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A Non-perturbative Solution of the Zero-Dimensional $\lambda\varphi^4$ Field Theory

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Abstract

We have done a study of the zero-dimensional $\lambda\varphi^4$ model. Firstly, we exhibit the partition function as a simple exact expression in terms of the Macdonald's function for $Re(\lambda) > 0$. Secondly, an analytic continuation of the partition function for $Re(\lambda) < 0$ is performed, and we obtain an expression defined in the complex coupling constant plane λ, for $|arg\,\lambda| < \pi$. Consequently, the partition function understood as an analytic continuation is defined for all values of λ, except for a branch cut along the real negative λ axis. We also evaluate the partition function on perturbative grounds, using the Borel summation technique and we found that in the common domain of validity for $Re(\lambda) > 0$, it coincides precisely with the exact expression.

Key-words: Field theory, Zero dimension, Partition function.

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1 Introduction

It is largely accepted that some insight on the behavior of Green's functions in Field theories and Statistical systems may be obtained by the analysis of zero-dimensional models [1] [2]. In particular it is expected that an exact understanding of some aspects of zero-dimensional field theories could leave trails about the non-perturbative behaviour of these theories in higher dimensions. In this sense, Bender et al. [3] for instance, have proposed an analytical approach to study non-perturbativelly quantum field theories, which requires to solve the corresponding zero-dimensional model. In the context of perturbative field theory, for the study of the summability of series giving evaluations of physical quantities, an exact analysis of a zero-dimensional field theory could help to obtain information about the large order terms behavior in the perturbative series of realistic models. In particular, we may have in mind the paper of Bender and Wu [4], who have obtained the precise asymptotic behavior for $n \to \infty$ of the n-th order Rayleigh-Schrödinger coefficient in the series for the energy levels of the anharmonic oscillator. A precise knowledge of these coefficients for arbitrary n is still missing, only bounds for their absolute values are available for finite generic values of n, even large.

Zinn-Justin [5] studying perturbation around instantons has used explicitly the zero dimensional $\lambda\varphi^4$ model in order to introduce some basic ideas to perform a detailed analysis based on numerical simulations, of the large order behavior of the perturbative expansion of various models. As noticed by Parisi [6] and also by Khuri [7] the nature of the large order estimates is strongly dependent on the analytic structure of the (presumably) summed perturbative series $F(\lambda) = \sum_n f_n \lambda^n$ as a function of the coupling constant λ. A pioneering work on the subject was done by Dyson [8], who remarked that for negative $\lambda = e^2$ in $(QED)_4$ the vacuum is metastable with a meanlife $\sim e^{-\frac{1}{|\lambda|}}$, and that a cut along the negative real λ-axis is present. Dyson emphasizes that since any physical quantity evaluated by perturbative methods is not analytic at vanishing value of the coupling constant, the asymptotic expansion is not sufficient to determine the quantity uniquely. Parisi, [6] using a functional representation of $F(\lambda)$ for $Re(\lambda) < 0$, still makes the important observation that a detailed knowledge of the imaginary part at negative values of λ would be necessary to improve Dyson's work. In other words, to obtain more detailed estimates for the coefficients in the perturbative series, a better control of the imaginary part of $F(\lambda)$ for negative $Re(\lambda)$ is required. It is worthwhile to remark that non-hermitian or unbounded Hamiltonians, in particular $(i\lambda\varphi^3)$ and $(-\lambda\varphi^4)$ models have been recently investigated by Bender et al. [9]. Arguing from the fact that the model $(-\lambda\varphi^4)$ is asymptotically free, these authors suggest that this theory should be useful for describing the Higgs boson. Historically, this idea of investigating the negative coupling scalar model in view of implementing asymptotic freedom is in fact present in the literature since the 70's [10].

