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A Family of Nonextensive Entropies

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Abstract

A generalized nonextensive two-parameter entropy is developed, along lines which unify current nonextensive frameworks. It recovers, as particular cases, the Tsallis and symmetric entropies, as well as the Boltzmann-Gibbs entropy. The properties of the new (q, q')-entropy are analysed.

Key-words: Statistical mechanics; Entropy; Nonextensivity.

Systems presenting long range interactions and/or long duration memory have been shown to be not well described by the Boltzmann-Gibbs statistics. Some examples may be found in gravitational systems, Lévy flights, fractals, turbulence physics, and even economics (see [1] and references therein). An attempt to deal with such systems was formulated by Tsallis [2]. He postulated a nonextensive entropy that generalizes the Boltzmann-Gibbs formalism through an entropic index q. The usual statistical mechanics is recovered as a particular case in the $q \to 1$ limit. Tsallis formalism has been applied to a variety of systems, such as Lévy anomalous diffusion [3], self-gravitating systems [4], peculiar velocities of galaxies [5], turbulence in pure electron plasma [6], solar neutrinos [7], linear response theory [8], perturbative and variational methods [9], Green's functions [10], phonon-electron interactions [11], low dimensional dissipative systems [12]. For an up-to-date bibliography, see [13]. Tsallis generalization is not unique. As a matter of fact, a generalization had already been suggested by Rényi [14] previously. These generalized entropies (and also the Boltzmann-Gibbs entropy) were *postulated* and then their properties investigated. It was recently proposed by Abe [15] a way to *generate* entropy functionals. The procedure is rather simple. Consider the probabilities $\{p_i\}$ associated with W microstates and consider the function $g(\alpha) = \sum_{i=1}^{W} p_i^{\alpha}$. Obviously $g(1) = 1$. It can be shown [15] that the Boltzmann-Gibbs entropy is obtained by the action of the derivative operator on $g(\alpha)$:

$$\begin{aligned} S_1 &= -k \left. \frac{dg(\alpha)}{d\alpha} \right|_{\alpha=1} \\ &= -k \sum\nolimits_{i=1}^{W} p_i \ln p_i \ , \end{aligned} \tag{1}$$

where k is a positive constant. Tsallis entropy is generated by the same procedure, but using the Jackson's q-derivative operator [16]

$$\frac{d_q f(x)}{d_q x} = \frac{f(qx) - f(x)}{qx - x} \ . \tag{2}$$

Applying it on $g(\alpha)$, yields

$$\begin{aligned} S_q^T &= -k \left. \frac{d_q g(\alpha)}{d_q \alpha} \right|_{\alpha=1} \\ &= k \frac{1 - \sum_{i=1}^{W} p_i^q}{q - 1} \ , \end{aligned} \tag{3}$$

where q is the entropic index and the limit $q \to 1$ recovers the Boltzmann-Gibbs entropy in the same way that the Jackson's derivative recovers the usual derivative.

Another particular case was analysed by Abe using this time the symmetric q-derivative

$$\frac{d_q^S f(x)}{d_q^S x} = \frac{f(qx) - f(q^{-1}x)}{(q - q^{-1})x} \tag{4}$$

that is invariant under $q \leftrightarrow q^{-1}$, a symmetry that plays a central role in the physical context of quantum groups [17]. The entropy generated is, thus,

$$\begin{aligned} S_q^S &= -k \left. \frac{d_q^S g(\alpha)}{d_q^S \alpha} \right|_{\alpha=1} \\ &= k \sum_{i=1}^{W} \frac{p_i^{q^{-1}} - p_i^q}{q - q^{-1}} \ . \end{aligned} \tag{5}$$

In this Letter, we show that it is possible to use $g(\alpha)$ in order to obtain a family of nonextensive entropies. We consider here the following derivative

$$\frac{d_{q,q'}f(x)}{d_{q,q'}x} = \frac{f(qx) - f(q'x)}{(q - q')x}, \quad q, q' \in \mathcal{R} \tag{6}$$

already proposed by Chakrabarti and Jagannathan [18], that is a generalization of the Jackson ($q' = 1$), symmetric ($q' = q^{-1}$) and McAnally ($q \to q^{1-\lambda}, q' \to q^{-\lambda}$, where q and λ are the parameters in McAnally [19] formulation) derivatives. We notice that equation (6) presents invariance under the exchange $q \leftrightarrow q'$. It follows naturally our proposal for a generalized (q, q')-entropy:

$$\begin{aligned} S_{q,q'} &= -k \left. \frac{d_{q,q'}g(\alpha)}{d_{q,q'}\alpha} \right|_{\alpha=1} \\ &= k \sum_{i=1}^{W} \frac{p_i^{q'} - p_i^{q}}{q - q'} \; . \end{aligned} \tag{7}$$

