Fundamental Polyhedron and Glueing Data for the Sixth Euclidean, Compact, Orientable 3-manifold*

by

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Abstract

Compact orientable 3-manifolds have been thoroughly used in probing a possible non-trivial topology of the space-like sections of spacetime and in the construction of mini- and midi-superspaces in quantum cosmology. However, there is in the physics literature a widespread misunderstanding concerning a class of compact orientable euclidean three-manifolds, namely the sixth class in Wolf's classification. This misunderstanding is rectified by constructing a fundamental polyhedron and glueing data for a manifold of this class using a method based on the action of discrete groups of isometries on the euclidean space E^3. The correctness of this construction is checked by computing its first homology group.

Key-words: Topology; Cosmology; Local homogeneous universes.

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* This work is an expanded version of a talk presented at the Workshop *A Estrutura Topológica do Universo* held at the Universidade Federal do Rio Grande do Norte (UFRN), Natal, RN-Brasil, from August 5th to August 9th of 1996.

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1 Introduction

Compact orientable 3-manifolds have been used in cosmology in mainly two different ways. On the one hand, it has become usual to try to establish correlations between observational data (CMBR, periodic distribution of cosmic objects, etc.) and the possible non-trivial topology of the space-like sections of spacetime. On the other hand, these manifolds have been used in the construction of mini- and midi-superspaces in models of quantum cosmology (See [1] and references therein). Thus, classifications of these manifolds and determination of their main properties are of primary importance in almost any study involving the topological structure of the space-time.

There exists a diffeomorphism classification for euclidean 3-manifolds due to Wolf [2]. Corresponding to the compact orientable case Wolf found only six classes denoted by $\mathcal{G}_1, \ldots, \mathcal{G}_6$ (Theorem 3.5.5 in [2]). Two manifolds of one class, say $\mathcal{G}_1$ (a 3-torus), are diffeomorphic but they may be non-isometric, for example one may be bigger than the other. Wolf's classification is given in terms of free actions of discrete groups of isometries on euclidean space E^3 and is by now well understood.

However, there has been a misunderstanding in the physics literature, concerning the fundamental polyhedron and glueing data of a specific euclidean 3-manifold, since the pioneering paper by Ellis on topology and cosmology [3]. This 3-manifold is of class $\mathcal{G}_6$ and has been described in [3] (therein listed as the fourth manifold) as being constructed by identifying opposite sides of a translation lattice *with all pairs rotated by* π. It has generally been inferred that the π-rotation axes pass through the center of the lattice cells [1, 4]. However, as will be shown in this paper, with these glueing data we do not obtain a manifold, but an orbifold [5, 6].

In this note we construct a fundamental polyhedron and glueing data for a manifold of class $\mathcal{G}_6$ using Wolf's results formulated in terms of isometry groups acting on the euclidean space E^3. This construction has already been used by Fagundes and Gausmann in the context of Cosmic Crystallography [7]. We have chosen this method of construction because when used in the five other euclidean cases it gives rise to the correct descriptions which appear in [3, 4] and [1].

To make our paper as clear and self contained as possible, in the next section we briefly review some important results in geometry and topology needed to establish our claims. In the third section we construct a fundamental polyhedron and glueing data for a manifold of class $\mathcal{G}_6$ according to Wolf's prescription (Theorem 3.5.5 in [2]). Then we show, using a theorem due to Poincaré, that the space so built is indeed a manifold, and we check that our construction gives rise to a manifold of class $\mathcal{G}_6$ by computing its first homology group. The final section is devoted to some discussion and comments.

2 Mathematical preliminaries

We begin by recalling two classical theorems on riemannian geometry. The first one characterizes the universal covering space of a constant curvature manifold and can be stated as follows. If M^n is a complete riemannian manifold with constant curvature k then its universal covering space $\widetilde{M}^n$, with the covering metric, is isometric to [2, 8]

- Euclidean space (E^n) if $k = 0$.
- Spherical space (S^n) if $k = 1$.
- Hyperbolic space (H^n) if $k = -1$.