In another branch or theoretical physics, zero-dimensional field models can have a direct interest to the study of disordered systems, in particular to systems presenting frustration, which is associated to negative couplings in field theory language. These systems have been studied, both from a diagrammatic lattice viewpoint (quenched random graphs) by for instance Bachas et al. [11] and Baillie et al. [12], or on more rigorous mathematical grounds by Derrida [13] and Aizeman et al. [14].

Perhaps an exact solution of the zero-dimensional $\lambda\varphi^4$ model could throw some light on the above described situations. In any case, it is clear that a main step to these kind of studies should be to understand how correlation functions behave for *complex* coupling constant, in particular for complex coupling constants having a *negative* real part. In this note we intend to go in this sense, by reducing an interacting system to its simplest possible form, the zero-dimensional $\lambda\varphi^4$ model. As a counterpart, an exact treatment is possible.

This paper is organized as follows. In section **2** the basic features of zero-dimensional field theory are reviwed. In section **3** we exhibit a non-perturbative (exact form) for the zero dimensional partition function and the analytic continuation in the coupling constant to the whole complex plane is performed. Also the exact form of the partition function is compared to the expression obtained from Borel summation of the perturbative series. In this paper we use the standard convention $\hbar = k_B = 1$. Concluding remarks are in section **4**.

2 Zero-dimensional field theory

Let $P(\varphi) \geq 0$ be a probability distribution over a random variable. The moments $< \varphi^n >$ of the probability distribution are obtained from the generating function,

$$Z(J) = \int d\varphi P(\varphi) e^{J\varphi}, \tag{1}$$

by successive derivatives,

$$< \varphi^n > = \frac{\int d\varphi\, \varphi^n P(\varphi)}{\int d\varphi P(\varphi)} = \frac{1}{Z(0)} \left[\frac{\partial^n Z(J)}{\partial J^n} \right]_{J=0} \tag{2}$$

Suppose that $P(\varphi)$ in Eq.(1) has the general form

$$P(\varphi) = e^{-\frac{1}{2}\varphi A^{-1}\varphi + f(\varphi,\lambda)}, \tag{3}$$

$f(\varphi, \lambda)$ being a regular function depending on some parameter λ (coupling constant). Then using the identity $f(\frac{\partial}{\partial J})e^{J\varphi} = f(\varphi)e^{J\varphi}$ the generating function may be written in the form,

$$Z(J) = Z(0) e^{f(\frac{\partial}{\partial J},\lambda)} e^{\frac{1}{2}JAJ} = Z(0) \sum_{n=0}^{\infty} \frac{1}{n!} \left[f(\frac{\partial}{\partial J}) \right]^n e^{\frac{1}{2}JAJ}, \tag{4}$$

which generates the diagrammatic expansion.

In this note we consider the model with a quartic probability distribution in which the partition function is given by,

$$Z(m^2, g) = \int_{-\infty}^{\infty} \frac{d\varphi}{\sqrt{2\pi}} e^{-\frac{m^2}{2}\varphi^2 - \frac{g}{4!}\varphi^4}. \tag{5}$$

The even order moments of this probability distribution can be obtained by successive derivatives respect to m^2,

$$< \varphi^{2n} > = \frac{\partial^n}{\partial (m^2)^n} \ln Z(m^2, g). \tag{6}$$

The partition function given by Eq.(5) has a contribution from the vacuum diagrams, and its perturbative expansion may be written in the form [15],

$$Z(m^2 = 1, g) = \sum_{n=0}^{\infty} (-g)^n z_n, \tag{7}$$

where the coefficients are given by

$$z_n = \frac{(4n-1)!!}{(4!)^n n!}. \tag{8}$$

Since the coefficient z_n increases as n^n, the series that defines the partition function is divergent. In this perturbative context, many authors claim that the point $g = 0$ is an essential singularity of $Z(m^2 = 1, g)$, with a cut on the negative real axis. In other words, the perturbative series may have zero radius of convergence. Nevertheless, resummation techniques can be used to deal with this non-convergent series. It is important to stress that in real models in field theory the same kind of problem appears when the perturbative series is asymptotic but divergent for finite values of the coupling constant. Actually, 't Hooft and Lautrup [16] showed that in the $\lambda\varphi^4$ model the Borel transform of the perturbative series has renormalons, which prevents Borel summability. In this case an alternative method which takes into account the existence of renormalons was developed by Khuri [7].

