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NOn Linear Non Local Theory of Gravity - II

by

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Abstract

We present a new non local theory of gravity based on the extension of an ancient work by Deser and Laurent. We show that our theory is compatible with all observations, ie, the classical tests of gravity. We suggest a new program to deal with the influence of matter on the gravitational field.

Key-words: Non local; No linear; Gravity.

1 INTRODUCTION

A. Introductory Remarks.

One of the cornerstones of the modern Field Theory relies in the acceptance of the Locality Principle. It seems worth to note that such hypothesis contains some apriorisms that are far beyond our observational means. Further than this, if we move into the quantum description of the world, there are many arguments (theoretically and observationally) that point against such principle. Thus, one could wonder if it should not be the case to contemplate the possibility of constructing theoretical models that go beyond this hypothesis, even at the classical level. A review on this subject concerning non local properties of Classical Eletrodynamics has been presented recently [4]. It should be natural, thus, to undertake a similar task to examine the interaction that is accepted to be the responsible for the characterization of the spacetime geometry, that is, gravity.

Such an exam is not a new one. Indeed, in the past (and for reasons that we will see later on) Deser and Laurent have undertaken the analysis of a specific non local theory for the gravitational interaction. Unfortunately these authors have limited their subject to a linear theory. In so doing they have faced difficulties that could not be surmounted at this level of the theory. As we will point out later on, the criticisms that one could make on the Deser and Laurent model are not specific of non local theories but mainly related to the simple linearization scheme they used.

The gravitational field, by reasons pointed out by many authors [13, 2] must be a self interaction process: the equations which describe this field are non linear. It is by now well known that observations have confirmed such characteristic non linearity[15].

Later and thanks to the effort of many people (Deser, Rosen, Grishchuk, etc.) General Relativity Theory, that deals with modifications of the geometrical structure of the space time could be formulated alternatively in the context of Classical Field Theory. The standard procedure is to add to the energy momentum tensor of matter, the source of the gravitacional field, aditional non linear terms of the field that whould represent the gravitational contribution to total energy. The reason for this is that gravity is not a ghost field and must also have a non vanishing energy-momentum distribution.

In a completely diferent context and using a new mathematical framework, a simple linear model to describe the gravitational field was constructed by Deser and Laurent, DL[2]. The scheme employed there conciliates the linearity of Fierz-Pauli equation with a complete theory of gravity, as we shall see in Section 2.2. The main drawback of DL formulation appears when gravitational waves are present. The reason for this is precisely related to the linearity of DL model, that treats gravity as transparent to gravitons.

In the present paper we would like to reexamine the DL proposal which concerns a non-local theory of gravity. The reason for this is related to a recent result [3] by which it is possible, with a slight modification of the scheme used by these authors, to obtain a theory of gravity that is least equivalent to Einstein's General Relativity, as far as the standard four classical tests are concerned and that could yield a new Physics in the highly non-linear regime. This will be done through a non-linear modification of DL proposal.

B. Synopsis.

In **Section 2** we present a short review of some mathematical properties of the divergenceless projection operator and summarize a non local theory formulated by Deser & Laurent.

Section 3 presents a non-linear extended model to describe the gravitational field in the context of the classical field theory. We describe the motion of particles and a particular solution is derived, namely the static spherically symmetric solution. Finally we show that the four standard classical tests of gravitation are satisfied by our model.

In **Section 4** we show a general procedure to construct non-local gravity theories considering higher orders in the gravity energy-momentum tensor.

Section 5 gives a summary of this paper and some future lines of investigation.

2 NON-LOCAL THEORY

The starting point of linear spin-two field theory is the Fierz equation that reads:

$$G^{(L)}_{\mu\nu} = -kT_{\mu\nu} \tag{1}$$

in which

$$G^{(L)}_{\mu\nu} \equiv \Box\phi_{\mu\nu} - \phi^{\alpha}{}_{\mu|\alpha\nu} - \phi^{\alpha}{}_{\nu|\alpha\mu} + \phi^{\alpha}{}_{\alpha|\mu\nu} - \eta_{\mu\nu}(\Box\phi^{\alpha}{}_{\alpha} - \phi^{\alpha\beta}{}_{|\alpha\beta}). \tag{2}$$

The quantity $G^{(L)}_{\mu\nu}$ is divergence-free. This implies that, in order to achieve compatibility, one must impose the condition that the energy-momentum tensor $T_{\mu\nu}$ of matter should also be divergenceless. Now, since the gravitational field contributes to the balance of the conservation law through its own energy, this imposition faces a difficulty, that is, the matter energy-momentum tensor cannot be conserved separately. Traditionally, there are two well-known solutions for this situation, that we will call generically the geometrical and the non-local field theory way.