The two-parameter (q, q')-entropy may be expressed in terms of Tsallis entropy as

$$S_{q,q'} = \frac{(1 - q')S_{q'}^T - (1 - q)S_q^T}{q - q'} \; . \tag{8}$$

Next we discuss some properties of this (q, q')-entropy.

i) Positivity. $S_{q,q'} \geq 0, \forall q, q'$. In the case of certainty ($p_i = 1, p_{j \neq i} = 0$), $S_{q,q'} = 0$, for both $q > 0$ and $q' > 0$.

ii) Expansibility. If we add events with vanishing probabilities, the entropy remains invariant, for both $q > 0$ and $q' > 0$.

iii) Nonadditivity. If we consider a system composed by two independent sub-systems A and B, with factorized probabilities $\{p_{i,A}\}$ and $\{p_{i,B}\}$, it is possible to express the entropy of the composed system in the following ways (hereafter we assume $k = 1$):

$$S_{q,q'}^{(A+B)} = S_{q,q'}^{(A)} + S_{q,q'}^{(B)} + (1 - q')(S_{q,q'}^{(A)} S_{q',1}^{(B)} + S_{q,q'}^{(B)} S_{q',1}^{(A)}) + (q' - q) S_{q,q'}^{(A)} S_{q,q'}^{(B)} \; , \tag{9}$$

and

$$S_{q,q'}^{(A+B)} = S_{q,q'}^{(A)} + S_{q,q'}^{(B)} + (1 - q) S_{q,q'}^{(A)} S_{q,1}^{(B)} + (1 - q') S_{q,q'}^{(B)} S_{q',1}^{(A)} \; . \tag{10}$$

Of course we have $S_{q,1} = S_q^T$. When we put $q' = 1$, these expressions yield the Tsallis nonadditivity rule (see equation (2) of [21]), $S_q^{T(A+B)} = S_q^{T(A)} + S_q^{T(B)} + (1 - q) S_q^{T(A)} S_q^{T(B)}$.

iv) Equiprobability. In the microcanonical ensemble ($p_i = 1/W, \forall i$) we obtain

$$S_{q,q'}[1/W] = \frac{W^{1-q} - W^{1-q'}}{(q' - q)} \; . \tag{11}$$

The (q, q')-entropy is monotonically increasing with W, $\forall q, q'$ except when both $q > 1$ and $q' > 1$, and consequently, in this case, it is not possible to have a physical meaning for $S_{q,q'}$.

v) Power-law behavior. When considering the canonical ensemble, Curado and Tsallis [20] introduced the generalized q-expectation value of the observable $\hat{O}$,

$$\langle \hat{O} \rangle_q = \sum\nolimits_{i=1}^{W} p_i^q \hat{O}_i \; , \tag{12}$$

and the canonical distribution is obtained with the constraint $\langle\hat{\mathcal{H}}\rangle_q$ =constant. In the work of Abe [15], the canonical ensemble was obtained with the usual $\langle\hat{\mathcal{H}}\rangle_1$ expectation value. Here we introduce a generalized (q,q')-expectation value of an observable $\hat{O}$:

$$\langle\hat{O}\rangle_{q,q'} \equiv \sum\nolimits_{i=1}^{W} p_i^{(q+q'-1)}\hat{O}_i \ . \tag{13}$$

With this definition, the (q,q')-entropy may be rewritten as $S_{q,q'} = -\left\langle \ln_{q,q'} p_i \right\rangle_{q,q'}$, where the (q,q')-logarithm follows from the functional form of the entropy for the microcanonical ensemble (equation (11)),

$$\ln_{q,q'} x \equiv \frac{x^{1-q} - x^{1-q'}}{q'-q} \ .$$

We find the following implicit canonical distribution

$$\begin{aligned} -\frac{q'}{q'-q} p_i^{q'-1}\left[1-(q-q')\beta\varepsilon_i p_i^{q-1}\right] &- \frac{q}{q-q'} p_i^{q-1}\left[1-(q'-q)\beta\varepsilon_i p_i^{q'-1}\right] \\ &+ \beta\varepsilon_i p_i^{q-1} p_i^{q'-1} - \alpha = 0 \end{aligned} \tag{14}$$

where $\{\varepsilon_i\}$ are the eigenvalues of the hamiltonian $\hat{\mathcal{H}}$ and α and β are the Lagrange multipliers associated with the constraints $\sum_i p_i = 1$ and $\langle\hat{\mathcal{H}}\,\rangle_{q,q'} = U_{q,q'}$ = constant ($U_{q,q'}$ is the generalized (q,q')-internal energy).