In the zero dimensional model, although for negative g the partition function is not defined, an analytic continuation from positive g can be performed by considering the contribution from the saddle points, as was remarked by Langer [17]. In this paper, we adopt an easier way to obtain information from the region $Re(g) < 0$. Although the partition function, when $Re(g) < 0$ is divergent, we are able to recover this divergence as singularities of a function defined on the complex coupling constant plane. In other words, we will obtain first an exact expression in terms of Bessel functions of the second kind for the partition function in the domain $Re(g) > 0$, and after this step, we analytically extend this function to the complex plane (i.e. also in the region where the original partition function diverges, $Re(g) < 0$). *We take this analytic continuation to the whole coupling constant complex plane as our definition of the partition function.*

3 The analytic continuation of the partition function

We have two different steps to accomplish: the first one is to find a representation in terms of special functions of Eq.(5), and the second one is to perform the analytic extension in the g-variable to the region $Re(g) < 0$. We accomplish our first step by a simple inspection in Gradhstein and Ryzhik [18]. Integrals of the type in Eq.(5) can be expressed as

$$\int_{-\infty}^{\infty} dx e^{-2\nu x^2 - \mu x^4} = \frac{1}{2}\sqrt{\frac{2\nu}{\mu}} e^{\frac{\nu^2}{2\mu}} K_{\frac{1}{4}}(\frac{\nu^2}{2\mu}), \tag{9}$$

for $Re(\mu) > 0$, which means that the partition function given by Eq. (5) may be exactly expressed in terms of the Bessel function of the second kind $K_{\frac{1}{4}}$ (Macdonald's function) in the form,

$$Z(m^2, g) = \sqrt{\frac{2}{\pi}}\sqrt{\frac{3m^2}{4g}} e^{\frac{3m^4}{4g}} K_{\frac{1}{4}}(\frac{3m^4}{4g}), \tag{10}$$

in the domain $Re(g) > 0$. Defining a rescaling of the coupling constant, $\lambda = \frac{4g}{3m^4}$, the partition function becomes

$$Z(m^2, \lambda) = \sqrt{\frac{2}{\pi}} \frac{1}{\sqrt{m^2\lambda}} e^{\frac{1}{\lambda}} K_{\frac{1}{4}}(\frac{1}{\lambda}), \tag{11}$$

valid for $Re(\lambda) > 0$. An inspection of Eq. (11) seems to indicate in the analytic structure of $Z(m^2, \lambda)$ the existence of an essential singularity at $\lambda = 0$. Actually, as we will see later on, this is only apparent, the only singularity present in the analytically continued partition function is a branch cut for values of λ lying along the negative real axis. We will perform our second step, by analytically extending the partition function $Z(m^2, \lambda)$ to the region $Re(\lambda) < 0$, i.e. to the whole complex λ-plane.

This analytical continuation may be done by simply starting from the following representation for the Bessel functions of the second kind [19],

$$K_\nu(z) = \sqrt{\frac{\pi}{2z}} \frac{e^{-z}}{\Gamma(\nu + \frac{1}{2})} \int_0^\infty ds\, e^{-s} s^{\nu - \frac{1}{2}} \left[1 + \frac{s}{2z}\right]^{\nu - \frac{1}{2}}, \tag{12}$$

valid for $|arg(z)| < \pi$, $Re(\nu) > -\frac{1}{2}$. Replacing the above equation in Eq.(11) we obtain an analytic continuation for $Z(m^2, \lambda)$ in the whole complex λ-plane except for a cut along the negative real λ-axis,

$$Z(m^2,\lambda) = \frac{1}{m\Gamma(\frac{3}{4})}\int_0^\infty ds\, e^{-s}s^{-\frac{1}{4}}\left[1+\frac{s\lambda}{2}\right]^{-\frac{1}{4}}, \tag{13}$$

for $|arg(\lambda)| < \pi$.

