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Superspace Descent Equations and Zero Curvature Formalism of the Four Dimensional N=1 Supersymmetric Yang-Mills Theories

L.C.Q.Vilar[1], *C.A.G.Sasaki*[1] and *S.P. Sorella*[2]

[1]Centro Brasileiro de Pesquisas Físicas - CBPF
Rua Dr. Xavier Sigaud, 150
22290-180 – Rio de Janeiro, RJ – Brasil

[2]UERJ, Universidade do Estado do Rio de Janeiro
Departamento de Física Teórica
Instituto de Física
Rua São Francisco Xavier, 524
20550-013, Maracanã, Rio de Janeiro, Brazil

Abstract

The supersymmetric descent equations in superspace are discussed by means of the introduction of two operators ζ^{α}, $\overline{\zeta}^{\dot{\alpha}}$ which allow to decompose the supersymmetric covariant derivatives D^{α}, $\overline{D}^{\dot{\alpha}}$ as BRS commutators.

Key-words: Supersymmetry; BRS; Gauge-theories.

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1 Introduction

It is well known nowadays that the problem of finding the anomalies and the invariant counterterms which arise in the renormalization of local field theories can be handled in a purely algebraic way by means of the BRS technique[1]. This amounts to look at the nontrivial solution of the integrated consistency condition

$$s \int \omega_D^g \ = \ 0 \ , \tag{1.1}$$

where s is the BRS operator and g and D denote respectively the ghost number and the space-time dimension. Condition (1.1) when translated at the nonintegrated level yields a system of equations usually called descent equations (see [1] and refs. therein)

$$\begin{aligned} & s\,\omega_D^g \ + \ d\,\omega_{D-1}^{g+1} \ = \ 0 \ , \\ & s\,\omega_{D-1}^{g+1} \ + \ d\,\omega_{D-2}^{g+2} \ = \ 0 \ , \\ & \qquad ..... \\ & \qquad ..... \\ & s\,\omega_1^{g+D-1} \ + \ d\,\omega_0^{g+D} \ = \ 0 \ , \\ & s\,\omega_0^{g+D} \ = \ 0 \ , \end{aligned} \tag{1.2}$$

where $d = dx^\mu \ \partial_\mu$ is the exterior space-time derivative and ω_i^{g+D-i} $(0 \le i \le D)$ are local polynomials in the fields of ghost number $(g+D-i)$ and form degree i. The cases $g = 0, 1$ correspond respectively to the invariant counterterms and to the anomalies. The operators s, d obey the algebraic relations

$$s^2 \ = \ d^2 \ = \ s\,d + d\,s \ = 0 \ . \tag{1.3}$$

The problem of solving the descent equations (1.2) is a problem of cohomology of s modulo d[2, 3], the corresponding cohomology classes being given by solutions of (1.2) which are not of the type

$$\begin{aligned} \omega_m^{g+D-m} \ &= \ s\,\hat{\omega}_m^{g+D-m-1} + d\,\hat{\omega}_{m-1}^{g+D-m} \ , \qquad 1 \le m \le D \ , \\ \omega_0^{g+D} \ &= \ s\,\hat{\omega}_0^{g+D-1} \ , \end{aligned}$$

with $\hat{\omega}$'s local polynomials. Notice also that at the nonintegrated level one loses the property of making integration by parts. This implies that the fields and their derivatives have to be considered as independent variables.

Of course, the knowledge of the most general nontrivial solution of the descent equations (1.2) yields the integrated cohomology classes of the BRS operator. Indeed, once the full system (1.2) has been solved, integration on space-time gives the general solution of the consistency condition (1.1).

Recently, a new method of obtaining nontrivial solutions of the tower (1.2) has been proposed[4] and successfully applied to a large number of field models such as Yang-Mills

[1] For a recent account on the so called *Algebraic Renormalization* see [1].

theories[4, 5], gravity[6], topological field theories[7, 8, 9], string[10] and superstring[11] theories as well as W_3-algebras[12]. The method relies on the introduction of an operator δ which allows to decompose the exterior derivative as a BRS commutator,

$$d \;=\; - \;[s \;,\; \delta] \;. \tag{1.4}$$

It is easily proven in fact that repeated applications of the operator δ on the cocycle ω_0^{g+D} which solves the last of the equations (1.2) will provide an explicit nontrivial solution for the higher cocycles ω_i^{g+D-i}.

One has to note that solving the last equation of the tower (1.2) is a problem of local cohomology instead of a modulo-d one. The former can be systematically analysed by using several methods as, for instance, the spectral sequences technique[13]. It is also worth to mention that in the case of the Yang-Mills type gauge theories the solutions of the descent equations (1.2) obtained via the decomposition (1.4) have been proven to be equivalent to those provided by the so called *Russian Formula*[14, 15].

Another important geometrical aspect related to the existence of the operator δ is the possibility of encoding all the relevant informations concerning the BRS transformations of the fields and the solutions of the system (1.2) into a unique equation which takes the form of a generalized zero curvature condition[16], *i.e.*

$$\tilde{\mathcal{F}} \;=\; \tilde{d}\,\tilde{\mathcal{A}} \;-\; \tilde{\mathcal{A}}^2 \;=\; 0 \;. \tag{1.5}$$

The operator $\tilde{d}$ and the generalized gauge connection $\tilde{\mathcal{A}}$ in eq.(1.5) turn out to be respectively the δ-transform of the BRS operator s and of the ghost field c corresponding to the Maurer-Cartan form of the underlying gauge algebra

$$\begin{aligned} \tilde{d} \;&=\; e^{\delta} s e^{-\delta} \;, \qquad \tilde{d}^2 \;=\; 0 \;, \\ \tilde{\mathcal{A}} \;&=\; e^{\delta} c \;. \end{aligned}$$

As discussed in detail in refs.[16] the zero curvature formulation allows to obtain straightforwardly the cohomology classes of the operator $\tilde{d}$. The latters are deeply related to the solutions of the descent equations (1.2).

The BRS algebraic procedure can be easily adapted to include the case of the renormalizable N=1 superspace supersymmetric gauge theories in four space-time dimensions, for which a set of superspace descent equations have been established[17, 18, 19]. The solution of these equations as much as in the nonsupersymmetric case yields directly all the manifestly supersymmetric gauge anomalies and the BRS invariant counterterms. One has to remark however that in the supersymmetric case both the derivation and the construction of a solution of the superspace version of the descent equations are more involved than the nonsupersymmetric case, due to the algebra of the spinorial covariant derivatives D_α and $\overline{D}_{\dot\alpha}$ and to the (anti)chirality constraints of some of the superfieds characterizing the theory.

In order to have an idea of the differences between the superspace and the ordinary case, let us briefly consider the integrated superspace N=1 BRS consistency condition corresponding to the supersymmetric chiral U(1) Yang-Mills axial anomaly[19]

$$s \int d^4x \; d^2\overline{\theta} \; K^0 \;=\; 0 \;, \tag{1.6}$$

with K^0 a local power series in the gauge vector superfield with ghost number zero and dimension two. It can be proven [18, 19] that condition (1.6) implies that the BRS variation of the integrand, *i.e* sK^0, is a total derivative in superspace

$$s\,K^0 \;=\; \overline{D}_{\dot\alpha}\overline{K}^{1\dot\alpha}\;, \tag{1.7}$$

with $\overline{K}^{1\dot\alpha}$ local power series with ghost number one[2]. Acting now on both sides of eq.(1.7) with the nilpotent BRS operator s we get

$$\overline{D}_{\dot\alpha}\; s\overline{K}^{1\dot\alpha} \;=\; 0\;.$$

This equation admits a superspace solution (see Sect.5 and App.A for the details) which, as in the standard nonsuperspace case (1.2), entails a set of new conditions which together with the equation (1.7) gives the whole set of the superspace descent equations for the U(1) axial anomaly[19], namely

$$\begin{aligned}
s\,K^0 \;&=\; \overline{D}_{\dot\alpha}\overline{K}^{1\,\dot\alpha}\;,\\
s\,\overline{K}^1_{\dot\alpha} \;&=\; \left(2D^\alpha\overline{D}_{\dot\alpha} \,+\, \overline{D}_{\dot\alpha}D^\alpha\right)\; K^2_{\alpha}\;,\\
s\,K^{2\,\alpha} \;&=\; D^\alpha\; K^3\;,\\
s\,K^3 \;&=\; 0\;,
\end{aligned} \tag{1.8}$$

with K^2_α and K^3 local power series of ghost number two and three.

In this work we shall extend the decomposition formula (1.4) to the case of the N=1 four dimensional supersymmetric Yang-Mills theory, yielding thus a simple way of solving the superspace descent equations. This means that we will introduce two operators ζ_α and $\overline{\zeta}_{\dot\alpha}$ which in analogy with the case of the operator δ of eq.(1.4) allow to decompose the supersymmetric covariant derivatives D_α and $\overline{D}_{\dot\alpha}$ as BRS commutators, according to

$$\left[\zeta_\alpha\;,\;s\right] \;=\; D_\alpha\,, \qquad \left[\overline{\zeta}_{\dot\alpha}\;,\;s\right] \;=\; \overline{D}_{\dot\alpha}\;, \tag{1.9}$$

with

$$D_\alpha\,\overline{D}_{\dot\alpha} \,+\, \overline{D}_{\dot\alpha}\,D_\alpha \;=\; 2i\sigma^\mu_{\alpha\dot\alpha}\,\partial_\mu\;\;, \tag{1.10}$$

$\sigma^\mu_{\alpha\dot\alpha}$ being the Pauli matrices.

Moreover as in the nonsupersymmetric case, the decomposition formulas (1.9) will allow us to cast both the supersymmetric BRS transformations and the superspace descent equations into a zero curvature formalism, providing thus a pure geometrical framework in superspace.

The work is organized as follows. In Section 2 we introduce the general notations and we discuss the supersymmetric decomposition (1.9). Section 3 is devoted to the analysis of the algebraic relations entailed by the operators ζ_α and $\overline{\zeta}_{\dot\alpha}$. In Section 4 we

[2] The absence of the term $D^\alpha K^1_\alpha$ in eq.(1.7) is actually due to the chirality nature of the consistency condition (1.6).

present the zero cuvature formulation of the superspace BRS transformations and of the descent equations corresponding to the invariant super Yang-Mills lagrangian. In Section 5 we discuss the descent equations for the superspace version of the U(1) axial anomaly. Section 6 deals with the case of the supersymmetric chiral gauge anomaly appearing in the quantum extension of the supersymmetric Slavnov-Taylor identity. In order to make the paper self contained the final Appendices A, B and C collect a short summary of the main results concerning the Yang-Mills superspace BRS cohomology as well as the solution of certain equations relevant for the superspace version of the descent equations.