The second theorem gives a procedure to construct a complete constant curvature manifold[1]. Let $\widetilde{M}^n$ be euclidean, spherical or hyperbolic space, then perform the following steps in order to obtain a manifold with constant curvature [2, 8]:

- Take a discrete subgroup Γ of isometries of $\widetilde{M}^n$ acting freely, i.e., such that the set $\{p \in \widetilde{M}^n \ / \ gp = p\}$ is empty for each $g \in \Gamma$, except for the identity element.
- Define an equivalence relation in $\widetilde{M}^n$ by putting $p \sim q$ iff there exists $g \in \Gamma$ such that $gp = q$, and take the quocient $M^n = \widetilde{M}^n/\Gamma$. This means that two points on $\widetilde{M}^n$ are looked upon as the same if they are related by the action of Γ. This group Γ turns out to be an embedding (a faithful representation) of the fundamental group of M^n in the group of isometries of $\widetilde{M}^n$.
- Define in the quocient space M^n the following distance function

  $$d([p],[q]) = \min_{g,h\in\Gamma}\{d(gp,hq)\}.$$

  Due to the discreteness of Γ this distance function is well defined, the line element in the quocient M^n is the same as that in $\widetilde{M}^n$ and, as a consequence, so is the curvature.

As a consequence of this second theorem the problem of constructing *all* complete riemannian manifolds of constant curvature k becomes a problem in group theory, namely, that of classifying all discrete subgroups of isometries acting freely on $\widetilde{M}^n$. This problem has been solved in the spherical case for all dimensions, and in the euclidean case for dimensions two and three (see [2] for a detailed mathematical exposition). The hyperbolic case remains unsolved even for dimension two [5, 6].

In order to give a useful description of the quocient manifold M^n it is usual to construct a fundamental domain for it, and a very common and simple procedure is that of Dirichlet [6, 10]. The Dirichlet domain of Γ with center $p \in \widetilde{M}^n$ is the set

$$\mathcal{D}_p = \{q \in \widetilde{M}^n \ / \ d(p,q) \leq d(gp,q), \ \forall g \in \Gamma\}.$$

The Dirichlet domain is a convex polyhedron and, if the quocient space M^n is compact, it is totally finite, that is, for each $k < n$ the number of faces of $\mathcal{D}_p$ of dimension k is finite. However the shape of the polyhedron, in general, depends on the center p. This is a manifestation of the global inhomogeneity of the quocient M^n, in contrast to the global homogeneity of the universal covering $\widetilde{M}^n$.

[1]This second theorem is valid in a more general setting (see [5]) of interest to cosmology, i.e., that of locally homogeneous spaces [9].

In the remainder of this section we shall consider only the 3-dimensional case although our statements can be generalized to any arbitrary dimension [11]. When dealing with 3-dimensional polyhedra it is usual to rename the faces according to their dimension, so a 0-dimensional face is called a vertex, a 1-dimensional face is an edge, and a 2-dimensional face is simply a face.

Two fundamental properties of $\mathcal{D}_p$ are [11, 12]

1) Glueing data. For each face F of $\mathcal{D}_p$ there exists another face F', and an isometry g_F of $\widetilde{M}^3$ such that

- $g_F(F) = F'$,
- g_F moves the interior of $\mathcal{D}_p$ out of $\mathcal{D}_p$,
- $g_{F'} = g_F^{-1}$,
- $\{g_F \in Isom(\widetilde{M}^3) \ / \ F \text{ is a face of } \mathcal{D}_p\}$ is a generating set for Γ.

2) Cycles of edges. Since every edge of $\mathcal{D}_p$ is shared by exactly two faces one can form the following sequences of elements: Take an edge C_1 and one of the faces that contain C_1, say F_1, let $F_1' = g_{F_1}(F_1)$; take $C_2 = g_{F_1}(C_1)$ and F_2 the other face of C_2 different from F_1', let $F_2' = g_{F_2}(F_2)$; take $C_3 = g_{F_2}(C_2)$ and so forth. It turns out that after a finite number r of steps we return to C_1 in such a way that $g_{F_r} g_{F_{r-1}} \dots g_{F_2} g_{F_1} = 1$ (that is $C_{r+1} = C_1$) and so the sum of the dihedral angles around the C_i's taken from F_{i-1}' to F_i is 2π.