vi) Concavity. Let us consider $\partial^2 S_{q,q'}/\partial p_i^2$ (or, alternatively, equation (8)). $S_{q,q'}$ presents a definite concavity if one of the parameters lies between 0 and 1 (say, $0 < q' < 1$). It is concave for $q > 1$ and convex for $q < 0$. $S_{q,q'}$ is also convex when both $q < 0$ and $q' < 0$. For the remaining regions ($q' < 0,\ q > 1$), ($0 < q < 1,\ 0 < q' < 1$) and ($q > 1,\ q' > 1$), there is a competition of effects and $S_{q,q'}$ does not present, in general, a definite concavity. If we fix one of the parameters equal to 1 ($q' = 1$), we are reduced to Tsallis entropy, and $S_{q,1}$ is concave (convex) for $q > 0$ ($q < 0$). Another particular case is when we fix one of the parameters equal to zero. Now, $S_{q,0}$ is concave (convex) for $q > 1$ ($q < 1$). The two limiting cases $S_{1,1}$ and $S_{0,0}$ are concave and convex, respectively (the former is the usual Boltzmann-Gibbs entropy).

vii) H-theorem. The time evolution of the probability distribution is given by the master equation

$$\frac{dp_i}{dt} = \sum_{j=1}^{W}(A_{ji}p_j - A_{ij}p_i) \ , \tag{15}$$

where A_{ji} is the probability of transition, per unit time, from the microscopic state j to the microscopic state i. Tsallis entropy satisfy the H-theorem, that is $dS_q^T/dt > 0$ for $q > 0$, $dS_q^T/dt = 0$ for $q = 0$, and $dS_q^T/dt < 0$ for $q < 0$. This result is obtained if one assumes [22] or not [23] the detailed balance ($A_{ij} = A_{ji}$). If we assume that the detailed balance holds for $S_{q,q'}$, we find

$$\frac{dS_{q,q'}}{dt} = \frac{1}{2}\frac{1}{q-q'}\sum_{i,j} A_{ij}(p_i - p_j)[q'(p_j^{q'-1} - p_i^{q'-1}) - q(p_j^{q-1} - p_i^{q-1})] \ . \tag{16}$$

We see that $dS_{q,q'}/dt > 0$ for $0 < q' < 1$ and $q > 1$, $dS_{q,q'}/dt = 0$ for $q' = 0$ and $q = 1$, and finally $dS_{q,q'}/dt < 0$ for $0 < q' < 1$ and $q < 0$. We find the same results if we do not assume the detailed balance, according to the lines given in [23]. The detailed balance also shows us that

$dS_{q,q'}/dt < 0$ for $q < 0$ and $q' < 0$. For the remaining regions, $dS_{q,q'}/dt$ does not have a definite sign. When we consider the particular case $q' = 1$, we are reduced to Tsallis entropy, already mentioned above. The particular case that one of the parameters is equal to zero (say, $q' = 0$), the detailed balance, then, gives

$$\frac{dS_{q,0}}{dt} = -\frac{1}{2}\sum_{i,j} A_{i,j}(p_i - p_j)(p_j^{q-1} - p_i^{q-1}) \ ,$$

and we find $dS_{q,0}/dt > 0$ for $q > 1$, $dS_{q,0}/dt = 0$ for $q = 1$ and $dS_{q,0}/dt < 0$ for $q < 1$. We therefore have a self-consistent result, *i.e.*, the regions of q and q' where $S_{q,q'}$ is increasing (decreasing) with time are coincident with those that it is concave (convex), and the regions that the (q, q')-entropy does not have a definite concavity, its time derivative does not have a definite sign.

We have thus shown that the present (q, q')-entropy exhibits the relevant properties for a generalized entropy and that for particular values of the parameters it is possible to obtain the usual Boltzmann-Gibbs entropy ($q = q' = 1$) and also Tsallis ($q' = 1$) and symmetric ($q' = q^{-1}$) Abe entropies. The symmetry $q \leftrightarrow q'$ put the entropic indexes on equal footing. We may conjecture, as a possible interpretation, that the indexes q and q' express two different sources of nonextensive behavior of the system, *e.g.*, two kinds of long range interactions, or a long range interaction and a long duration memory. It is reasonable to expect that these features lead to different nonextensive behaviors that are somehow superimposed. The description of these systems may request two different parameters to deal properly with space nonextensivity and time nonextensivity.

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