2 General Notations and Decomposition Formulas

In order to present the general algebraic set up let us begin by fixing the notations[3]. We shall work in a four dimensional space-time with N=1 supersymmetry. The superfield content which will be used throughout is the standard set of the superfields of the pure N=1 super Yang-Mills theories, *i.e.* the vector superfield ϕ and the gauge superconnections φ_α and $\overline{\varphi}_{\dot\alpha}$. They are defined as

$$\varphi_\alpha \equiv e^{-\phi} D_\alpha e^{\phi} , \qquad \overline{\varphi}_{\dot\alpha} \equiv e^{\phi} \overline{D}_{\dot\alpha} e^{-\phi} , \tag{2.11}$$

where D_α and $\overline{D}_{\dot\alpha}$ are the usual supersymmetric derivatives:

$$\begin{aligned} \{D_\alpha, D_\beta\} &= \left\{\overline{D}_{\dot\alpha}, \overline{D}_{\dot\beta}\right\} = 0 , \\ D_\alpha \overline{D}_{\dot\alpha} + \overline{D}_{\dot\alpha} D_\alpha &= 2i\sigma^\mu_{\alpha\dot\alpha} \partial_\mu . \end{aligned} \tag{2.12}$$

Introducing now the chiral and antichiral Faddeev-Popov ghosts c and $\overline{c}$

$$\overline{D}_{\dot\alpha} c = D_\alpha \overline{c} = 0 ,$$

for the superspace nilpotent BRS transformations one has

$$\begin{aligned} se^{\phi} &= e^{\phi} c - \overline{c} e^{\phi} , \\ sc &= -c^2 , \\ s\overline{c} &= -\overline{c}^2 , \\ s\varphi_\alpha &= -D_\alpha c - \{c, \varphi_\alpha\} , \\ s\overline{\varphi}_{\dot\alpha} &= -\overline{D}_{\dot\alpha}\overline{c} - \{\overline{c}, \overline{\varphi}_{\dot\alpha}\} . \end{aligned} \tag{2.13}$$

and

$$\{s, D_\alpha\} = \left\{s, \overline{D}_{\dot\alpha}\right\} = 0 .$$

Let us also give, for further use, the BRS transformations of the chiral and antichiral superfield strengths F_α and $\overline{F}_{\dot\alpha}$

$$\begin{aligned} &F_\alpha \equiv \overline{D}^2 \varphi_\alpha , && \overline{D}_{\dot\alpha} F_\alpha = 0 , \\ &\overline{F}_{\dot\alpha} \equiv D^2 \overline{\varphi}_{\dot\alpha} , && D_\alpha \overline{F}_{\dot\alpha} = 0 , \\ &sF_\alpha = -\{c, F_\alpha\} , && s\overline{F}_{\dot\alpha} = -\{\bar{c}, \overline{F}_{\dot\alpha}\} . \end{aligned} \tag{2.14}$$

The quantum numbers, *i.e.* the dimensions, the ghost numbers and the $\mathcal{R}$-weights of all the fields are assigned as follows

[3]The superspace conventions used here are those of ref. [20].

R−weights, dim. and ghost numb.

	s	D_α	$\overline{D}_{\dot\alpha}$	ϕ	c	$\overline{c}$	φ_α	$\overline{\varphi}_{\dot\alpha}$	F_α	$\overline{F}_{\dot\alpha}$
dim	0	$\frac{1}{2}$	$\frac{1}{2}$	0	0	0	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{3}{2}$	$\frac{3}{2}$
N_g	1	0	0	0	1	1	0	0	0	0
$\mathcal{R}$	0	1	-1	0	0	0	1	-1	-1	1

Table 1

The fields will be treated as commuting or anticommuting according to the fact that their total degree, here chosen to be the sum of the ghost number and of the spinorial indices, is even or odd. Otherwise stated all the fields are Lie-algebra valued, the gauge group $\mathcal{G}$ being assumed to be a semisimple Lie group with antihermitian generators T^a.

The set of fields $(c, \overline{c}, \phi, \varphi_\alpha, \overline{\varphi}_{\dot\alpha})$ and their covariant derivatives will define therefore the basic local space for studying the superspace descent equations. Let us also observe that due to the fact that $D, \overline{D}$ have dimension $\frac{1}{2}$, the number of covariant derivatives turns out to be limited by power counting requirements. For instance, as we shall see in the explicit examples considered in the next sections, the analysis of the superspace consistency condition for both the U(1) axial anomaly and the gauge anomaly requires the use of local formal power series in the variables $(c, \overline{c}, \phi, \varphi_\alpha, \overline{\varphi}_{\dot\alpha})$ of dimension 2. We recall here that the non polynomial character of certain N=1 superspace expressions is due to the fact that the vector superfield ϕ is dimensionless. Finally, whenever the space time derivatives ∂_μ appear they are meant to be replaced by the covariant derivatives $D, \overline{D}$, according to the supersymmetric algebra (2.12).

Let us introduce now the two operators ζ_α and $\overline{\zeta}_{\dot\alpha}$ of ghost number -1, defined by

$$
\begin{aligned}
\zeta_\alpha c \;=\; \varphi_\alpha\,, \qquad \overline{\zeta}_{\dot\alpha}\overline{c} \;=\; \overline{\varphi}_{\dot\alpha}\,,\\
\zeta_\alpha \overline{c} \;=\; \overline{\zeta}_{\dot\alpha} c \;=\; \zeta_\alpha \phi \;=\; \overline{\zeta}_{\dot\alpha}\phi \;=0\,,\\
\zeta_\alpha\,\varphi_\beta \;=\; \overline{\zeta}_{\dot\alpha}\,\varphi_\beta \;=\; 0\,.
\end{aligned}
\tag{2.15}
$$

It is almost immediate thus to check that they are of total degree zero and that they obey the following algebraic relations

$$
\begin{aligned}
\left[\zeta_\alpha, s\right] \;&=\; D_\alpha\,,\\
\left[\overline{\zeta}_{\dot\alpha}, s\right] \;&=\; \overline{D}_{\dot\alpha}\,,\\
\left[\zeta_\alpha, \zeta_\beta\right] \;&=\; \left[\zeta_\alpha, \overline{\zeta}_{\dot\beta}\right] \;=\; \left[\overline{\zeta}_{\dot\alpha}, \overline{\zeta}_{\dot\beta}\right] \;=\; 0\,,
\end{aligned}
\tag{2.16}
$$

yielding then the supersymmetric decomposition (1.9) we are looking for. As we shall see later on the operators ζ_α and $\overline{\zeta}_{\dot\alpha}$ will turn out to be very useful in order to solve the superspace descent equations. Let us focus for the time being on the analysis of the algebraic consequences stemming from the eqs.(2.16).

3 Algebraic Relations

To study the algebra entailed by the two operators ζ_α and $\overline{\zeta}_{\dot\alpha}$ let us first observe that they do not commute with the supersymmetric covariant derivatives D, $\overline{D}$. Instead as one can easily check by using the equations (2.15) we have, in complete analogy with the nonsupersymmetric case [4],

$$\left[\overline{\zeta}_{\dot\beta}, D_\alpha\right] \;=\; \left[\zeta_\alpha, \overline{D}_{\dot\beta}\right] \;=\; -\,G_{\alpha\dot\beta}\,, \tag{3.17}$$

$$\left[\overline{\zeta}_{\dot\alpha}, \overline{D}_{\dot\beta}\right] \;=\; \left[\zeta_\alpha, D_\beta\right] \;=\; 0 \tag{3.18}$$

where the new operator $G_{\alpha\dot\beta}$ has negative ghost number -1 and acts on the fields as

$$\begin{array}{c} G_{\alpha\dot\alpha}c \;=\; \overline{D}_{\dot\alpha}\varphi_\alpha\,, \qquad G_{\alpha\dot\alpha}\overline{c} \;=\; D_\alpha\overline{\varphi}_{\dot\alpha}\,, \\ G_{\alpha\dot\alpha}\,\phi \;=G_{\alpha\dot\alpha}\,\varphi_\beta \;\;=G_{\alpha\dot\alpha}\,\overline{\varphi}_{\dot\beta} =\; 0\;, \end{array} \tag{3.19}$$

and

$$\begin{array}{c} \left\{G_{\alpha\dot\alpha}, s\right\} \;=\; \left\{D_\alpha, \overline{D}_{\dot\alpha}\right\}\,, \\ \left[\zeta_\alpha, G_{\beta\dot\beta}\right] \;=\; \left[\overline{\zeta}_{\dot\alpha}, G_{\beta\dot\beta}\right] \;=\; \left\{G_{\alpha\dot\alpha}, G_{\beta\dot\beta}\right\} \;=\; 0\,. \end{array} \tag{3.20}$$

Again, the operator $G_{\alpha\dot\beta}$ does not anticommute with the covariant derivatives D, $\overline{D}$. It yields in fact

$$\left\{G_{\alpha\dot\alpha}, D_\beta\right\} \;=\; -\tfrac{1}{2}\epsilon_{\alpha\beta}\overline{R}_{\dot\alpha}\,,\;\; \left\{G_{\alpha\dot\alpha}, \overline{D}_{\dot\beta}\right\} \;=\; \tfrac{1}{2}\epsilon_{\dot\alpha\dot\beta}R_\alpha\,, \tag{3.21}$$

with R_α and $\overline{R}_{\dot\alpha}$ of ghost number -1 and defined as

$$\begin{array}{c} R_\alpha c \;=\; F_\alpha\,, \qquad \overline{R}_{\dot\alpha}\overline{c} \;=\; \overline{F}_{\dot\alpha}\,, \\[2ex] R_\alpha\overline{c} \;=\; 2\overline{D}_{\dot\alpha}D_\alpha\overline{\varphi}^{\dot\alpha} + D_\alpha\overline{D}_{\dot\alpha}\overline{\varphi}^{\dot\alpha} + \left(D_\alpha\overline{\varphi}_{\dot\alpha}\right)\overline{\varphi}^{\dot\alpha} + \overline{\varphi}_{\dot\alpha}\left(D_\alpha\overline{\varphi}^{\dot\alpha}\right)\,, \\[2ex] \overline{R}_{\dot\alpha}c \;=\; 2D^\alpha\overline{D}_{\dot\alpha}\varphi_\alpha + \overline{D}_{\dot\alpha}D^\alpha\varphi_\alpha + \left(\overline{D}_{\dot\alpha}\varphi^\alpha\right)\varphi_\alpha + \varphi^\alpha\left(\overline{D}_{\dot\alpha}\varphi_\alpha\right)\,, \\[2ex] R_\alpha\,\phi = \;R_\alpha\,\varphi_\beta \;=\; R_\alpha\,\overline{\varphi}_{\dot\beta} =\; R_\alpha\,F_\beta =\; R_\alpha\,\overline{F}_{\dot\beta} =\; 0\,, \\[2ex] \overline{R}_{\dot\alpha}\,\phi = \;\overline{R}_{\dot\alpha}\,\varphi_\beta \;=\; \overline{R}_{\dot\alpha}\,\overline{\varphi}_{\dot\beta} =\; \overline{R}_{\dot\alpha}\,F_\beta =\; \overline{R}_{\dot\alpha}\,\overline{F}_{\dot\beta} =\; 0. \end{array} \tag{3.22}$$

In addition, we have

$$\begin{array}{c} \left[R_\alpha, s\right] \;=\; \left[R_\alpha, D_\beta\right] \;=\; \left[R_\alpha, \overline{D}_{\dot\beta}\right] \;=\; \left[R_\alpha, G_{\alpha\dot\beta}\right] \;=\; 0\,, \\ \left[R_\alpha, \zeta_\beta\right] \;=\; \left[R_\alpha, \overline{\zeta}_{\dot\beta}\right] \;=\; \left[R_\alpha, R_\beta\right] \;=\; \left[R_\alpha, \overline{R}_{\dot\beta}\right] \;=\; 0\,. \end{array} \tag{3.23}$$

Let us finally display the quantum numbers of the operators entering the algebraic relations (2.16), (3.17), (3.21)

R−weights, dim. and ghost numb.