We have thus as a starting point a formula for the partition function defined for $Re(\lambda) > 0$. It happens that this function has a representation defined as an analytic function on the domain $|arg\,\lambda| < \pi$. Hence, we have an analytic extension of the partition function for the whole complex plane of the coupling constant λ, except for $|arg\,\lambda| = \pi$. In other words, we have an exact expression for the partition function valid in the whole coupling constant complex plane except for a branch cut on the real negative axis. Actually, we may obtain in a closed form an expression for the partition function. From the representation of Macdonald's function in terms of the confluent hypergeometric function,

$$K_\nu(z) = \sqrt{(\pi)}(2z)^\nu e^{-z}\Psi(\nu+\frac{1}{2}, 2\nu+1, 2z), \tag{14}$$

we obtain replacing the above representation in Eq. (11) the simple expression,

$$Z(m^2,\lambda) = \frac{1}{m}(\frac{2}{\lambda})^{\frac{3}{4}}\Psi(\frac{3}{4},\frac{3}{2},\frac{2}{\lambda}). \tag{15}$$

We remark that the $\Psi(a,c,z)$ is a many-valued function of z, and we shall consider in the above equation its principal branch in the plane cut along the negative real axis. The analytic continuation of $K_{\frac{1}{4}}(\frac{1}{\lambda})$ corresponds to the definition of $Z(m^2,\lambda)$ on the whole complex λ-plane except for a branch cut for $|arg\,\lambda| = \pi$.

The plots of the real and imaginary parts of the analytically continued partition function given by Eq. (13) are in (fig.1) and (fig.2). We see from these figures that the real part of the partition function is perfectly regular for any complex values of λ. The branch cut for λ on the negative real axis appears only in the imaginary part of the partition function. It does not appear in those graphics an essential singularity at $\lambda = 0$, as claimed by many authors. Indeed, using the expansion for the Bessel function of the second kind for small λ, we get from Eqs. (13), (12) and (11) a Taylor series for the partition function,

$$Z(m^2,\lambda)|_{m^2=1} = 1-\frac{3}{32}\,\lambda+\frac{105}{2048}\,\lambda^2-\frac{3465}{65536}\,\lambda^3+\frac{675675}{8388608}\,\lambda^4-\frac{43648605}{268435456}\,\lambda^5+O(\lambda^6) \tag{16}$$

valid for $|arg(\lambda)| < \pi$, which clearly shows the absence of an essential singularity of the partition function at $\lambda = 0$.

It is interesting to compare our exact result in Eq.(15) or Eq.(13) with the partition function obtained from perturbative methods as in Eqs.(7) and (8). Since, as argued in [15] the series in Eq.(7) is asymptotic, we define its Borel transform as

$$B(b) = \sum_{n=0}^{\infty} \frac{z_n}{n!}(-b)^n, \tag{17}$$

and replacing z_n from Eq.(8) in the above expression we can show that the Borel transformed series $B(b)$ is convergent, given by an hypergeometric function,

$$B(b) = F(\frac{1}{4}, \frac{3}{4}; 1; -\frac{2b}{3}). \tag{18}$$

Then, from the Watson-Nevanlinna-Sokal theorem [21], the divergent series in Eq.(7) is Borel summable and, remembering that $\lambda = \frac{4g}{3}$ and using Eq. (18), its Borel sum is given by

$$Z_{pert}(m^2 = 1, \lambda) = \frac{4}{3\lambda} \int_0^{\infty} db\, e^{-\frac{4b}{3\lambda}} B(b) = \sqrt{\frac{2}{\pi}} \frac{1}{\sqrt{\lambda}} e^{\frac{1}{\lambda}} K_{\frac{1}{4}}(\frac{1}{\lambda}). \tag{19}$$