The triplets (F, g_F, F') are called the glueing data, and the quocient manifold M^3 is described as the polyhedron $\mathcal{D}_p$ with the faces F and F' identified (glued) under the action of g_F, i.e., a point $q \in F$ is identified with a point $q' \in F'$ iff $g_F q = q'$. We call $\mathcal{D}_p$ a fundamental polyhedron for M^3.

There exists a reciprocal, known as the Poincaré's Polyhedron Theorem [13], which ensures that if a polyhedron P on $\widetilde{M}^3$ has the above two properties then the group Γ generated by the g_F's is discrete, acts freely on $\widetilde{M}^3$, and so $M^3 = \widetilde{M}^3/\Gamma$ is a complete riemannian manifold with constant curvature. Relaxing a little the second property by demanding the sum of the dihedral angles of any cycle of edges to be $2\pi/m$ for some positive integer m, it turns out that the group Γ is discrete but does not act freely on $\widetilde{M}^3$ if $m > 1$, the quocient $\widetilde{M}^3/\Gamma$ is thus not a manifold but what Thurston has called an orbifold [5, 6, 11, 12, 14].

3 A manifold of class $\mathcal{G}_6$

In this section we will present a fundamental polyhedron and glueing data that correctly describe a manifold of class $\mathcal{G}_6$. We begin by showing that what in [3, 4] is claimed to be a manifold of class $\mathcal{G}_6$ is not actually a manifold, but an euclidean orbifold. Referring to Fig.1, which shows a unit cube centered at the origin and with identifications of faces given as in [3, 4], it is easy to see that any cycle of edges one can construct is of order two. Since each dihedral angle is $\pi/2$, the sum of the dihedral angles in each cycle is π, and so, according to Poincaré's Polyhedron Theorem, these glueing data do not give rise to a manifold, but to an orbifold.

We shall now discuss the problem of constructing a correct descriptions for manifolds of class $\mathcal{G}_6$. According to Wolf [2] let $\overline{X}$, $\overline{Y}$ and $\overline{Z}$ be three mutually orthogonal vectors in euclidean space E^3 with possibly different units of length, then any manifold of class $\mathcal{G}_6$ is obtained by performing the quocient of E^3 by the group Γ generated by the following three elements:

- α being translation $\overline{X}$ followed by rotation of π around the $\overline{X}$ axis,
- β being translation $\overline{Y}+\overline{Z}$ followed by rotation of π around the $\overline{Y}$ axis,
- γ being translation $\overline{X}+\overline{Y}+\overline{Z}$ followed by rotation of π around the $\overline{Z}$ axis.

A particularly simple fundamental polyhedron results when we take the origin of E^3 as the center of our Dirichlet polyhedron, and the vector $\overline{X}$ has a unit length, while $\overline{Y}$ and $\overline{Z}$ have length $1/\sqrt{2}$; the Dirichlet domain for this particular manifold of class $\mathcal{G}_6$ then becomes a cube of volume 1, and should be known as the *standard* $\mathcal{G}_6$ in analogy with what is known as the standard 3-torus (or $\mathcal{G}_1$ in Wolf's notation [2]). It should be noticed that with a different choice of a center we generally obtain a different polyhedron for the same manifold, while with a different choice for the length of the vectors we generally obtain a different manifold of class $\mathcal{G}_6$ (and therefore a different polyhedron).

To write down simple explicit formulas for the generators we introduce an orthonormal basis first suggested by Fagundes [15] (see Fig.2)

$$\begin{aligned} \overline{x} &= \overline{X}\,, \\ \overline{y} &= \overline{Y}-\overline{Z}\,, \\ \overline{z} &= \overline{Y}+\overline{Z}\,. \end{aligned}$$