	ζ^α	$\overline{\zeta}^{\dot\alpha}$	$G^{\alpha\dot\alpha}$	R^α	$\overline{R}^{\dot\alpha}$
dim	$\frac{1}{2}$	$\frac{1}{2}$	1	$\frac{3}{2}$	$\frac{3}{2}$
N_g	-1	-1	-1	-1	-1
$\mathcal{R}$	1	-1	0	-1	1

Table 2

4 The Zero Curvature Condition

Having characterized all the relevant operators entailed by the consistency of the supersymmetric decomposition (2.16), let us pay attention to the geometrical aspects of the algebraic relations so far obtained. To this purpose it is useful to introduce a set of global parameters e^{α}, $\overline{e}^{\dot\alpha}$ and $\tilde{e}^{\alpha\dot\alpha}$, naturally associated to the operators ζ_α, $\overline{\zeta}_{\dot\alpha}$ and $G_{\alpha\dot\beta}$, of ghost number one and obeying the relations

$$
\begin{aligned}
e^{\alpha}\, e^{\beta} \;&=\; \overline{e}^{\dot\alpha}\, \overline{e}^{\dot\beta} \;=\; \tilde{e}^{\alpha\dot\alpha}\, \tilde{e}^{\beta\dot\beta} \;=\; 0 \;, \\
\left[e^{\alpha}, \overline{e}^{\dot\beta}\right] \;&=\; \left[e^{\alpha}, \tilde{e}^{\beta\dot\beta}\right] \;=\; \left[\tilde{e}^{\alpha\dot\alpha}, \overline{e}^{\dot\beta}\right] \;=\; 0 \;,
\end{aligned}
\tag{4.24}
$$

R−weights, dim. and ghost numb.

	e^{α}	$\overline{e}^{\dot\alpha}$	$\tilde{e}^{\alpha\dot\alpha}$
dim	$-\frac{1}{2}$	$-\frac{1}{2}$	-1
N_g	1	1	1
$\mathcal{R}$	0	0	0

Table 3

In addition, the global parameters $(e^{\alpha}, \overline{e}^{\dot\alpha}, \tilde{e}^{\alpha\dot\alpha})$ will be required to obey the following conditions

$$
\begin{aligned}
e^{\alpha}\, \tilde{e}^{\beta\dot\alpha} \;&=-\tfrac{1}{2}\epsilon^{\alpha\beta}\, e^{\gamma}\, \tilde{e}^{\dot\alpha}_{\gamma} \;, \\
\tilde{e}^{\alpha\dot\alpha}\, \overline{e}^{\dot\beta} \;&=\; \tfrac{1}{2}\epsilon^{\dot\alpha\dot\beta}\, \tilde{e}^{\alpha}_{\dot\gamma}\, \overline{e}^{\dot\gamma} \;, \\
e^{\alpha}\, \tilde{e}^{\beta\dot\alpha}\, \overline{e}^{\dot\beta} \;&=-\tfrac{1}{4}\epsilon^{\alpha\beta}\epsilon^{\dot\alpha\dot\beta}\, e^{\gamma}\, \tilde{e}_{\gamma\dot\gamma}\, \overline{e}^{\dot\gamma} \;,
\end{aligned}
$$

fixing the symmetry properties of the product of two parameters with respect to their spinorial indices. Defining now the nilpotent dimensionless operators ζ, $\overline{\zeta}$ and G as

$$
\zeta \;=\; \zeta^{\alpha}\, e_{\alpha} \;, \quad \overline{\zeta} \;=\; \overline{\zeta}_{\dot\alpha}\, \overline{e}^{\dot\alpha} \;, \quad G \;=\; G^{\alpha}_{\dot\alpha}\, \tilde{e}^{\dot\alpha}_{\alpha} \;,
$$

it is straightforward to verify that they have zero ghost number and $\mathcal{R}$ weight respectively 1, -1, 0, and that the subalgebra generated by ζ_α, $\overline{\zeta}_{\dot\alpha}$ and $G_{\alpha\dot\beta}$, *i.e.*

$$
\begin{aligned}
\left[\zeta_\alpha, \zeta_\beta\right] \;&=\; \left[\zeta_\alpha, \overline{\zeta}_{\dot\beta}\right] \;=\; \left[\overline{\zeta}_{\dot\alpha}, \overline{\zeta}_{\dot\beta}\right] \;=\; 0 \;, \\
\left[\zeta_\alpha, G_{\beta\dot\beta}\right] \;&=\; \left[\overline{\zeta}_{\dot\alpha}, G_{\beta\dot\beta}\right] \;=\; \left\{G_{\alpha\dot\alpha}, G_{\beta\dot\beta}\right\} \;=\; 0 \;,
\end{aligned}
$$

can be simply rewritten as

$$
\left[\zeta, \overline{\zeta}\right] \;=\; \left[\zeta, G\right] \;=\; \left[\overline{\zeta}, G\right] \;=\; 0 \;.
$$

Analogously, introducing the nilpotent operators $\tilde{G}$, D, $\overline{D}$, R, $\overline{R}$, ∂, $\tilde{\partial}$

$$
\begin{aligned}
\tilde{G} \;&=\; G^{\alpha}_{\dot\alpha}\, e_{\alpha}\, \overline{e}^{\dot\alpha} \;, & G \;&=\; G^{\alpha}_{\dot\alpha}\, \tilde{e}^{\dot\alpha}_{\alpha} \;, \\
D \;&=\; D^{\alpha}\, e_{\alpha} \;, & \overline{D} \;&=\; \overline{D}_{\dot\alpha}\, \overline{e}^{\dot\alpha} \;, \\
R \;&=\; R^{\alpha}\, \tilde{e}_{\alpha\dot\alpha}\, \overline{e}^{\dot\alpha} \;, & \overline{R} \;&=\; \overline{R}_{\dot\alpha}\, e^{\alpha}\, \tilde{e}^{\dot\alpha}_{\alpha} \;, \\
\tilde{\partial} \;&=\; \left\{D^{\alpha}, \overline{D}_{\dot\alpha}\right\}\, e_{\alpha}\, \overline{e}^{\dot\alpha} \;, & \partial \;&=\; \left\{D^{\alpha}, \overline{D}_{\dot\alpha}\right\}\, \tilde{e}^{\dot\alpha}_{\alpha} \;,
\end{aligned}
\tag{4.25}
$$

it is immediate to check that all the algebraic relations and field transformations of eqs.(2.15)-eqs.(3.23) may be cast into the following free index notation

$$
\begin{array}{llll}
[\zeta, s] = D\,, & \left[\overline{\zeta}, s\right] = \overline{D}\,, & \left\{\tilde{G}, s\right\} = \tilde{\partial}\,, & [G, s] = \partial\,, \\
\{s, D\} = 0\,, & \left\{s, \overline{D}\right\} = 0\,, & \left[s, \tilde{\partial}\right] = 0\,, & \{s, \partial\} = 0\,, \\
\left[D, \tilde{\partial}\right] = 0\,, & \left[\overline{D}, \tilde{\partial}\right] = 0\,, & \{D, \partial\} = 0\,, & \left\{\overline{D}, \partial\right\} = 0\,, \\
[D, \zeta] = 0\,, & \left[\overline{D}, \overline{\zeta}\right] = 0\,, & \left[\overline{D}, \zeta\right] = \tilde{G}\,, & \left[D, \overline{\zeta}\right] = \tilde{G}\,, \\
\left[\partial, \tilde{\partial}\right] = 0\,, & \left[G, \tilde{G}\right] = 0\,, & \left[\zeta, \tilde{G}\right] = 0\,, & \left[\overline{\zeta}, \tilde{G}\right] = 0\,, \\
[G, \partial] = 0\,, & \left[\tilde{G}, \tilde{\partial}\right] = 0\,, & \left[G, \tilde{\partial}\right] = 0\,, & \left\{\tilde{G}, \partial\right\} = 0\,, \\
\left\{\tilde{G}, D\right\} = 0\,, & \left\{\tilde{G}, \overline{D}\right\} = 0\,, & 2\,[D, G] = \overline{R}\,, & 2\left[G, \overline{D}\right] = R\,, \\
\left[\zeta, \tilde{\partial}\right] = 0\,, & \left[\overline{\zeta}, \tilde{\partial}\right] = 0\,, & 2\,[\zeta, \partial] = \overline{R}\,, & 2\left[\partial, \overline{\zeta}\right] = R\,, \\
[\zeta, R] = 0\,, & \left[\zeta, \overline{R}\right] = 0\,, & \left[\overline{\zeta}, R\right] = 0\,, & \left[\overline{\zeta}, \overline{R}\right] = 0\,, \\
\left[R, \tilde{\partial}\right] = 0\,, & \left[\overline{R}, \tilde{\partial}\right] = 0\,, & \{R, \partial\} = 0\,, & \left\{\overline{R}, \partial\right\} = 0\,, \\
\{D, R\} = 0\,, & \left\{\overline{D}, R\right\} = 0\,, & \left\{D, \overline{R}\right\} = 0\,, & \left\{\overline{D}, \overline{R}\right\} = 0\,, \\
\left\{\tilde{G}, R\right\} = 0\,, & \left\{\tilde{G}, \overline{R}\right\} = 0\,, & [G, R] = 0\,, & \left[G, \overline{R}\right] = 0\,, \\
\{s, R\} = 0\,, & \left\{s, \overline{R}\right\} = 0\,, & \left\{R, \overline{R}\right\} = 0\,, &
\end{array}
\tag{4.26}
$$

