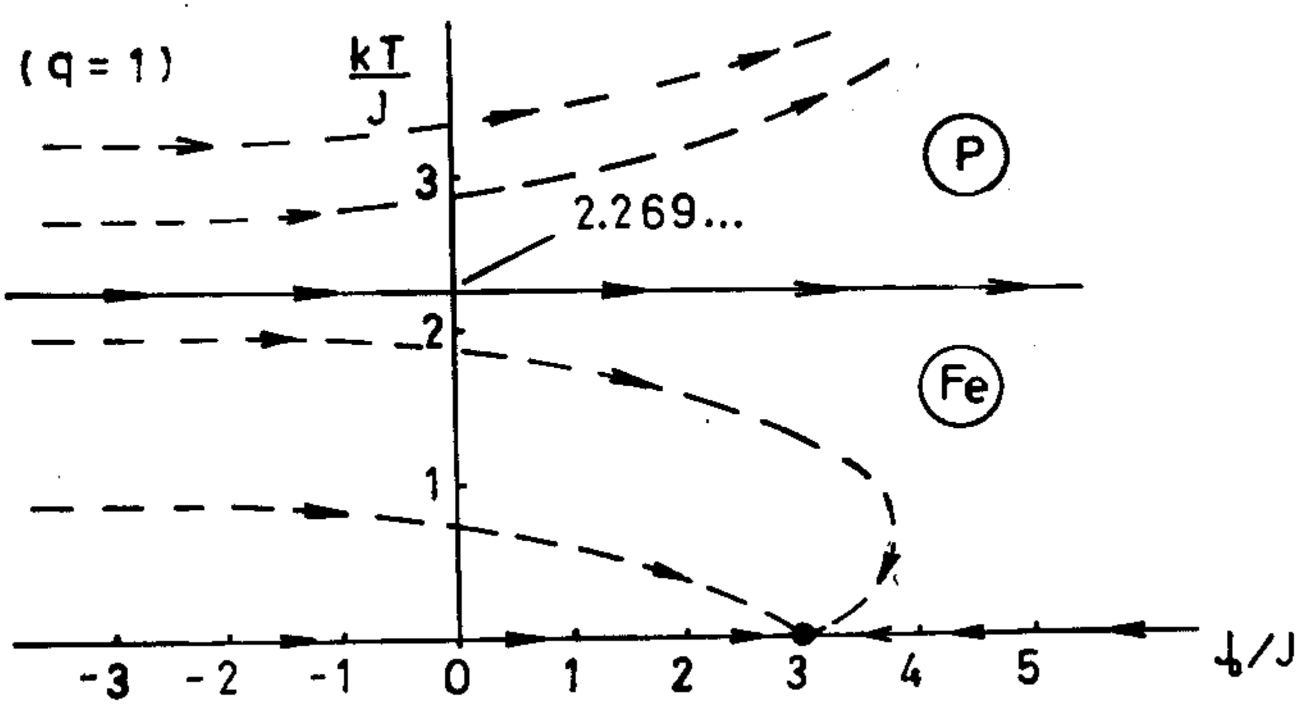

FIG. 2

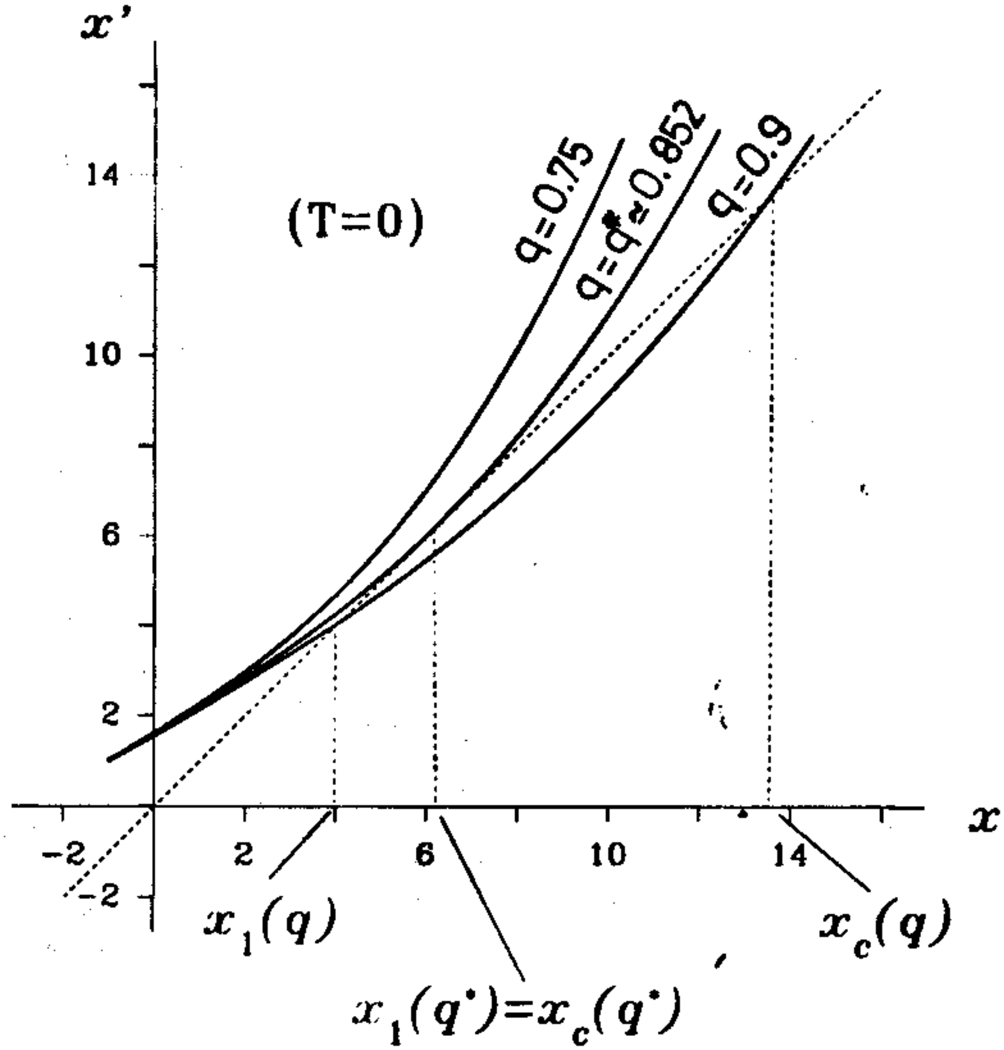