In terms of these vectors we find

$$\begin{aligned} \alpha(\overline{a}) &= R_{\overline{X}}(\overline{a}+\overline{x})\,, \\ \beta(\overline{a}) &= R_{\overline{Y}}(\overline{a}+\overline{z})\,, \\ \gamma(\overline{a}) &= R_{\overline{Z}}(\overline{a}+\overline{x}+\overline{z})\,, \end{aligned}$$

where $R_{\overline{X}}$, $R_{\overline{Y}}$ and $R_{\overline{Z}}$ are π-rotations around the $\overline{X}$, $\overline{Y}$ and $\overline{Z}$ axes respectively, and $\overline{a}\in E^3$. It turns out convenient to replace the generator γ by the new one $\delta=\alpha\gamma^{-1}$, explicitly

$$\delta(\overline{a}) = R_{\overline{Y}}(\overline{a}+\overline{y})\,.$$

The actions of α, β and δ on a point $\overline{a}=(x,y,z)$ are then

$$\begin{aligned} \alpha(\overline{a}) &= (x+1,-y,-z)\,, \\ \beta(\overline{a}) &= (-x,z+1,y)\,, \\ \delta(\overline{a}) &= (-x,z,y+1)\,, \end{aligned}$$

so the generators α, β and δ and their inverses bring the center $(0,0,0)$ to new positions that lie uniformly distributed on the unit sphere centered at the origin. Moreover, the next neighbors lie uniformly distributed on a sphere of radius $\sqrt{2}$ centered at the origin, so a Dirichlet domain of this group is the cube shown in Fig.2. The identifications of faces are shown in a self consistent way in Fig.3.

Fig.4 shows how the edges are identified by the glueing data. It turns out that each cycle of edges is of order 4, so the sum of dihedral angles for each cycle of edges is 2π, thus we have ended up with a manifold.

Finally we shall compute the first homology group of the manifold so constructed in order to check that it is indeed of class $\mathcal{G}_6$. First we compute its fundamental group using the Seifert-van Kampen theorem (see Fig.4) to be

$$\pi_1(M^3) = <a,\ b,\ c\ ;\ ac^{-1}bc^{-1},\ a^2b^{-2},\ acbc>,$$

and next the first homology group is computed abelianizing $\pi_1(M^3)$ (see [16]). A simple calculation shows that $b^{-1} = ac^2$ and from this we immediately obtain

$$H_1(M^3) = <a,\ c\ ;\ a^4,\ c^4,\ aca^{-1}c^{-1}> \cong Z_4 \times Z_4.$$

This concludes our verification (see Corollary 3.5.10 in [2]).

4 Discussion

In this paper we have clarified and rectified a widespread misunderstanding concerning a specific class of euclidean compact orientable 3-manifolds. We have constructed a fundamental polyhedron and correct glueing data for one manifold of this class and we have checked our construction by using Poincaré's Polyhedron Theorem and computing its first homology group.

The method of construction used here, that of Dirichlet domains, can also be used to construct fundamental polyhedra and glueing data for any other euclidean compact orientable 3-manifold. Indeed, it can be used to construct fundamental polyhedra and glueing data for every locally homogeneous 3-manifold M^3, all we need to know is the universal covering manifold $\widetilde{M}^3$ and the group of isometries Γ of $\widetilde{M}^3$ used to obtain M^3. There exist exactly eight simply connected 3-manifolds that can be universal coverings for compact locally homogeneous 3-manifolds, this is the content of a celebrated theorem due to Thurston (see [5, 17]), and, with the exception of the hyperbolic case, for all these cases we know all subgroups of isometries that yield compact manifolds. So this method is of extreme generality.

Two other remarks are in order here. First, the Dirichlet polyhedron for a manifold is in general not unique and depends on the center we have chosen. There is only one situation in which the Dirichlet polyhedron is independent of the center, it is when the manifold is *globally* homogeneous. As is known the vast majority of 3-manifolds that admit a geometric structure (in the sense of Thurston, see [5, 17]) are not globally homogeneous, so the form of the Dirichlet polyhedron for most compact locally homogeneous 3-manifolds is center dependent, and thus a judicious choice of it would be of central importance when studying their geometrical and topological properties.