Let us proceed now by showing that, as announced in the introduction, the supersymmetric BRS transformations (2.13), (2.14) can be obtained by means of a generalized zero curvature condition. To this aim let us introduce the operator δ

$$
\delta = \zeta + \overline{\zeta} - G\,, \tag{4.27}
$$

from which one easily obtains the following decomposition

$$
[s, \delta] = -D - \overline{D} - \partial\,.
$$

Defining now the δ-transform of the BRS operator s as

$$
\tilde{d} = e^{\delta}\, s\, e^{-\delta}\,, \tag{4.28}
$$

one gets

$$\begin{aligned}\widetilde{d} &= s + D + \overline{D} + \partial - \widetilde{G} + \frac{1}{2}\overline{R} - \frac{1}{2}R\,, \\ \widetilde{d}\,\widetilde{d} &= 0\,,\end{aligned} \tag{4.29}$$

so that, calling $\widetilde{A}$ and $\widetilde{\overline{A}}$ the δ-transform of the chiral and antichiral ghosts $(c, \overline{c})$

$$\widetilde{A} = e^{\delta}\, c = c + \varphi + \overline{D}\,\varphi\,, \qquad \varphi = \varphi^{\alpha} e_{\alpha}\,, \tag{4.30}$$

$$\widetilde{\overline{A}} = e^{\delta}\, \overline{c} = \overline{c} + \overline{\varphi} + D\,\overline{\varphi}\,, \qquad \overline{\varphi} = \overline{\varphi}_{\dot{\alpha}} \overline{e}^{\dot{\alpha}}\,, \tag{4.31}$$

it follows that the BRS transformations of $(c, \overline{c})$ imply the zero curvature equations

$$e^{\delta}\, s\, e^{-\delta}\, e^{\delta}\, c = -e^{\delta}\, c^2 \Longrightarrow \widetilde{d}\,\widetilde{A} + \widetilde{A}^2 = 0\,, \tag{4.32}$$

and

$$e^{\delta}\, s\, e^{-\delta}\, e^{\delta}\, \overline{c} = -e^{\delta}\, \overline{c}^2 \Longrightarrow \widetilde{d}\,\widetilde{\overline{A}} + \widetilde{\overline{A}}^2 = 0\,. \tag{4.33}$$

Equations (4.32) and (4.33) are easily checked to reproduce all the BRS transformations (2.13), (2.14) as well as the whole set of the equations (3.19)-(3.22). One sees thus that, in complete analogy with the nonsupersymmetric case [16], the zero curvature equations (4.32) and (4.33) deeply rely on the existence of the operators ζ_{α} and $\overline{\zeta}_{\dot{\alpha}}$. Let us underline here that the nilpotent operator $\widetilde{d}$ in eq.(4.29) will play a rather important role in the discussion of the superspace descent equations. For instance, as we shall see explicitly in the example given in the next subsection, it turns out that the superspace descent equations corresponding to the BRS invariant counterterms can be remarkably obtained from the single equation

$$\widetilde{d}\,\widetilde{\omega} = 0\,, \tag{4.34}$$

where $\widetilde{\omega}$ is an appropriate cocycle of dimension zero and ghost number three, whose components are the superspace field polynomials of the Taylor expansion of $\widetilde{\omega}$ in the global parameters $(e^{\alpha}, \overline{e}^{\dot{\alpha}}, \widetilde{e}^{\alpha\dot{\alpha}})$. Equation (4.34) can also be applied to characterize the descent equations of the U(1) anomaly. In Sect.6 we shall see that a slight modification of the eq.(4.34) will allow to treat the case of the Yang-Mills gauge anomaly as well. In all these cases the components of $\widetilde{\omega}$ will not exceed dimension two, this dimension being taken as the upper limit of our superspace analysis of the descent equations. In other words in what follows we shall limit ourselves to the study of the solutions of the superspace descent equations in the space of local functionals with dimension less or equal to two. In particular, according to the Table 1, this implies that the maximum number of covariant derivatives $D, \overline{D}$ present in each component of $\widetilde{\omega}$ is four.

Let us conclude this section with the following important remark. Being interested in the descent equations involving superspace functionals of dimension less or equal to two, we should have checked the closure of the algebra (4.26) built up by the operators $(s, \zeta, \overline{\zeta}, G, \widetilde{G}, D, \overline{D}, R, \overline{R}, \partial, \widetilde{\partial})$ on all the fields and their covariant derivatives up to reaching dimension two. It is not difficult to convince oneself that actually there is a breakdown of the closure of this algebra in the highest level of dimension two. However, as it usually happens in supersymmetry, the breaking terms turn out to be nothing but the equations of motion corresponding to the pure N=1 susy Yang-Mills action, implying thus an on-shell closure of the algebra. Evaluating in fact the commutator between the operators ζ and s on the superfield strength F one gets

$$[\zeta\ ,\ s]\ F\ =\ -\ [\varphi\ ,\ F]\ . \tag{4.35}$$

The right hand side of the equation (4.35) can be rewritten as

$$[\zeta\ ,\ s]\ F\ = D\ F\ -\ (\ D\ F\ +\ [\varphi\ ,\ F]\ )\ ,$$

so that, recalling that

$$D\ F\ \ +\ [\varphi\ ,\ F]\ = -\frac{1}{2} e^{\gamma} \tilde{e}_{\gamma\dot{\gamma}} \overline{e}^{\dot{\gamma}}\ (D^{\alpha}\ F_{\alpha}\ +\ \{\varphi^{\alpha}\ ,\ F_{\alpha}\})\ =\ 0\ , \tag{4.36}$$

are precisely the equations of motion of the pure N=1 susy Yang-Mills action, one obtains

$$[\zeta\ ,\ s]\ F\ = D\ F\ -\ \text{eq. of motion}\ .$$

It is worth to emphasize here that the on-shell closure of the algebra does not actually represent a real obstruction in order to solve the superspace consistency conditions. In fact one can observe from the eq.(4.36) and from the Table 1 that the Yang-Mills equations of motion are of dimension two. Therefore they could eventually contribute only to the highest level of the descent equations. Rather, the aforementioned on-shell closure of the algebra (4.26) relies on the fact that our analysis has been carried out without the introduction of the BRS external fields (the so-called Batalin-Vilkoviski antifields) which, as it is well known, allow to properly take care of the equations of motion, reestablishing thus the desired off-shellnes closure. However, as shown by [17, 18, 20], these external fields do not contribute to the superspace BRS cohomology in the cases considered here of the U(1) chiral anomaly, of the gauge anomaly as well as of the invariant action. This is the reason why we have discarded them. In the App.C we will show how the introduction of an appropriate external field takes care in a simple way of the Yang-Mills equations of motion, closing thus the algebra off-shell.

4.1 Nonchiral Descent Equations for the Invariant Action

In order to apply the supersymmetric decomposition (2.16) to the analysis of the superspace descent equations, let us begin by considering the BRS consistency condition corresponding to the nonchiral Yang-Mills invariant action, *i.e.*

$$s \int d^4x\ d^2\theta\ d^2\overline{\theta}\ \mathcal{L}^0\ =\ 0\ \Longrightarrow\ s\ \mathcal{L}^0\ =\ D^{\alpha}\ \mathcal{L}^1_{\alpha}\ +\ \overline{D}_{\dot{\alpha}}\ \mathcal{L}^{1\dot{\alpha}}\ , \tag{4.37}$$

where $\mathcal{L}^0$ is a local power series of dimension two and ghost number zero. According to what mentioned in the previous section, the full set of the superspace descent equations characterizing $\mathcal{L}^0$ can be obtained directly from the generalized equation

$$\tilde{d}\ \tilde{\omega}\ =\ 0\ , \tag{4.38}$$

with $\tilde{\omega}$ a generalized cocycle of ghost number three and dimension zero, whose Taylor expansion in the global parameters $(e^{\alpha}, \tilde{e}^{\alpha\dot{\alpha}}, \overline{e}^{\dot{\alpha}})$ reads

$$\begin{aligned} \tilde{\omega} &= \omega^3 + \omega^{2\,\alpha} e_{\alpha} + \overline{\omega}^2_{\dot{\alpha}} \overline{e}^{\dot{\alpha}} + \tilde{\omega}^{2\,\alpha}_{\dot{\alpha}} \tilde{e}^{\dot{\alpha}}_{\alpha} + \tilde{\omega}^{1\,\alpha}_{\dot{\alpha}} e_{\alpha} \overline{e}^{\dot{\alpha}} \\ &\quad + \omega^{1\,\alpha} \tilde{e}_{\alpha\dot{\alpha}} \overline{e}^{\dot{\alpha}} + \overline{\omega}^1_{\dot{\alpha}} e^{\alpha} \tilde{e}^{\dot{\alpha}}_{\alpha} + \omega^0 e^{\alpha} \tilde{e}_{\alpha\dot{\alpha}} \overline{e}^{\dot{\alpha}} . \end{aligned} \tag{4.39}$$

The coefficients $(\omega^3, \omega^{2\,\alpha}, \overline{\omega}^2_{\dot{\alpha}}, \tilde{\omega}^{2\,\alpha}_{\dot{\alpha}}, \tilde{\omega}^{1\,\alpha}_{\dot{\alpha}}, \omega^{1\,\alpha}, \overline{\omega}^1_{\dot{\alpha}}, \omega^0)$ are local power series in the superfields with the following quantum numbers

R−weights, dim. and ghost numb.

	ω^3	$\omega^{2\,\alpha}$	$\overline{\omega}^2_{\dot{\alpha}}$	$\tilde{\omega}^{2\,\alpha}_{\dot{\alpha}}$	$\tilde{\omega}^{1\,\alpha}_{\dot{\alpha}}$	$\omega^{1\,\alpha}$	$\overline{\omega}^1_{\dot{\alpha}}$	ω^0
dim	0	$\frac{1}{2}$	$\frac{1}{2}$	1	1	$\frac{3}{2}$	$\frac{3}{2}$	2
N_g	3	2	2	2	1	1	1	0

Table 4

In particular one observes that the coefficient ω^0 in the expression (4.39) has the same dimension of the invariant action we are looking for, justifying thus the choice of the quantum numbers of $\tilde{\omega}$ in eq.(4.38).