The above representation for the partition function obtained using the Borel summation technique is valid for $Re(\lambda) > 0$, λ belonging to a disc $C_R = \{\lambda : Re(\lambda^{-1}) > \frac{1}{R}\}$, that is, the Borel summed expression for the partition function is restricted to positive values of the real part of the coupling constant. In this region the Borel summed expression for the partition function coincides *precisely* with the exact partition function Eq. (11) for $Re(\lambda) > 0$. For $Re(\lambda) < 0$ the Borel summed partition function is no longer valid. The analytically continued exact expression given by Eq.(15) or by E.(13) should then be used.

The even order moments φ^{2n} for all complex values of λ except for λ lying on the real negative axis can be obtained exactly from Eq.(15) or from Eq.(13) (remembering that $\lambda = \frac{4g}{3m^4}$) by direct application of Eq.(6). We note, in connection to the idea of perturbation around some kinds of non-Gaussian probability distributions (instantons) in the paper by Zinn-Justin [5], that we can work out perturbation theory around the quartic probability distribution Eq.(5) using its exact analytically extended expression (15) or (13), not necessarily restricted to positive real λ values. Then a possible extension of this work is to introduce a $\sigma\varphi^6$ term in the partition function to discuss the tricritical singularity in this oversimplyfied model [22].

4 Conclusions

In quantum field theory it is well known that the separation of the Hamiltonian into the free and the interaction part leads to conceptual problems in many models, since the perturbative expansion based on the free part is divergent. In these situations the interaction part should not be used as a small perturbation, because at the origin of the interaction parameter an essential singularity would be present. The main idea is to include the interaction part in the new definition of a unperturbated Hamiltonian. It is expected that if it is possible to implement such a program, it would be equivalent to the resummation of the perturbative series taking into account non-perturbative effects. In this paper we have used this idea to solve the zero dimensional $\lambda\varphi^4$ model. Using the principle of analytic continuation, we have obtained an exact expression for the partition function of the model defined on the complex coupling constant plane, which presents no essential singularity at the origin. We compare the Borel summed form $Z_{pert}(m^2 = 1, \lambda)$ of the partition function and our exact expression and we have a perfect agreement between the two functions in their common domain of validity.

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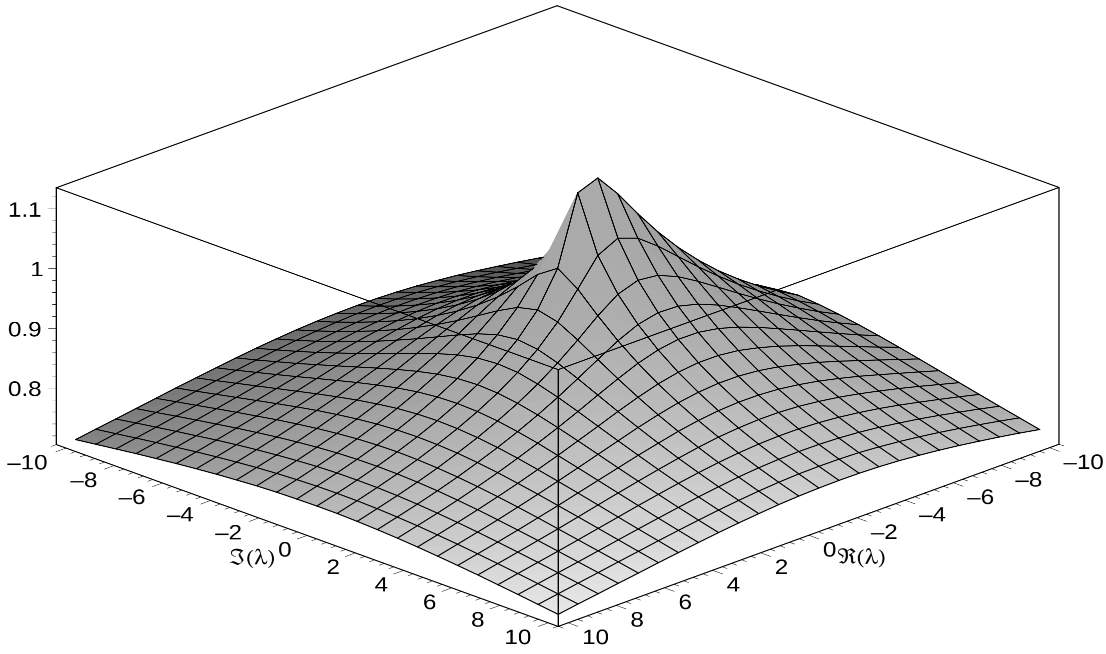