As our second and final remark, recall that compact euclidean 3-manifolds are classified by diffeomorphism classes and so in the $\mathcal{G}_6$ class, for instance, all 3-manifolds are diffeomorphic but not necessarily isometric. This *degeneracy* is due to the existence of different realizations of its fundamental group as discrete subgroups of $Isom(E^3)$, or said in other words, to the possibility of choosing different lengths for the orthogonal vectors $\overline{X}$, $\overline{Y}$ and $\overline{Z}$. This freedom of choice forms what is known as the parameter space of $\mathcal{G}_6$, and it is clear that for each set of parameters we have a different manifold of class $\mathcal{G}_6$. We have constructed *one* fundamental polyhedron with glueing data for *one* manifold of class $\mathcal{G}_6$, but the path is given for the construction of many other examples.

Acknowledgments

It is a pleasure to thank A.F.F. Teixeira, who first found the problem discussed in this paper, for useful discussions and close collaboration in parts of this work. I would also like to thank A. Bernui for introducing me into this problem and for his collaboration with the figures, and M.J. Rebouças for reading the manuscript and for many useful comments. I also thank CAPES for financial support.

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Figures

Fig.1 Fundamental polyhedron and glueing data for Ellis' orbifold. It is shown one cycle of edges of order two. It is easy to see from the glueing data that every other cycle of edges is of order two also.

Fig.2 Basis vectors and fundamental polyhedron for a manifold of class $\mathcal{G}_6$.

Fig.3 Glueing data for the fundamental polyhedron of Fig.2. The L-faces are identified by the generator α, the F-faces by the generator β, and the K-faces by δ.

Fig.4 Identification of edges by the glueing data of Fig.3.