FIG. 3

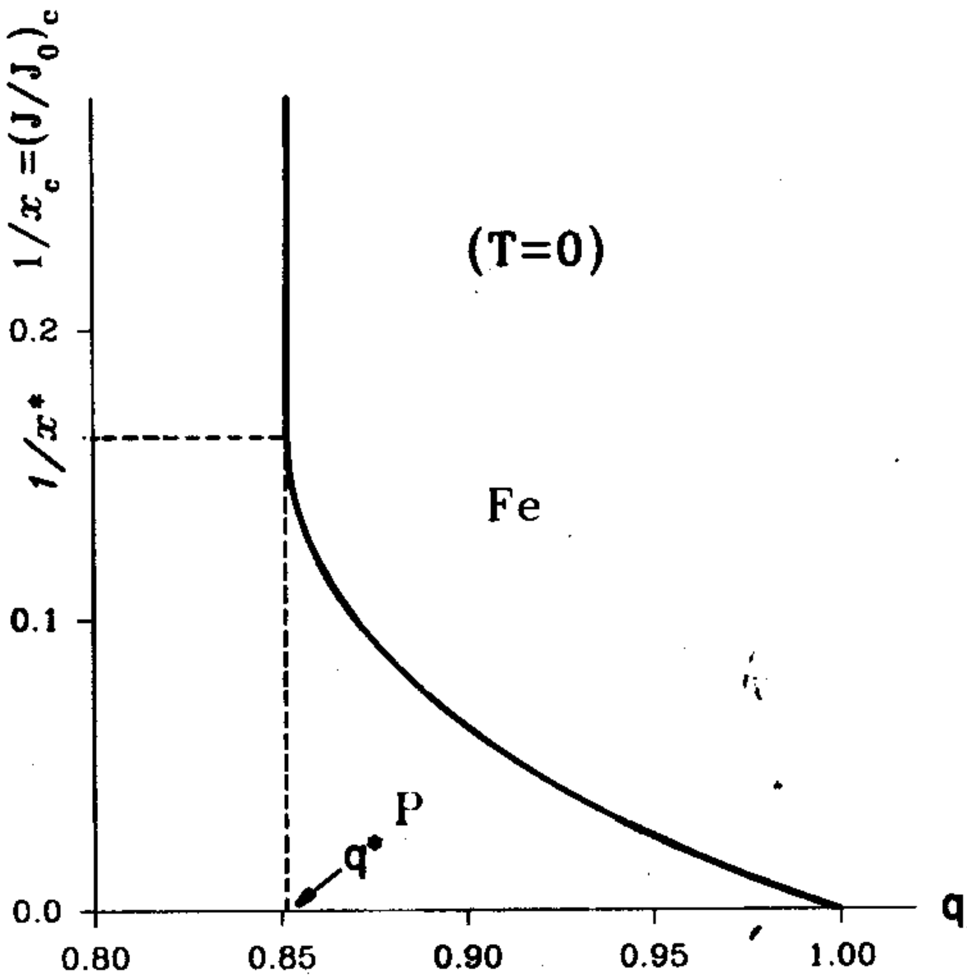