The generalized condition (4.38) is easily worked out and yields the following set of equations

$$\begin{aligned} s\,\omega^0 &= -\tfrac{1}{2} D^{\alpha}\omega^1{}_{\alpha} + \tfrac{1}{2}\overline{D}_{\dot{\alpha}}\overline{\omega}^{1\,\dot{\alpha}} + \tfrac{1}{4}\overline{R}_{\dot{\alpha}}\overline{\omega}^{2\,\dot{\alpha}} + \tfrac{1}{4}R^{\alpha}\omega^2_{\alpha} \\ &\quad -\tfrac{1}{4}\left\{D^{\alpha}, \overline{D}_{\dot{\alpha}}\right\}\tilde{\omega}^{1\,\dot{\alpha}}_{\alpha} - \tfrac{1}{4}G^{\alpha}_{\dot{\alpha}}\,\tilde{\omega}^{2\,\dot{\alpha}}_{\alpha} , \\ s\,\overline{\omega}^1_{\dot{\alpha}} &= -\tfrac{1}{2}\left\{D^{\alpha}, \overline{D}_{\dot{\alpha}}\right\}\omega^2_{\alpha} - \tfrac{1}{2} D^{\alpha}\,\tilde{\omega}^2_{\alpha\dot{\alpha}} + \tfrac{1}{2}\overline{R}_{\dot{\alpha}}\,\omega^3 , \\ s\,\tilde{\omega}^{1\,\alpha}_{\dot{\alpha}} &= -D^{\alpha}\,\overline{\omega}^2_{\dot{\alpha}} - \overline{D}_{\dot{\alpha}}\,\omega^{2\,\alpha} + G^{\alpha}_{\dot{\alpha}}\,\omega^3 , \\ s\,\omega^{1\,\alpha} &= \tfrac{1}{2}\left\{D^{\alpha}, \overline{D}_{\dot{\alpha}}\right\}\overline{\omega}^{2\,\dot{\alpha}} + \tfrac{1}{2}\overline{D}_{\dot{\alpha}}\,\tilde{\omega}^{2\,\alpha\dot{\alpha}} - \tfrac{1}{2}R^{\alpha}\,\omega^3 , \\ s\,\overline{\omega}^2_{\dot{\alpha}} &= -\overline{D}_{\dot{\alpha}}\,\omega^3 , \\ s\,\tilde{\omega}^{2\,\alpha}_{\dot{\alpha}} &= \left\{D^{\alpha}, \overline{D}_{\dot{\alpha}}\right\}\omega^3 , \\ s\,\omega^{2\,\alpha} &= -D^{\alpha}\,\omega^3 , \\ s\,\omega^3 &= 0 . \end{aligned} \tag{4.40}$$

These equations do not yet represent the final version of the superspace descent equations, due to the presence of the operators $(G_{\alpha\dot{\alpha}}, R_{\alpha}, \overline{R}_{\dot{\alpha}})$ in their right hand sides. However we shall prove that these undesired terms can be rewritten as pure BRS cocycles or as total superspace derivatives, meaning that they can be eliminated by means of a redefinition of the ω 's cocycles entering the equations (4.40). Let us first observe that a particular solution of the tower (4.40) can be fully expressed in terms of the BRS invariant cocycle

ω^3. In fact owing to the zero curvature equations (4.28), (4.32) and (4.33) it is apparent that the system (4.40) is solved by

$$\widetilde{\omega} \;=\; e^{\delta}\, \omega^3 \;, \tag{4.41}$$

which when written in components yields the following expressions

$$\begin{aligned}
\omega^{2\,\alpha} \;&=\; \zeta^{\alpha}\, \omega^3 \;, \\
\widetilde{\omega}^{2\,\alpha}_{\;\dot\alpha} \;&=\; G^{\alpha}_{\dot\alpha}\, \omega^3 \;, \\
\overline{\omega}^{2}_{\;\dot\alpha} \;&=\; \overline{\zeta}_{\dot\alpha}\, \omega^3 \;, \\
\omega^{1\,\alpha} \;&=\; \tfrac{1}{2}\, G^{\alpha}_{\dot\alpha}\, \overline{\zeta}^{\dot\alpha}\, \omega^3 \;, \\
\widetilde{\omega}^{1\,\alpha}_{\;\dot\alpha} \;&=\; \zeta^{\alpha}\, \overline{\zeta}_{\dot\alpha}\, \omega^3 \;, \\
\overline{\omega}^{1}_{\;\dot\alpha} \;&=\; -\tfrac{1}{2}\, G^{\alpha}_{\dot\alpha}\, \zeta_{\alpha}\, \omega^3 \;, \\
\omega^{0} \;&=\; \tfrac{1}{4}\, \zeta^{\alpha}\, G_{\alpha\dot\alpha}\, \overline{\zeta}^{\dot\alpha}\, \omega^3 \;.
\end{aligned} \tag{4.42}$$

In particular, from the results on the superspace BRS cohomology [18, 19, 21](see the App.B), it turns out that the most general form for ω^3 can be identified with the invariant ghost monomial

$$Tr \left(\frac{c^3}{3} \right) \;, \tag{4.43}$$

which, of course, is determined modulo a trivial exact BRS cocycle. Recalling then (App.B) that the difference $(Tr\; c^3 - Tr\; \overline{c}^3\;)$ is cohomologically trivial, *i.e.*

$$Tr\; c^3 \;-\; Tr\; \overline{c}^3 \;=\; s\,(...)\;,$$

we can choose for ω^3 the following symmetric expression[4]

$$\omega^3 \;=\; Tr \left(\frac{c^3}{3} \right) \;+\; Tr \left(\frac{\overline{c}^3}{3} \right) \;. \tag{4.44}$$

On the other hand it is easily established that all the terms $R^{\alpha}\, \omega^3$, $\overline{R}_{\dot\alpha}\, \omega^3$, $R^{\alpha}\, \omega^2_{\;\alpha}$, $\overline{R}_{\dot\alpha}\, \overline{\omega}^{2\,\dot\alpha}$ in the right hand side of eqs.(4.40) are trivial BRS cocycles. Considering for instance the first term, we have from the eqs.(3.23)

$$s\; R^{\alpha}\, \omega^3 \;=\; R^{\alpha}\; s\, \omega^3 \;=\; 0 \;, \tag{4.45}$$

which implies that $R^{\alpha}\, \omega^3$ belongs to the cohomology of s in the sector of ghost number two and dimension one half. Therefore, being the BRS cohomology empty in this sector, it follows that

$$R^{\alpha}\, \omega^3 \;=\; s\; \Lambda^{1\,\alpha} \;, \tag{4.46}$$

[4]One should observe that due to the anti-hermiticity property of the group generators T^a the cocycle $(Tr\; c^3 + Tr\; \overline{c}^3)$ is real.

as well as

$$\overline{R}^{\dot\alpha}\,\omega^3 \;=\; s\,\overline{\Lambda}^{1\,\dot\alpha}\;. \tag{4.47}$$

In fact, from

$$\begin{aligned} R^{\alpha}\,Tr\left(\frac{c^3}{3}\right) &= s\,Tr\left(cR^{\alpha}c\right) \;=\; s\,Tr\left(cF^{\alpha}\right)\;, \\ \overline{R}^{\dot\alpha}\,Tr\left(\frac{c^3}{3}\right) &= s\,Tr\left(c\overline{R}^{\dot\alpha}c\right)\;, \end{aligned}$$

and

$$\begin{aligned} R^{\alpha}\,Tr\left(\frac{\overline{c}^3}{3}\right) &= s\,Tr\left(\overline{c}R^{\alpha}\overline{c}\right)\;, \\ \overline{R}^{\dot\alpha}\,Tr\left(\frac{\overline{c}^3}{3}\right) &= s\,Tr\left(\overline{c}\overline{R}^{\dot\alpha}\overline{c}\right) = \;s\,Tr\left(\overline{c}\overline{F}^{\dot\alpha}\right)\;, \end{aligned}$$

we have that $\Lambda^{1\,\alpha}$ and $\overline{\Lambda}^{1\,\dot\alpha}$ can be identified, modulo trivial terms, with

$$\Lambda^{1\,\alpha} \;=\; Tr\left(cF^{\alpha}\right) + Tr\left(\overline{c}R^{\alpha}\overline{c}\right)\;, \qquad \overline{\Lambda}^{1\,\dot\alpha} \;=\; Tr\left(\overline{c}\overline{F}^{\dot\alpha}\right) \;+\; Tr\left(c\overline{R}^{\dot\alpha}c\right)\;, \tag{4.48}$$

where $\overline{R}^{\dot\alpha}c$, $R^{\alpha}\overline{c}$ are given in eqs.(3.22).

In the same way we have

$$\begin{aligned} R^{\alpha}\,\omega^2_{\;\alpha} &= R^{\alpha}\,\zeta_{\alpha}\,\omega^3 \;=\; \zeta_{\alpha}\,R^{\alpha}\,\omega^3 \;= \zeta_{\alpha}\,s\,\Lambda^{1\,\alpha} \\ &= s\,\left(\zeta_{\alpha}\,\Lambda^{1\,\alpha}\right) \;+\; D_{\alpha}\,\Lambda^{1\,\alpha}\;, \end{aligned} \tag{4.49}$$

showing that $R^{\alpha}\,\omega^2_{\;\alpha}$ is a trivial BRS cocycle plus a total superspace derivative. The same conclusions hold for $\overline{R}_{\dot\alpha}\,\omega^3$ and $\overline{R}_{\dot\alpha}\,\overline{\omega}^{2\,\dot\alpha}$ and can be extended by similar arguments to include the G-terms $G^{\alpha}_{\dot\alpha}\,\omega^3$ and $G^{\alpha}_{\dot\alpha}\,\tilde{\omega}^{2\,\dot\alpha}_{\;\;\alpha}$.

