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SOME QUESTIONS REGARDING CHIRAL SOLITONS AS BARYONS*

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ABSTRACT

A short description is made of the meaning and application of chiral solitons as baryons. The hedgehog proposal for a unitary chiral field is discussed and it is emphasized the appearance of an arbitrary parameter in the soliton solution which may conflict with the customary argument on (in)stability.

Key-words: Skyrmeons; Chiral solitons; Sigma model.

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I - INTRODUCTION

The description of hadrons and of low energy strong interaction physics is assumed to be contained in the quantum theory of gauge colour-dynamics known as QCD. In fact, the systematic building of a set of strongly interacting particles at low energies is performed by adding appropriately the quantum numbers of a quark-antiquark pair or of three quarks.

This description, however, is so far not yet quantitative. On the other hand, a satisfactory explanation of low energy hadronic properties is partially obtained through the use of current algebra methods, which in turn can be expressed in terms of effective chiral lagrangeans [1]. Actually, chiral lagrangeans are written in terms of hadron fields, and no memory is kept of the quark or gluon degrees of freedom of the fundamental theory.

High energy, or more properly, high momentum transfer collisions involving small numbers of quarks and gluons, have been sufficiently well described from perturbative QCD [2].

This state of affairs might be summarized saying that while QCD and the counting of quark properties are relevant to successfully describe quantitatively phenomena at high energies, at low energies they are not as crucial, except for bookeeping of quantum numbers.

A step to link these two regimes was given in recent years by several authors [3,4,5], who pretended to abstract from QCD information related to chiral lagrangeans and their properties. This lead in a natural way to introduce the so called Wess-Zumino terms, which were originally established for the chiral non-linear σ-model become consistent.

These facts are supported by theoretical experience with two dimensional models [6] of non-abelian gauge theories with fermions. There, non-linear σ-models, Wess-Zumino terms and current algebras result from the anomalous character of the axial current in gauge invariant lagrangeans.

In short, then, it is reasonable to expect that chiral non-linear σ-model lagrangeans with suitable extension would be the effective theory emerging from QCD at low energies.

In these theories, baryons would be semiclassical solitons, as proposed by Skyrme long time ago [7]. A term with four derivatives of the unitary chiral field is believed to be needed to stabilize the solutions, which introduces a parameter in the description of their properties. The interaction with mesons would result from introducing fluctuations of the chiral field around the soliton solution [8].

In this note we analyze some mathematical aspects of this proposal and introduce some new aspects of the problem which deserve further consideration, and at the same time may open interesting possibilities for development.

II - CHIRAL SOLITONS AS NUCLEONS

As mentioned before, the proposal of a description of baryons by chiral solitons could be summarized as follows:

Nucleon at low energies ≡ Soliton from [(non-linear σ-model) + + (Skyrme term) + (Wess-Zumino term)]

In lagrangean form, we write:

$$L = - \int d^3 x \, \{\tfrac{1}{2} f_\pi^2 \, Tr[\partial_\mu U^\dagger) \cdot (\partial_\mu U)] + \frac{1}{32e^2} \, Tr([U^+(\partial_\mu U), U^+(\partial_\nu U)]^2)\}$$

$$+ \frac{iN_c}{240\pi^2} \int_0^1 dt \int dx^4 \; \varepsilon^{ijklm} \, Tr \, [(U^+(\partial_i U)(U^+(\partial_j U)(U^+(\partial_k U))$$

$$(U^+(\partial_l U)|) \; (U^+(\partial_m U))] \qquad (1)$$

where U is a unitary chiral field which maps a chiral flavour group, which remains after symmetry breaking onto some 3-dimensional manifold ($\mathbb{R}^3$ or S^3, the 3-dimensional sphere). We use the notation such that repeated indices means summation and the complete antisymmetric symbols are $\varepsilon^{n_i \ldots n_m}$ in m indices.

The first term in the lagrangean contains the physical information that permits to describe low energy pion dynamics. This term may have a soliton solution, as we shall discuss later. It is proportional to the square of the pion decay constant, f_π.

The second is the Skyrme term, which "stabilizes" the soliton solution for the first term. We shall comment on it. It introduces an additional parameter, e.