FIG. 4

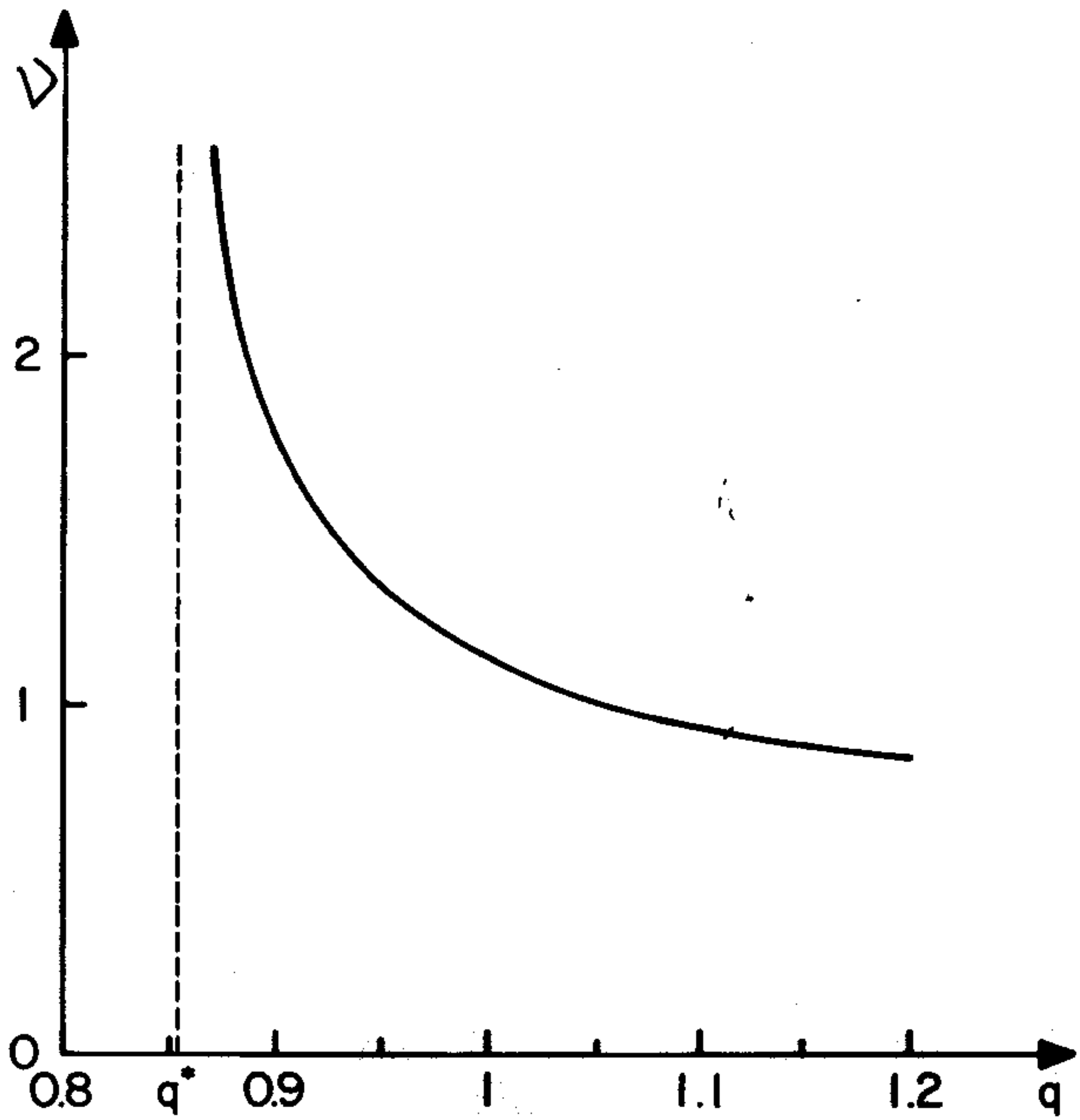

FIG. 5