The final result is that the equations (4.40) can be rewritten without the explicit presence of the operators R and G, yielding thus the final version of the superspace descent equations for the invariant action, *i.e.*

$$\begin{aligned} s\,\left(\omega^0 + \tfrac{1}{4}\overline{\zeta}_{\dot\alpha}\,\overline{\Lambda}^{1\,\dot\alpha} + \tfrac{1}{4}\zeta^{\alpha}\,\Lambda^1_{\;\alpha}\right) &= -\tfrac{1}{2}\,D^{\alpha}\,\left(\omega^1_{\;\alpha} \;+\; \tfrac{1}{2}\Lambda^1_{\alpha}\right) \\ &\quad +\tfrac{1}{2}\,\overline{D}_{\dot\alpha}\,\left(\overline{\omega}^{1\,\dot\alpha} \;-\; \tfrac{1}{2}\overline{\Lambda}^{1\,\dot\alpha}\right)\;, \\ s\,\left(\overline{\omega}^1_{\;\dot\alpha} \;-\; \tfrac{1}{2}\overline{\Lambda}^1_{\;\dot\alpha}\right) &= -\tfrac{1}{2}\,\overline{D}_{\dot\alpha}D^{\alpha}\,\omega^2_{\;\alpha} \;-\; D^{\alpha}\overline{D}_{\dot\alpha}\,\omega^2_{\;\alpha} - \tfrac{1}{2}\,D^2\,\overline{\omega}^2_{\;\dot\alpha}\;, \\ s\,\left(\omega^{1\,\alpha} \;+\; \tfrac{1}{2}\Lambda^{1\,\alpha}\right) &= \tfrac{1}{2}\,D^{\alpha}\overline{D}_{\dot\alpha}\,\overline{\omega}^{2\,\dot\alpha} \;+\; \overline{D}_{\dot\alpha}D^{\alpha}\,\overline{\omega}^{2\,\dot\alpha} + \tfrac{1}{2}\,\overline{D}^2\,\omega^{2\,\alpha}\;, \\ s\,\overline{\omega}^2_{\;\dot\alpha} &= -\overline{D}_{\dot\alpha}\,\omega^3\;, \\ s\,\omega^{2\,\alpha} &= -D^{\alpha}\,\omega^3\;, \\ s\,\omega^3 &= 0\;. \end{aligned} \tag{4.50}$$

In particular the first equation of the above system explicitly shows that the invariant action $\mathcal{L}^0$ can be identified with

$$\mathcal{L}^0 \;=\; \omega^0 + \frac{1}{4}\overline{\zeta}_{\dot\alpha}\,\overline{\Lambda}^{1\,\dot\alpha} + \frac{1}{4}\zeta^{\alpha}\,\Lambda^1_{\;\alpha}\,. \tag{4.51}$$

The above expression has to be understood modulo an exact BRS cocycle or a total superspace derivative. Its nontriviality relies on the nontriviality of the ghost cocycle (4.44), as one can show by using a well known standard cohomological argument[14, 15]. Recalling then the expressions (4.42), (4.48), for $\mathcal{L}^0$ we get

$$\mathcal{L}^0 \;=\; \frac{1}{4}Tr\,\left(\varphi^{\alpha}\,F_{\alpha}\right) \;+\; \frac{1}{4}Tr\,\left(\overline{\varphi}_{\dot\alpha}\,\overline{F}^{\dot\alpha}\right)\,,$$

which when integrated on the full superspace $d^4x\;d^2\theta\;d^2\overline{\theta}$ yields the familiar N=1 supersymmetric invariant Yang-Mills lagrangian[5]:

$$S_{YM} = \int d^4x\;d^2\theta\;d^2\overline{\theta}\;\mathcal{L}^0 = \frac{1}{4}\int d^4x\;d^2\theta\;Tr\;F^{\alpha}F_{\alpha} \;+\; \frac{1}{4}\int d^4x\;d^2\overline{\theta}\;Tr\;\overline{F}_{\dot\alpha}\overline{F}^{\dot\alpha}.$$

5 Descent Equations for the U(1) Anomaly

As already remarked in the Introduction the BRS consistency condition for the chiral U(1) axial anomaly reads[19, 20]

$$s\int d^4x\;d^2\overline{\theta}\;K^0 = \;0 \;\Longrightarrow\; s\;K^0 \;=\; \overline{D}_{\dot\alpha}\,\overline{K}^{1\dot\alpha}\,, \tag{5.52}$$

where K^0 and $\overline{K}^{1\dot\alpha}$ have dimensions two and three half and ghost numbers zero and one respectively. K^0 has thus the same quantum numbers of the invariant action considered in the previous section, the only difference lying in the fact that the superspace measure, *i.e.* $d^4x\;d^2\overline{\theta}$, is now chiral instead of the vector one $d^4x\;d^2\overline{\theta}\;d^2\theta$. Therefore the descent equations for K^0 are obtained by performing the chiral limit of the vector equations (4.40). Acting indeed with the BRS operator on the second equation of the condition (5.52), we obtain

$$\overline{D}_{\dot\alpha}\;s\;\overline{K}^{1\dot\alpha} \;=\; 0\,. \tag{5.53}$$

Using then the results given in the App.A, it follows that the general solution of the equation (5.53) is given by

$$\begin{aligned} s\,\overline{K}^{1\dot\alpha} \;&=\; \left(\overline{D}^{\dot\alpha}\;D^{\alpha} \;+\; 2\;\;D^{\alpha}\;\overline{D}^{\dot\alpha}\right)K^2_{\alpha}\,, \\ \overline{D}^2\;K^2_{\alpha} \;&=\; 0\,, \end{aligned} \tag{5.54}$$

where K^2_{α} is of dimension one half and ghost number two. Again, acting with the BRS operator on the eq.(5.54) one gets

$$\left(\overline{D}^{\dot\alpha}\;D^{\alpha} \;+\; 2\;\;D^{\alpha}\;\overline{D}^{\dot\alpha}\right)\;s\;K^2_{\alpha} \;=\; 0\,, \tag{5.55}$$

[5]We recall here the useful superspace identity $\int d^4x\;d^2\theta\;d^2\overline{\theta} \;=\; \int d^4x\;d^2\theta\;\overline{D}^2$.

which according to the App.A implies

$$
\begin{aligned}
s\ K^2_\alpha &= D^\alpha\ K^3\ , \\
\overline{D}^2\ D^\alpha\ K^3 &= 0\ , \qquad D^2\ \overline{D}^{\dot\alpha}\ K^3\ =\ 0\ ,
\end{aligned}
$$

with K^3 of dimension zero and ghost number three. Finally, from

$$
D^\alpha\ s\ K^3\ =\ 0\ ,
$$

it follows that

$$
s\ K^3\ =\ 0\ .
$$

Summarizing, the superspace descent equations for the U(1) chiral axial anomaly are

$$
\begin{aligned}
s\ K^0 &= \overline{D}_{\dot\alpha}\overline{K}^{1\ \dot\alpha}\ , \\
s\ \overline{K}^1_{\dot\alpha} &= \left(2D^\alpha\overline{D}_{\dot\alpha}\ +\ \overline{D}_{\dot\alpha}D^\alpha\right)\ K^2_\alpha\ , \\
s\ K^{2\ \alpha} &= D^\alpha\ K^3\ , \\
s\ K^3 &= 0\ ,
\end{aligned}
\tag{5.56}
$$

with the constraints

$$
\begin{aligned}
\overline{D}^2\ K^2_\alpha\ &=\ 0\ , \\
\overline{D}^2\ D^\alpha\ K^3\ =\ D^2\ \overline{D}^{\dot\alpha}\ K^3\ &=\ 0\ .
\end{aligned}
\tag{5.57}
$$

Recalling then the result of the previous section, for K^3 we have

$$
K^3\ =\ \left(Tr\frac{c^3}{3}\ +\ Tr\frac{\overline{c}^3}{3}\right)+\ s\ \Delta^2\ , \tag{5.58}
$$

for some local power series Δ^2. It is interesting to observe that in this case the constraints (5.57) fix completely the trivial part of K^3, giving for instance

$$
s\ \Delta^2\ =\ 0\ .
$$

Acting with the operator ζ_α on both sides of the last of the eqs.(5.56) and making use of the decomposition (2.16), for K^2_α one gets

$$
K^2_\alpha\ =\ -\zeta_\alpha\ K^3\ +\ s\ \Delta^1_\alpha\ .
$$

Once more, it is not difficult to prove that the imposition of the constraints (5.57) yields a unique expression for Δ^1_α, *i.e.*

$$
\Delta^1_\alpha\ =\ Tr\left(c\ \varphi_\alpha\right)\ ,
$$

so that for K^2_α we get

$$
K^2_\alpha\ =\ Tr\left(c\ D_\alpha\ c\right)\ .
$$

One sees thus that in the chiral case, due to the constraints (5.57), the trivial BRS contributions are uniquely fixed at the lowest levels of the descent equations. Repeating now the same procedure and making use of the relations (3.17) for $\overline{K}^{1\dot{\alpha}}$ one obtains

$$\overline{K}^{1\dot{\alpha}} = G^{\alpha\dot{\alpha}} K^2_{\alpha} - \overline{\Lambda}^{1\dot{\alpha}} + D^{\alpha}\overline{D}^{\dot{\alpha}} Tr\,(c\,\varphi_{\alpha}) + Tr\left(\overline{c}\,\overline{F}^{\dot{\alpha}}\right) + s\,\Delta^{0\dot{\alpha}} \,, \qquad (5.59)$$

where the cocycle $\overline{\Lambda}^{1\dot{\alpha}}$ is the same as in eq.(4.45), *i.e.*

$$\overline{\Lambda}^{1\dot{\alpha}} = Tr\left(c\,\overline{R}^{\dot{\alpha}}\,c\right) + Tr\left(\overline{c}\,\overline{F}^{\dot{\alpha}}\right) .$$

It follows thus that

$$\overline{K}^{1\dot{\alpha}} = -\,2\,Tr\left(D^{\alpha}\,c\,\overline{D}^{\dot{\alpha}}\,\varphi_{\alpha}\right) + s\,\Delta^{0\dot{\alpha}} . \qquad (5.60)$$

Finally, acting with the operator $\overline{\zeta}_{\dot{\alpha}}$ on both sides of the equation

$$s\,\overline{K}^1_{\dot{\alpha}} = \left(2D^{\alpha}\overline{D}_{\dot{\alpha}} + \overline{D}_{\dot{\alpha}}D^{\alpha}\right) K^2_{\alpha}\,,$$

for the last level K^0 we find

$$K^0 = -\,\overline{\zeta}_{\dot{\alpha}}\,\overline{K}^{1\dot{\alpha}} + Tr\left(2\varphi^{\alpha}\,F_{\alpha} + \overline{D}_{\dot{\alpha}}\,\varphi^{\alpha}\,\overline{D}^{\dot{\alpha}}\,\varphi_{\alpha}\right) ,$$

reproducing the well known expression for the U(1) supersymmetric chiral anomaly

$$K^0 = Tr\left(2\varphi^{\alpha}\,F_{\alpha} - \overline{D}_{\dot{\alpha}}\,\varphi^{\alpha}\,\overline{D}^{\dot{\alpha}}\,\varphi_{\alpha}\right) - \overline{D}_{\dot{\alpha}}\,\Delta^{0\dot{\alpha}} .$$

Let us conclude by remarking that the expressions of the cocycles K^3, K^2_{α}, $\overline{K}^{1\dot{\alpha}}$ and K^0 found here are completely equivalent to those of [19], *i.e.* the difference is an exact BRS cocycle or a total superspace derivative.