The third term is the Wess-Zumino term, and is absent for the $SU(2)_L$ x $SU(2)_R$ case, it only appears for $SU(3)_L$ x $SU(3)_R$. Its coefficient, N_c, turns out to be the possible number of colours or varieties of quarks with a given flavour [4].

It is currently stated that a soliton for the first term cannot be stable. The argument, in its simplest version, is as follows [9,10]. Assume that the mass of the soliton is given, in the static limit, by:

$$M^{\sigma} = \int d^3 x \frac{1}{2} f_{\pi}^2 \, Tr[(\partial_{\mu}U_o^{+})(\partial^{\mu}U_o)] \tag{2}$$

The subscript in U_o indicates a soliton. If, inside the integral symbol, we make the change of variable:

$$x \rightarrow \lambda x \tag{3}$$

we have

$$M_{\lambda}^{\sigma} = 1/\lambda \, M^{\sigma} \tag{4}$$

so, by taking λ big enough we make M_{λ} small at will. The argument can be more sophisticated, in terms of the so called Bogomol'ny's bound [8] as used originally by Skyrme.

The solution is to stabilize via the Skyrme term. If we apply the same reasoning for the first two terms in eq. (1) we get:

$$M_{\lambda}^{\sigma} = \frac{1}{\lambda} \; M^{\sigma} + \lambda M^{sk.} \tag{5}$$

which has a stable solution at a value:

$$\lambda^2 = - \frac{M^{\sigma}}{M^{sk}} \tag{6}$$

Having overcome this difficulty, applications have been made [11] substituting the "hedgehog" form for the chiral U field on SU(2):

$$U = \cos F(r) + i\vec{\tau}.\vec{n} \sin F(r) \tag{7}$$

with $\vec{n}=\vec{r}/|\vec{r}|$ and F a function depending only on the radial coordinate.

With this solution, the winding number given by:

$$B = \frac{1}{24\pi^2} \varepsilon^{klm} \int d^3 x \, Tr[U^+(\partial_k U))(U^+(\partial_l U))(U^+(\partial_m U))] \quad (8)$$

is:

$$B = \frac{1}{2\pi^2} \int_0^\infty \frac{\sin^2 F}{r^2} \frac{dF}{dr} d^3x = \frac{2}{\pi} \int_{F(o)}^{F(\infty)} \sin^2 F dF = \frac{1}{\pi} \{F(\infty) - F(o)\} \quad (9)$$

It is, as a result, an integer whenever:

$$F(o) = -n\pi \quad , \quad F(\infty) = o \quad (10)$$

This baryonic soliton, nontheless, is not yet a particle with spin, nor an object with isotopic properties. To introduce them, it is usual to recur to collective coordinates which rotate it:

$$U(\vec{x},t) = A(t)U(\vec{x})A^{-1}(t) = \cos F(r) + i\tau_i D_{ik}(t) n_k \sin F(r) \quad (11)$$

with $D_{ik}(t)$ SU(2) rotation matrices, and A an SU(2) matrix.

It is shown that the Casimir operators both for the isotopic spin and angular momentum are the same [9,10]:

$$\vec{T}^2 = \vec{J}^2 \quad (12)$$

and the hamiltonian is the one of a rotating top:

$$H = \frac{1}{2\theta} \vec{J}^2 + M \quad (13)$$

with:

$$\theta = \frac{8\pi}{3} \int_0^\infty r^2 dr \sin^2 F \left\{ \frac{1}{4} f_\pi^2 + \frac{1}{e^2} \left(\left(\frac{dF}{dr}\right)^2 + \frac{\sin^2 F}{r^2} \right) \right\} \quad (14a)$$

$$M = 4\pi \int_0^\infty dr\, r^2 \left\{ f_\pi^2 \left[\left(\frac{dF}{dr}\right)^2 + \frac{2}{r^2} \sin^2 F(r) \right] + \frac{\sin^2 F(r)}{e^2 r^2} \left[\left(\frac{dF}{dr}\right)^2 + \frac{\sin^2 F}{2r^2} \right] \right\} \quad (14b)$$

Notice, however, than in $SU(3)_L \times SU(3)_R$ the WZ-term makes the soliton (the baryon) to be a fermion[5,9].