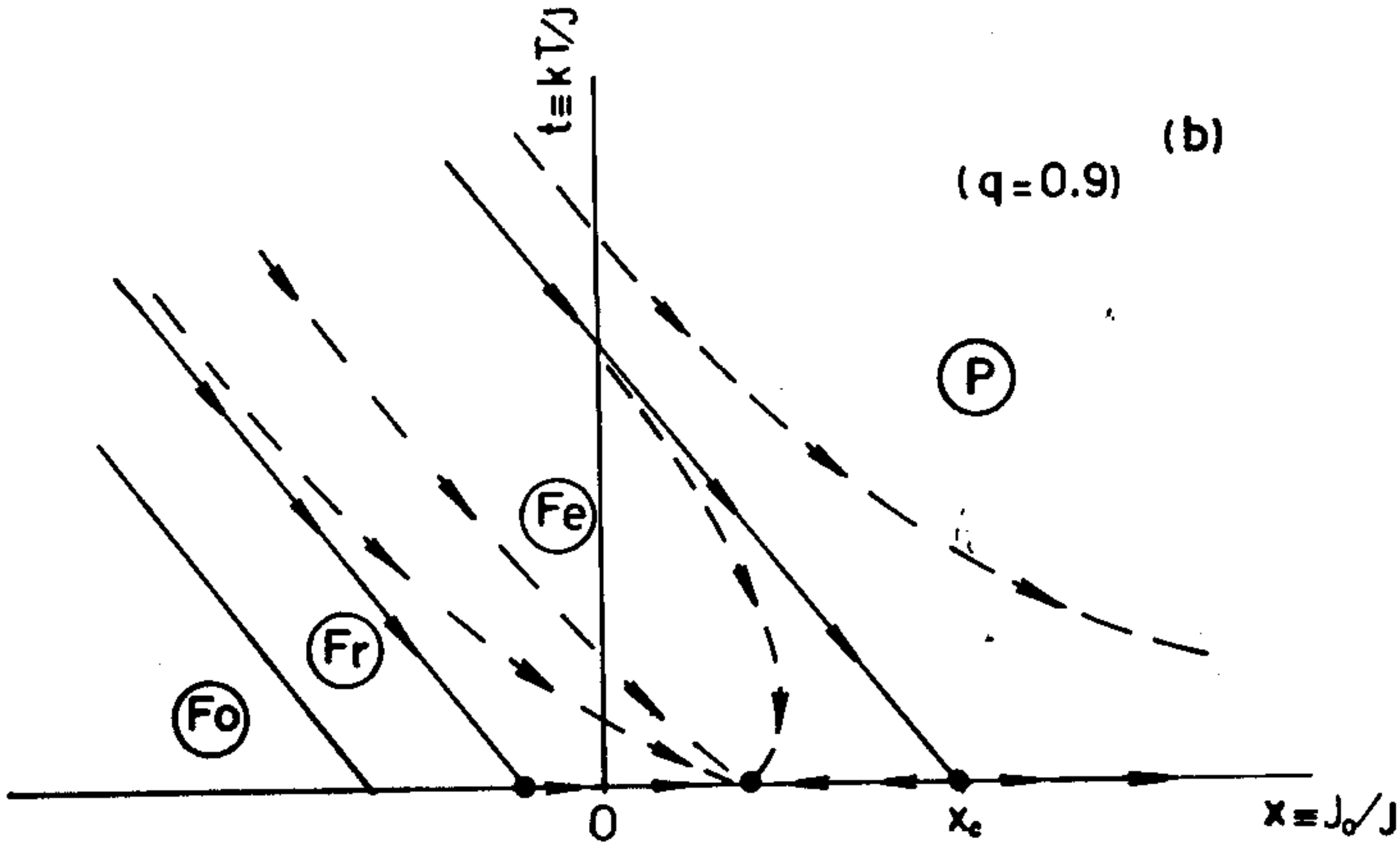

FIG. 6

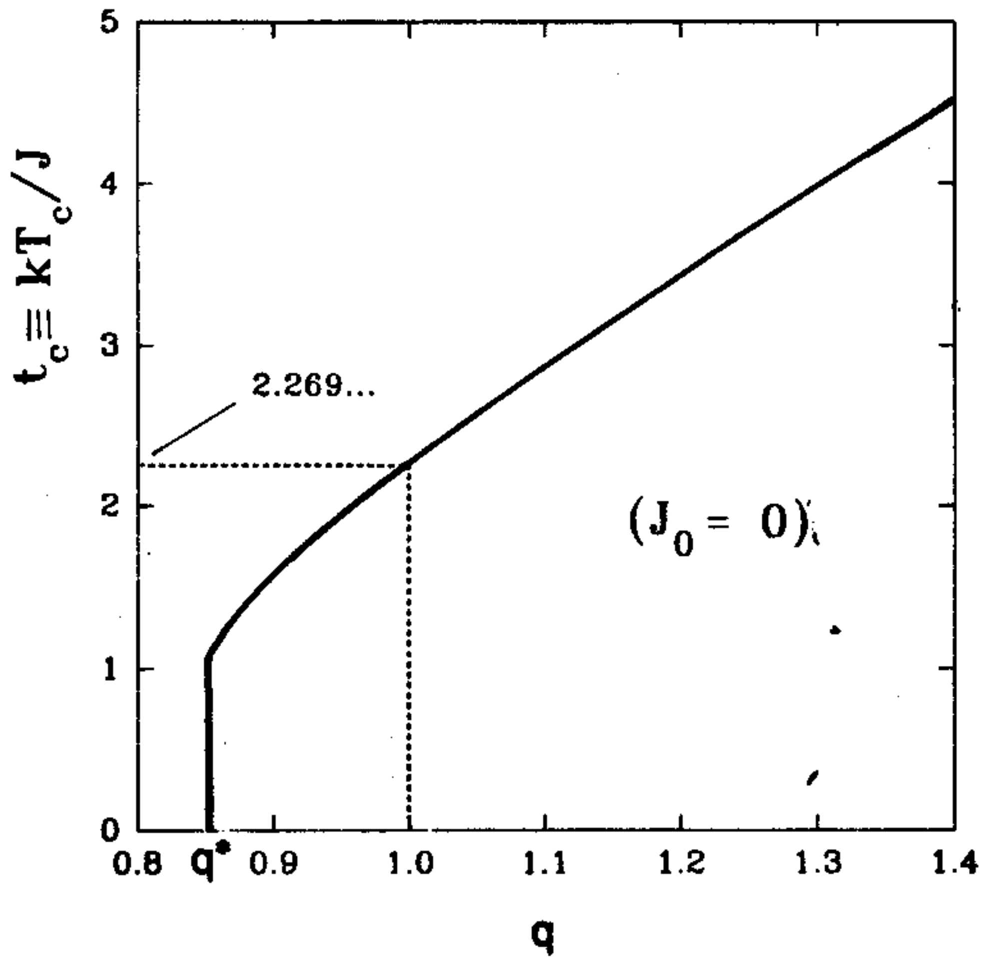