Figure 1: Plot of the real part of the partition function $Z(m^2, \lambda)$ in the complex coupling constant plane. We take $m^2 = 1$.

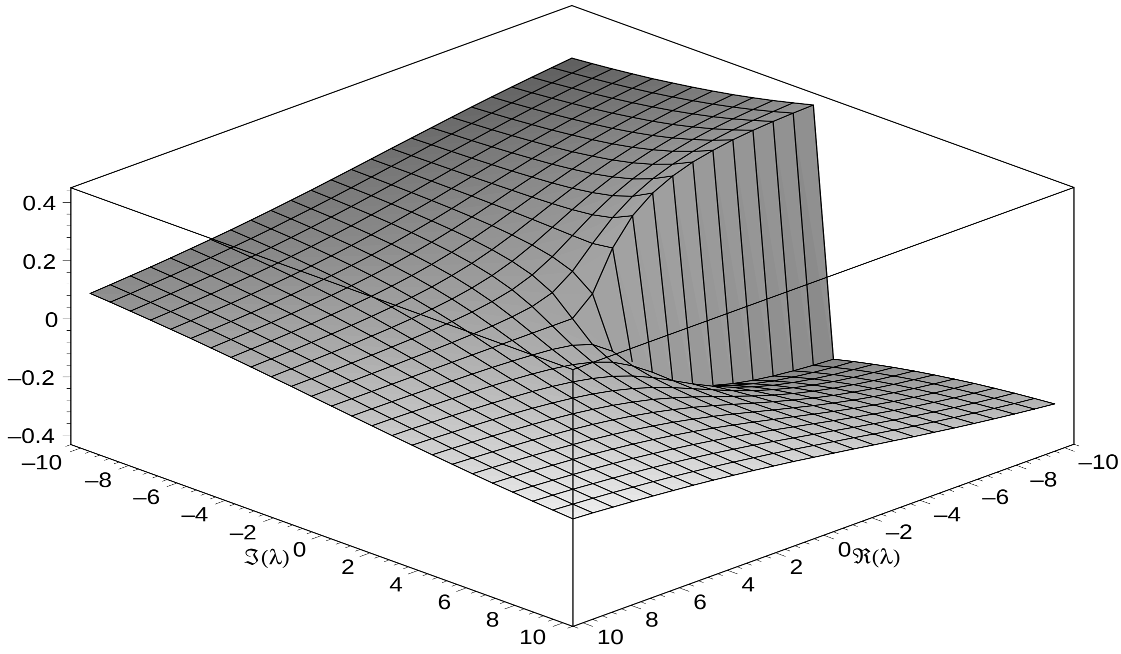

Figure 2: Plot of the imaginary part of the partition function $Z(m^2, \lambda)$ in the complex coupling constant plane. We take $m^2 = 1$. Note the branch cut for $Re(\lambda) < 0$.