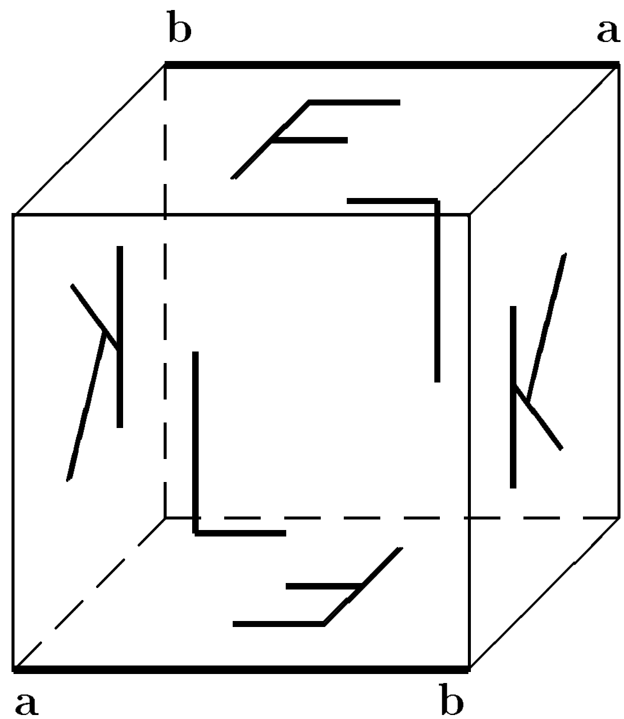

Fig.1

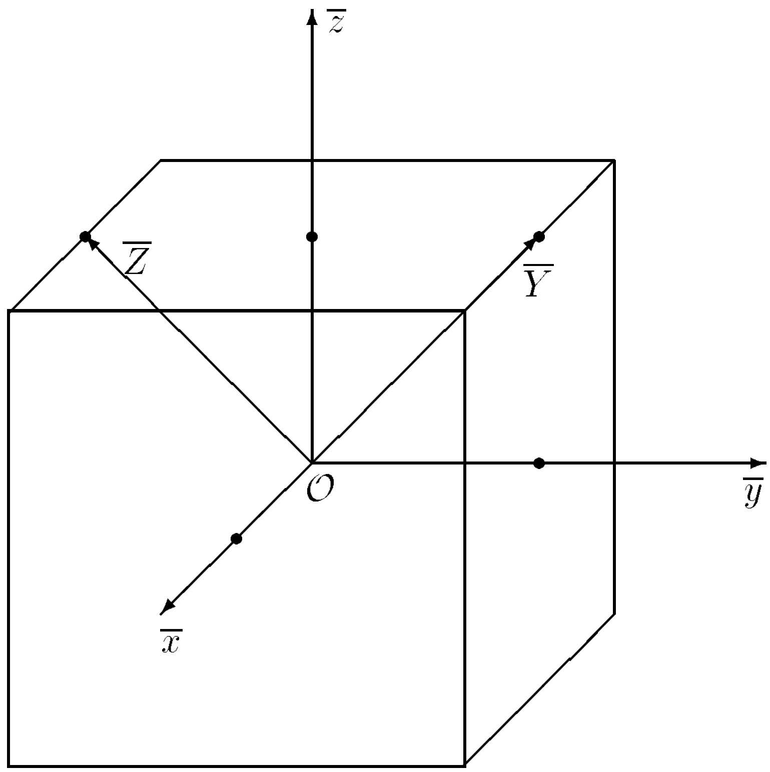
$\overline{z}$
$\overline{Z}$
$\overline{Y}$
$\mathcal{O}$
$\overline{y}$
$\overline{x}$

Fig.2

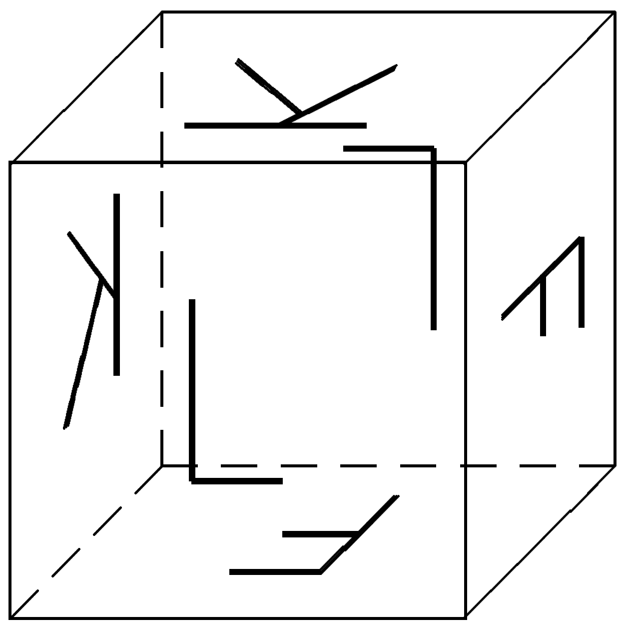

Fig.3

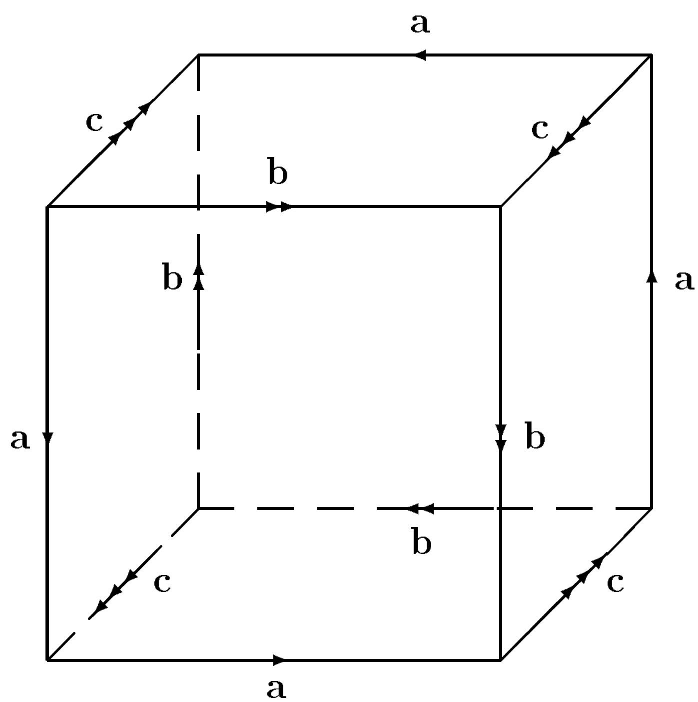

Fig.4