To study low energy pion-nucleon interaction a pion fluctuation is introduced via a chiral rotation [8]:

$$U_{\pi N} = U_\pi U_o U_\pi \tag{15a}$$

$$U_\pi = \exp\left[\frac{1}{f_\pi} i\vec{\tau} \cdot \vec{\pi}(x)\right] \tag{15b}$$

(where $\vec{\pi}$ is the pion isovector field) and when this is taken into the expression for the lagrangean (1), it turns out that up to order $\vec{\pi}^2$ one recovers a chiral effective lagrangean provided the identification is (Goldberger-Treiman [1]):

$$1/f_\pi = \frac{g_V}{g_A} \frac{g_{NN\pi}}{2M_N} \tag{16}$$

where (g_V/g_A) is the ratio of couplings for the vector and axial-vector currents, $g_{NN\pi}$ is the strong pion-nucleon coupling constant and M_N is the mass of the nucleon (obtained as given in (14b)).

With all these tools at hand, several applications have been developed in the realm of static or nearly static hadronic parameters (11, 12). The success of the numerical predictions appears to be reasonable.

The value of e, the Skyrme parameter, is fixed by the nucleon mass in eq. (14b). However, the form of the pion nucleon interaction obtained from the introduction of the $\vec{\pi}$ field does not depend on the parameter e.

It has been recently attempted to fix e by the use of QCD sum rules, [13], and it turned to be approximately equal to $^1/f_\pi$.

In the next section, we analyze mainly the contribution of the non--linear σ-model term to the soliton.

III - THE NON-LINEAR σ-MODEL SOLITON REVISITED

The aim of this section is twofold: Starting from the lagrangean arrived at from eq. (1) by the substitution of the hedgehog soliton (eq.(7)), we show that:

(i) The stability argument developed before (eqs. (3)-(6)) may not apply to the soliton obtained from the hedgehog, since an arbitrary parameter, with dimension (lenght)$^{-1}$ is necessary because of the singularity of the Euler-Lagrange equation;

(ii) It is hard to see from a variational calculation that the soliton solution gets stability from the introduction of the Skyrme term.

We have, for the static hedgehog

$$L = -4\pi \quad f_\pi^2 \int_0^\infty dr \; r^2 [\frac{dF(r)}{dr}]^2 + 2 \sin^2 F] \tag{17}$$

The Euler-Lagrange equation is:

$$r^2 \frac{d^2F}{dr^2} + 2r \frac{dF}{dr} = \sin 2F \tag{18}$$

which can be reduced to the form:

$$\chi''(x) = \frac{d^2\chi(x)}{dx^2} = \frac{2}{x} \sin (\frac{\chi(x)}{x}) \tag{19}$$

$$\chi(r) = rF(r), \quad x = \frac{1}{2} r \tag{20}$$

This equation is singular at $x = 0$ ($r=0$). This means that it doesn't satisfy the Lifshitz condition [14], which ensures the uniqueness of the solution as determined by the boundary conditions.* Instead, they may depend continuously on an undetermined parameter. Let us show how this happens in our case.

Assume that we develop a regular solution to (19) in power series near the origin

$$\chi(x) \underset{x \sim 0}{=} \chi(0) + \chi'(0)x + \frac{1}{2!} x^2 \chi''(0) + \dots \tag{21}$$

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* We thank Prof. Jair Koiller, Inst. de Matemática, UFRJ, for having directed our attention to this point.

Substituing it into (19), we need that

$$\chi(0) = 0 \tag{22a}$$

$$\chi'(0) = -2n\pi \ , \qquad n \in \mathbb{Z}^{+} \tag{22b}$$

The first result is the condition that F(r) be finite at the origin. The second, which turns to be necessary for eq. (19) to have a regular solution, is precisely the same needed to guarantee that the (winding) baryon number is an integer. Proceeding further, we get a trivial identity for $\chi''(0)$, and from there on all coefficients are expressed in terms of powers of this quantity. The result is:

$$\chi(x) = -2n\pi x + \frac{1}{2}\chi''(0)x^2 \Big[1 - \frac{1}{5!}\chi''^2(0)x^2 + \frac{1}{6!}\frac{3^2}{2^4.7}\chi''^4(0)x^4 - $$

$$ - \frac{1}{8!}\frac{17}{2^5.3.5}\chi''^6(0)x^6 + \frac{1}{10!}\frac{3.7.73}{2^8.5.11}\chi''^8(0)x^8 - \ldots\Big] \tag{23}$$