5.1 The Supersymmetric Gauge Anomaly

As the last example of our superspace analysis let us consider the case of the supersymmetric gauge anomaly. As usual let us first focus on the derivation of the corresponding descent equations. The latters, as mentioned in Sect.4 , can be obtained by adding to the right hand side of the generalized equation (4.38) an appropriate extra term. The presence of this term actually stems from the BRS triviality[18] of the pure ghost cocycles $(Tr\ c^{2n+1} - Tr\ \overline{c}^{2n+1})$, $n \geq 1$,

$$s\ \Omega^{2n} = Tr\,\frac{c^{2n+1}}{2n+1} - Tr\,\frac{\overline{c}^{2n+1}}{2n+1}\,, \qquad (5.61)$$

$\Omega^{\ 2n}$ being a local dimensionless functional of $(\phi, c, \overline{c})$ with ghost number $2n$. Acting in fact with the operator e^{δ} on both sides of eq.(5.61) and recalling the definitions (4.30) and (4.31) we get the desired modified version of the generalized superspace equation (4.38) we are looking for,

$$\begin{aligned} \tilde{d}\,\tilde{\Omega} &= Tr\,\frac{\tilde{A}^{2n+1}}{2n+1} - Tr\,\frac{\overline{\tilde{A}}^{2n+1}}{2n+1}\ , \qquad (5.62)\\ \tilde{\Omega} &= e^{\delta}\ \Omega^{2n} . \end{aligned}$$

The descent equations for the gauge anomaly follows then from eq.(5.62) when $n = 2$, *i.e.*

$$
\begin{aligned}
\tilde{d}\,\tilde{\Omega} &= \frac{1}{5}\,Tr\left(\tilde{A}^5 \;-\; \overline{\tilde{A}}^5\right)\;,\\
\tilde{\Omega} &= e^{\delta}\,\Omega^4\;.
\end{aligned}
\tag{5.63}
$$

To see that the above equation characterizes indeed the gauge anomaly let us write it in components. Expanding $\tilde{\Omega}$ in the global parameters $(e^{\alpha}, \overline{e}^{\dot\alpha}, \tilde{e}^{\dot\alpha}_{\alpha})$

$$
\begin{aligned}
\tilde{\Omega} \;=\; & \Omega^4 \;+\; \Omega^{3\,\alpha}\,e_{\alpha} \;+\; \overline{\Omega}^3_{\dot\alpha}\,\overline{e}^{\dot\alpha} \;+\; \tilde{\Omega}^{3\,\alpha}_{\dot\alpha}\,\tilde{e}^{\dot\alpha}_{\alpha} \;+\; \tilde{\Omega}^{2\,\alpha}_{\dot\alpha}\,e_{\alpha}\,\overline{e}^{\dot\alpha}\\
& + \Omega^{2\,\alpha}\,\tilde{e}_{\alpha\dot\alpha}\,\overline{e}^{\dot\alpha} \;+\; \overline{\Omega}^2_{\dot\alpha}\,e^{\alpha}\,\tilde{e}^{\dot\alpha}_{\alpha} \;+\; \Omega^1\,e^{\alpha}\,\tilde{e}_{\alpha\dot\alpha}\,\overline{e}^{\dot\alpha}\;,
\end{aligned}
\tag{5.64}
$$

and eliminating the G and R terms as done in Subsect. 4.1 we get the known descent equations for the superspace gauge anomaly[18, 21]

$$
\begin{aligned}
s\;\Omega^1 &= D^{\alpha}\,\Omega^2_{\alpha} \;+\; \overline{D}_{\dot\alpha}\,\Omega^{2\dot\alpha}\;,\\
s\;\Omega^2_{\alpha} &= -\overline{D}^2\Omega^3_{\alpha} \;+\; \left(2\overline{D}_{\dot\alpha}D_{\alpha} \;+\; D_{\alpha}\overline{D}_{\dot\alpha}\right)\,\overline{\Omega}^{3\dot\alpha}\\
&\quad + 2\,Tr\left(\left(D_{\alpha}\overline{D}_{\dot\alpha}\overline{c}\right)\left(\overline{c}\overline{D}^{\dot\alpha}\overline{c} \;+\; \overline{D}^{\dot\alpha}\overline{c}\,\overline{c}\right)\right)\;,\\
s\,\overline{\Omega}^{2\dot\alpha} &= D^2\overline{\Omega}^{3\dot\alpha} \;-\; \left(2D^{\alpha}\overline{D}_{\dot\alpha} \;+\; \overline{D}_{\dot\alpha}D^{\alpha}\right)\,\Omega^3_{\alpha}\\
&\quad - 2\,Tr\left(\left(\overline{D}^{\dot\alpha}D^{\alpha}c\right)\left(cD_{\alpha}c \;+\; D_{\alpha}c\,c\right)\right)\;,\\
s\,\Omega^3_{\alpha} &= D_{\alpha}\,\Omega^4 \;+\; Tr\left(c^3D_{\alpha}c\right)\;,\\
s\,\overline{\Omega}^{3\dot\alpha} &= -\overline{D}^{\dot\alpha}\,\Omega^4 \;+\; Tr\left(\overline{c}^3\overline{D}^{\dot\alpha}\overline{c}\right)\;,\\
s\,\Omega^4 &= \tfrac{1}{5}Tr\left(c^5 \;-\; \overline{c}^5\right)\;.
\end{aligned}
\tag{5.65}
$$

One sees in particular that integrating the first equation of (5.65) on superspace, the cocycle Ω^1 obeys exactly the BRS consistency condition corresponding to the possible gauge breakings

$$
s\int d^4x\;d^2\theta\;d^2\overline{\theta}\;\Omega^1 = \;0\;,
$$

identifying therefore Ω^1 with the supersymmetric Yang-Mills anomaly.

In order to find a solution of the descent equations (5.65) we use the same climbing procedure of the previous examples obtaining the following nontrivial expressions

$$
\begin{aligned}
\Omega^3_\alpha &= -\,\zeta^\alpha\,\Omega^4 \;-\; Tr\,(\varphi^\alpha\, c^3)\;, \\
\overline{\Omega}^{3\dot\alpha} &= \overline{\zeta}_{\dot\alpha}\,\Omega^4 \;-\; Tr\,(\overline{\varphi}_{\dot\alpha}\,\overline{c}^3)\;, \\
\Omega^2_\alpha &= G_{\alpha\dot\alpha}\,\overline{\zeta}^{\dot\alpha}\,\Omega^4 \;+\; \overline{D}_{\dot\alpha}\,\overline{\zeta}^{\dot\alpha}\,\zeta_\alpha\,\Omega^4 \\
&\quad -\,Tr\left(\overline{\varphi}_{\dot\alpha}\,\left(D_\alpha\overline{\varphi}^{\dot\alpha}\right)\overline{c}^2 \;-\; \overline{\varphi}_{\dot\alpha}\,\overline{c}\left(D_\alpha\overline{\varphi}^{\dot\alpha}\right)\overline{c} \;+\; \overline{\varphi}_{\dot\alpha}\,\overline{c}^2\,D_\alpha\overline{\varphi}^{\dot\alpha}\right) \\
&\quad +2\,Tr\left(\left(D_\alpha\overline{\varphi}_{\dot\alpha}\right)\left(\overline{c}\overline{D}^{\dot\alpha}\overline{c} \;+\; \overline{D}^{\dot\alpha}\overline{c}\,\overline{c}\right)\right)\;, \\
\overline{\Omega}^{2\dot\alpha} &= G^{\alpha\dot\alpha}\,\zeta_\alpha\,\Omega^4 \;+\; D^\alpha\,\overline{\zeta}^{\dot\alpha}\,\zeta_\alpha\,\Omega^4 \\
&\quad +\,Tr\left(\varphi^\alpha\,\left(\overline{D}^{\dot\alpha}\varphi_\alpha\right)\overline{c}^2 \;-\; \varphi^\alpha\,\overline{c}\left(\overline{D}^{\dot\alpha}\varphi_\alpha\right)\overline{c} \;+\; \varphi^\alpha\,\overline{c}^2\,\overline{D}^{\dot\alpha}\varphi_\alpha\right) \\
&\quad -2\,Tr\left(\left(\overline{D}^{\dot\alpha}\varphi_\alpha\right)\left(cD_\alpha c \;+\; D_\alpha c\,c\right)\right)\;,
\end{aligned}
\tag{5.66}
$$

and for the gauge anomaly

$$
\begin{aligned}
\Omega^1 &= 2\,\zeta^\alpha\,G_{\alpha\dot\alpha}\,\overline{\zeta}^{\dot\alpha}\,\Omega^4 \\
&\quad +2\,Tr\left(F^\alpha\,c\,\varphi_\alpha \;-\; F^\alpha\,\varphi_\alpha\,c \;+\; \left(\overline{D}_{\dot\alpha}\,\varphi^\alpha\right)\left(\overline{D}^{\dot\alpha}\,\varphi_\alpha\right)c\right) \\
&\quad -2\,Tr\left(\overline{F}_{\dot\alpha}\,\overline{c}\,\overline{\varphi}^{\dot\alpha} \;-\; \overline{F}_{\dot\alpha}\,\overline{\varphi}^{\dot\alpha}\,\overline{c} \;+\; \left(D^a\,\overline{\varphi}_{\dot\alpha}\right)\left(D_\alpha\,\overline{\varphi}^{\dot\alpha}\right)\overline{c}\right)\;.
\end{aligned}
\tag{5.67}
$$

One should observe that the explicit final expression for the gauge anomaly depends on the knwoledge of the cocycle Ω^4 solution of the last of the descent equations (5.65). This point is particularly important and deserves some further clarifying remarks.