FIG. 7

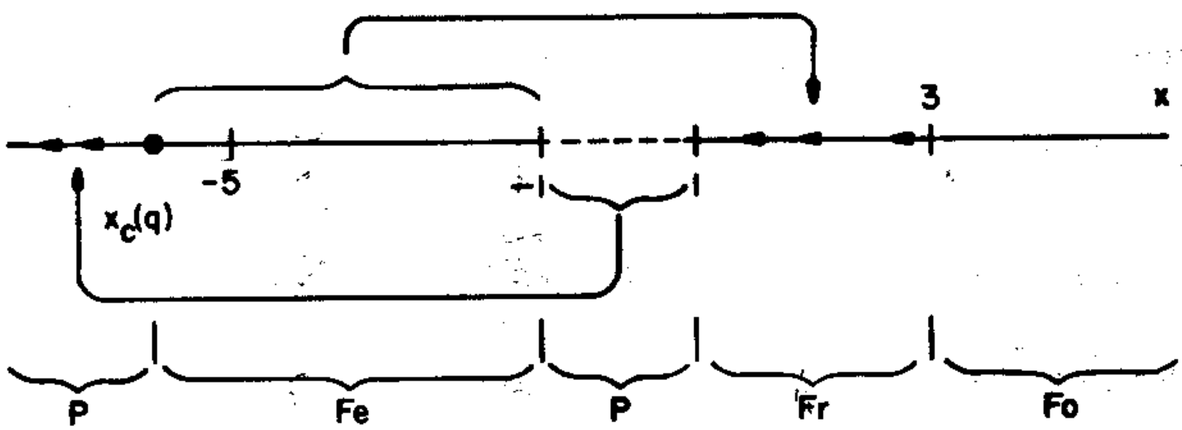

FIG. 8