To study the solution at infinity, we perform the usual transformations

$$y = 1/x \quad , \quad 1/y\ K(y) = \chi(x) \tag{24}$$

and the equation looks:

$$\frac{d^2K(y)}{dy^2} = \frac{2}{y^2}\sin K(y) \tag{25}$$

with

$$K(y) = \frac{1}{2!}K'(0)y^2 + \frac{1}{6!}\left(-\frac{15}{14}\right)K'^3(0)y^6 + \frac{1}{10!}\left(\frac{3^4.5}{11}\right)K'^5(0)y^{10} + \ldots \tag{26}$$

Again, $K'(0)$ is a parameter with undetermined value. This is right, since confirms the property manifested from eq. (19).

Let us remark that the addition of a Skyrme term does not modify the situation, since the contribution of the Skyrme term is less singular than the σ-model one. We shall add a few comments in the next section.

Let us turn our attention to the stability problem. The second variation of the lagrangean should correspond to a minimum, and have a definite (positive) sign. To check this, let us write:

$$F(r) \cong F_0(r) + \varepsilon u_1(r) + \frac{1}{2}\varepsilon^2 u_2(r) \tag{27}$$

where $u_1(r)$ and $u_2(r)$ are arbitrary functions and ε a small parameter. We then obtain

$$L = L_0 + \varepsilon L_1 + \frac{1}{2}\varepsilon^2 L_2 + O(\varepsilon^3) \tag{28}$$

where

$$L_0 = -4\pi f_\pi^2 \int_0^\infty dr\, r^2 \left\{ \left(\frac{dF_0}{dr}\right)^2 + \frac{2}{r^2} \sin^2 F_0 \right\} \tag{28a}$$

$$L_1 = L_1\{u_1\} = -8\pi f_\pi^2 \int_0^\infty dr\, u_1\left\{-r^2 \frac{d^2F_0}{dr^2} - 2r\frac{dF_0}{dr} + \sin 2F_0\right\} \tag{28b}$$

$$L_2 = L_2^{(1)}\{u_1\} + L_2^{(2)}\{u_2\}$$

$$L_2^{(2)}\{u_2\} = L_1\{u_2\} \tag{28c}$$

$$L_2^{(1)}\{\mu_1\} = -8\pi f_\pi^2 \int_0^\infty dr \left\{ r^2 \left(\frac{du_1}{dr}\right)^2 + 2\cos 2F_0\, u_1^2 \right\} \tag{28d}$$

The contribution of u_2 vanishes when the Euler-Lagrange equation is satisfied. So, in the vicinity of the solution, it is (28d) that gives the sign of the second variation. It is not definite. The inclusion of the Skyrme term does not improve this.

IV - SUMMARY AND CONCLUSION

In this note we have sketched the fundamentals of the "skyrmeon" concept as recently used rather widely in hadronic physics [15]. We have emphasized the role that chiral dynamics, as suggested from current algebra, must have in the choice of this way to try to reconcile QCD and hadron physics.

On the other hand, we have stressed that chiral non-linear static, σ-model lagrangeans in the "hedgehog" approximations, needs the introduction of an arbitrary parameter to make sense of the Euler-Lagrange equation at its singularity.

The appearance of this parameter may spoil the argument against the stability of the soliton. Several scenarios appear to be possible: the less attractive is that physically interesting quantities, like the mass of the soliton, depend monotonically on this parameter. That is, there is no solution for the equation.

$$\frac{dM}{d\chi''(o)} = 0$$

except at a maximum. The other possibilities which may be speculated (work is in progress on them) are certainly attractive.

As said before, there is no need of the Skyrme term in order to recover formally a chiral-dynamics lagrangean from the static soliton.

On the other hand, work with fermionic determinant [6] and effective actions show that whereas the non-linear σ-model and the Wess-Zumino term appear often naturally, it is hard to obtain a kind of Skyrme term.

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