5.2 Nonpolynomial Character of The Gauge Anomaly

It is known that due to a theorem by Ferrara, Girardello, Piguet and Stora [22], the superspace gauge anomaly cannot be expressed as a polynomial in the variables $(\varphi_\alpha, \lambda_\alpha \equiv e^\varphi D_\alpha\; e^{-\varphi})$ and their covariant derivatives. In fact all the known superspace closed expressions of the gauge anomaly so far obtained by means of homotopic transgression procedures[24, 25, 26] show up an highly nonpolynomial character in the gauge superconnetion. On the other hand in our approach the simple knowledge of the cocycle Ω^4 would produce a closed expression for the supersymmetric gauge anomaly without any homotopic integral. Of course this would imply a deeper understanding of this anomaly. It is not difficult however to convince oneself that solving the equation

$$
s\,\Omega^4 \;=\; \tfrac{1}{5}Tr\,(c^5 \;-\; \overline{c}^5) \tag{5.68}
$$

is not an easy task. This is actually due to the BRS transformation of the vector superfield ϕ

$$s\, e^{\phi} \;=\; e^{\phi}\, c \;-\; \overline{c}\, e^{\phi} \;.$$

which when written in terms of ϕ takes the highly complex form[20]

$$s\,\phi \;=\; \frac{1}{2}\mathcal{L}_{\phi}\;(c+\overline{c}) \;+\; \frac{1}{2}\mathcal{L}_{\phi}\;\left[\coth\;\left(\frac{\mathcal{L}_{\phi}}{2}\right)\right]\;(c-\overline{c})\;, \tag{5.69}$$

where

$$\mathcal{L}_{\phi}\,\cdot \;=\; [\phi,\;\cdot\;]\;,$$

and

$$\coth\;\left(\frac{\mathcal{L}_{\phi}}{2}\right) \;=\; \frac{e^{\frac{\mathcal{L}_{\phi}}{2}} \;+\; e^{-\frac{\mathcal{L}_{\phi}}{2}}}{e^{\frac{\mathcal{L}_{\phi}}{2}} \;-\; e^{-\frac{\mathcal{L}_{\phi}}{2}}}\;.$$

The formula (5.69) can be expanded in powers of ϕ, allowing to solve the equation (5.68) order by order in the vector superfield ϕ. For instance, in the first approximation which corresponds to the abelian limit of retaining only the linear terms of the BRS transformations, *i.e.*

$$s \;\rightarrow\; s_{ab}$$

with

$$\begin{aligned} s_{ab}\,\phi \;&=\; c \;-\; \overline{c}\;, \\ s_{ab}\,c \;&=\; s_{ab}\,\overline{c} \;=\; 0\;, \end{aligned}$$

one easily checks that

$$Tr\left(c^5 \;-\; \overline{c}^5\right) \;=\; s_{ab}\;Tr\left(\phi\left(c^4 \;+\; c^3\,\overline{c} \;+\; c^2\,\overline{c}^2 \;+\; c\,\overline{c}^3 \;+\; \overline{c}^4\right)\right)\;, \tag{5.70}$$

which shows indeed the BRS triviality[1] of $Tr\,(c^5 \;-\; \overline{c}^5)$.

Up to our knowledge a closed exact form for Ω^4 has not yet been established. In other words, due to the theorem of Ferrara, Girardello, Piguet and Stora[22], the nonpolynomiality of the supersymmetric gauge anomaly directly relies on the nonpolynomial nature of the cocycle Ω^4. Any progress in this direction will be reported as soon as possible.

Let us conclude this section by giving the explicit expression of the gauge anomaly (5.67) up to the second order in the vector field ϕ, *i.e.*

$$\begin{aligned} \Omega^1 \;&=\; -2\;Tr\left(D^{\alpha}\phi\;\overline{D}^2 D_{\alpha}\phi\;c \;+\; \overline{D}^2 D^{\alpha}\phi\;D_{\alpha}\phi\;c \;-\; \left(\overline{D}_{\dot\alpha}D^{\alpha}\phi\right)\left(\overline{D}^{\dot\alpha}D_{\alpha}\phi\right)c\right) \\ &+\;\; 2\;Tr\left(\overline{D}_{\dot\alpha}\phi\;D^2\overline{D}^{\dot\alpha}\phi\;\overline{c} \;+\; D^2\overline{D}_{\dot\alpha}\phi\;\overline{D}^{\dot\alpha}\phi\;\overline{c} \;-\; \left(D^{a}\overline{D}_{\dot\alpha}\phi\right)\left(D_{\alpha}\overline{D}^{\dot\alpha}\phi\right)\overline{c}\right)\;, \end{aligned} \tag{5.71}$$

which is easily recognized to be equivalent to that of ref.[18]. One should also observe that the above expression do not receive contributions from the term Ω^4 since they are at least of the order three in ϕ, as it can be checked by applying the combination $\zeta^{\alpha}\;G_{\alpha\dot\alpha}\;\overline{\zeta}^{\dot\alpha}$ on the cocycle of the eq.(5.70).

6 Conclusion

The supersymmetric version of the descent equations for the four dimensional N=1 Super-Yang-Mills gauge theories can be analysed by means of the introduction of two operators ζ^{α} and $\overline{\zeta}^{\dot{\alpha}}$which decompose the supersymmetric derivatives D^{α} and $\overline{D}^{\dot{\alpha}}$ as BRS commutators. These operators provide an algebraic setup for a systematic derivation of the superspace descent equations. In addition they allow to cast both the supersymmetric BRS transformations and the descent equations into a very suggestive zero curvature formalism in superspace.

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A Appendix

We list here the superspace algebraic solutions [18, 19, 20, 27] of some equations needed for the analysis of the supersymmetric descent equations. All these solutions are built up by superfields. They have always to be understood modulo terms which automatically solve the corresponding equations but cannot be written in the same algebraic form as the solutions.The existence of such particular terms strongly depends on the superfield content of the particular model under consideration.

The first result states that the solution of the superspace equation

$$\overline{D}^2\, Q \;=\; 0\;,$$

can be generically written as

$$Q \;= \overline{D}_{\dot\alpha}\, \mathcal{M}^{\dot\alpha}\;,$$

for some superfield $\mathcal{M}^{\dot\alpha}$.

The second important result concerns the solution of the following equation

$$\left(2\overline{D}_{\dot\alpha}D_\alpha \;+\; D_\alpha\overline{D}_{\dot\alpha}\right)\,\overline{Q}^{\dot\alpha} \;=\; \overline{D}^2\, Q_\alpha\;.$$

For the superfields $\overline{Q}^{\dot\alpha}$ and Q_α we have now

$$\begin{aligned}\overline{Q}^{\dot\alpha} \;&=\; \overline{D}^{\dot\alpha}\,\mathcal{M}\;,\\ Q_\alpha \;&=\; -D_\alpha\,\mathcal{M}\;,\end{aligned}$$

with $\mathcal{M}$ an arbitrary superfield. Let us observe that in this case the term $Tr\,(cD_\alpha c)$, due to the fact that the ghost c is a chiral superfield, is automatically annihilated by the operator $\overline{D}^2$. Therefore it must be included in the expression given for Q_α although it cannot be written as a total superspace derivative.

Considering now the equation

$$D^\alpha\, Q_\alpha \;=\; \overline{D}_{\dot\alpha}\, \overline{Q}^{\dot\alpha}\;,$$

we have

$$\begin{aligned}Q_\alpha \;&=\; -\overline{D}^2\,\mathcal{P}_\alpha \;+\; \left(2\overline{D}_{\dot\alpha}D_\alpha \;+\; D_\alpha\overline{D}_{\dot\alpha}\right)\,\overline{\mathcal{P}}^{\dot\alpha} \;+\; D^\beta\,\mathcal{N}_{(\alpha\beta)}\;,\\ \overline{Q}^{\dot\alpha} \;&=\; -D^2\,\overline{\mathcal{P}}^{\dot\alpha} \;+\; \left(2D^\alpha\overline{D}^{\dot\alpha} \;+\; \overline{D}^{\dot\alpha}D^\alpha\right)\,\mathcal{P}_\alpha \;+\; \overline{D}_{\dot\beta}\overline{\mathcal{N}}^{(\dot\alpha\dot\beta)}\;,\end{aligned} \tag{A.72}$$

with $\mathcal{P}_\alpha$ and $\mathcal{N}_{(\alpha\beta)}$ appropriate superfields. Of course the existence of the symmetric superfield $\mathcal{N}_{(\alpha\beta)}$ depends on the dimension and on the ghost number of Q_α. For instance in the case of the vector descent equations (4.50) in which Q_α corresponds to $s(\omega^1_\alpha+\frac{1}{2}\Lambda^1_\alpha)$, it is not difficult to check that $\mathcal{N}_{(\alpha\beta)}$ is automatically absent due to the quantum numbers of the problem.

In particular, in the case of the chiral descent equations considered in Sect.5, eq.(A.72) imply that the most general solution of the eq.(5.53) is given indeed by

$$s\,\overline{K}^{1\dot\alpha} \;=\; \left(\overline{D}^{\dot\alpha}\;D^{\alpha}\;+\;2\;\;D^{\alpha}\,\overline{D}^{\dot\alpha}\right)K^2_{\alpha}\;,$$

with the constraint

$$\overline{D}^2\;K^2_{\alpha}\;=\;0\;.$$

B Appendix

In this appendix we summarize some useful results concerning the BRS superspace cohomology for the N=1 supersymmetric Yang-Mills gauge theories. The various BRS cohomolgy classes are labelled by the ghost number g and by the spinor indices.

The following results hold[18, 19, 21]:

1. The BRS cohomology is empty in the space of the invariant local power series A^g with dimension 2 and positive ghost number g.
2. The cohomology classes corresponding to local BRS invariant cocycles A^g_{α} or $\overline{A}^g_{\dot\alpha}$ with dimension $\frac{3}{2}$ and ghost number $g=1$, 2 or 3 are empty.
3. The cohomology classes in the space of the BRS invariant local power series A^g_{α} or $\overline{A}^g_{\dot\alpha}$ with dimension $\frac{1}{2}$ and ghost number g greater than zero are empty.
4. The BRS cohomology classes in the space of the local power series A^g with dimension 0, ghost number g and at least of order $g+1$ in the fields are empty.
5. Any invariant object A^g with dimension 0 and even ghost number g greater than zero and of order g in the fields is BRS trivial.

In particular it turns out that in the pure ghost sector the BRS cohomology classes are given by polynomials built up with monomials of the type

$$Tr\frac{c^{2n+1}}{2n+1}\;,\qquad n\geq 1\;, \tag{B.73}$$

or

$$Tr\frac{\overline{c}^{2n+1}}{2n+1}\;,\qquad n\geq 1\;. \tag{B.74}$$

We remark also that the two expressions above (B.73) and (B.74) do not actually define different cohomology classes. Instead they are equivalent, due to the triviality[18, 20] of the combination

$$Tr\frac{c^{2n+1}}{2n+1}\;-\;Tr\frac{\overline{c}^{2n+1}}{2n+1}\;=\;s\;\Omega^{2n}\;,$$

for some local power series Ω^{2n}. This result implies that the expression (B.73) and (B.74) are related each other by means of an exact BRS cocycle.

C Appendix

In this last appendix we show that the off-shell closure of the algebra (4.26) can be recovered in a simple way by introducing an appropriate external field η. Let indeed be η a superfield with dimension 2 and ghost number -1, whose BRS transformation reads

$$\begin{aligned} s\,\eta &= [\eta\,,\,c] + 2\,(D\,F + [\varphi\,,\,F])\ , \\ s^2\,\eta &= 0\ . \end{aligned}$$

Modifying now the operator ζ in such a way that

$$\zeta\,F = -\frac{1}{2}\,\eta\ ,$$

it is easily verified that the commutator (4.35)

$$[\zeta\,,\,s]\ F = -\zeta\ [c\,,\,F] + \frac{1}{2}\,s\,\eta = D\,F\ ,$$

gives now the covariant derivative of F without making use of the equations of motion, closing therefore the algebra (4.26) off-shell. Let us conclude by also remarking that the external field η cannot contribute to the BRS cohomology classes relevant for the examples considered in the previous Sections due to its ghost number and to its dimension.

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