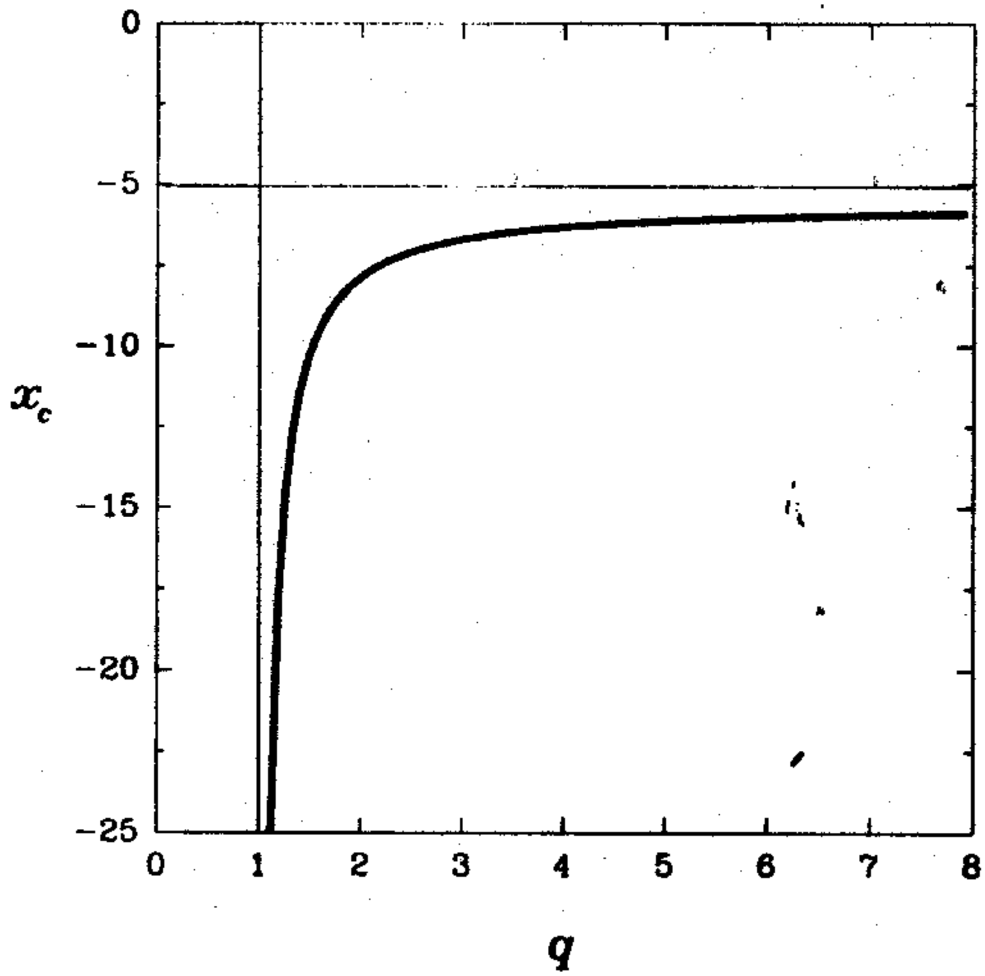

FIG. 9

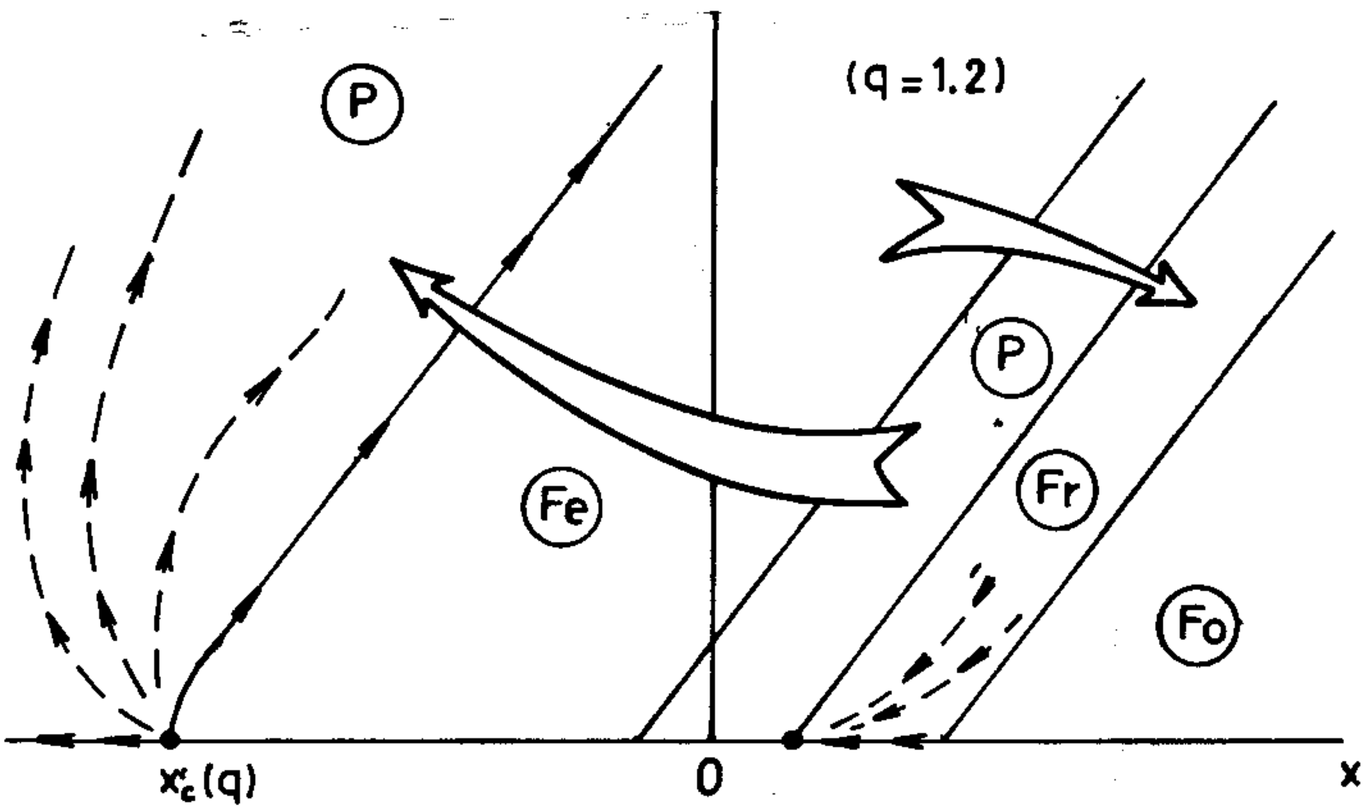

FIG. 10

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