We will not discuss here the well-known Einstein (i. e., geometrical) approach, but we synthesize it just for completeness as follows. If we add to the right-hand-side of the equation (1) the energy momentum tensor of gravity, then an infinite series appears [13]. This is a recurrence procedure and a direct consequence of the non-localizability of the gravitational energy.

Nevertheless several proposals concerning the description of the form of the energy-momentum tensor of the gravitational field have been examined through the years. Unfortunately they all suffer from a lethal disease: they are not true tensors. Recently Grishchuk, Petrov and Popova [9] has made an apparent improvement in such situation[1] by setting

$$G^{(L)}_{\mu\nu} = T^{M}_{\mu\nu} + t^{g}_{\mu\nu} \tag{3}$$

in which, contrary to all previous proposals, the gravitational energy momentum tensor $t^{g}_{\mu\nu}$ is a true tensor. However it contains an intrinsic gauge degree of freedom that singularizes it from the standard energy momentum tensors of other fields and inhibits its conventional treatment.

[1] Note that the quoted paper of GPP seems apparently out of this difficulty. However, the energy-momentum tensor proposed by these authors have an internal gauge freedom that seems to be a reminiscent of the same problem. In any way, they do not exhibit a $t^{g}_{\mu\nu}$ free of ambiguities.

Besides such geometrical paradigma of General Relativity, there is an equivalent way to deal with this problem in a very ingenious procedure. This was presented by Deser et al. This method (contrary to the previous ideas on this subject) does not intend to provide gravity with a local energy-momentum tensor but instead it operates on the right hand side just by eliminating from it the undesirable non-conserving part.

The way to undertake such an enterprise is very simple. We will describe its main properties in the next section. Before this, however, let us introduce some mathematical machinery.

2.1 THE NONLOCAL OPERATOR

The nonlocal projector $P_{\mu\nu}$ is defined by:

$$P_\mu{}^\alpha \equiv \delta_\mu^\alpha - \Box^{-1}\partial_\mu\partial^\alpha,$$

in which the quantity $\Box^{-1}$ represents the inverse of the d'Alembertian operator, that is

$$\Box\,\Box^{-1} = 1.$$

Given an arbitrary vector V_α we construct an associated quantity, represented by $\hat{V}_\alpha$, thus defined:

$$\hat{V}_\mu \equiv P_\mu{}^\alpha V_\alpha, \tag{4}$$

and such that $\hat{V}_\mu$ is divergence-free. We are interested here not only on vectors that are divergence-free but specially tensors.

We can use the above operator $P_\mu{}^\alpha$ to construct the most general form of divergenceless second order tensor. This is provided by the relation:

$$\hat{T}^{\mu\nu} \equiv Q_{\alpha\beta}{}^{\mu\nu}\, T^{\alpha\beta}, \tag{5}$$

in which the operator $Q_{\alpha\beta}{}^{\mu\nu}$ is defined as

$$Q_{\alpha\beta}{}^{\mu\nu} \equiv P_\alpha{}^\mu\, P_\beta{}^\nu + p\, P^{\mu\nu}\,\eta_{\alpha\beta} + q\, P^{\mu\nu} P_{\alpha\beta}.$$

The constants p and q are free parameters[2]. To support our claim that the above second order projector $Q_{\mu\nu\alpha\beta}$ is the most general form, it is enough to examine a product of these objects. Indeed, we have

$$Q_{\mu\nu}{}^{\alpha\beta} Q_{\alpha\beta}{}^{\rho\sigma} = P_\mu{}^\rho P_\nu{}^\sigma + \xi P_{\mu\nu}\eta^{\rho\sigma} + \lambda P_{\mu\nu}P^{\rho\sigma}. \tag{6}$$

The right hand side of this expression can be identified with the same operator $Q'_{\mu\nu}{}^{\rho\sigma}$, that is

$$Q'_{\mu\nu}{}^{\rho\sigma} \equiv P_\mu{}^\rho P_\nu{}^\sigma + \xi P_{\mu\nu}\eta^{\rho\sigma} + \lambda P_{\mu\nu}P^{\rho\sigma}, \tag{7}$$

with a simple redefinition of the new free parameters ξ and λ in terms of the primary ones p and q given by

$$\xi \equiv 3p^2 + 3pq + p \tag{8}$$

$$\lambda \equiv 3q^2 + 3pq + 2q + p. \tag{9}$$

We can thus see from this expression that by further multiplication of these operators we do not obtain any new form of second order projector operator.

[2]We shall see later on that these parameters are to be fixed for each specific model to conform with observational data.

2.2 DESER-LAURENT THEORY

Deser and Laurent have explored the possibility of considering gravity as a spin two field without self-interaction. The scheme that they employed conciliates the linearity of Fierz-Pauli equation with a complete theory of gravity that circunvents the divergence-free problem by making appeal to the non-local property contained in the projected operators defined above.

Indeed, they take as the true source of the spin-2 field the quantity $\hat{T}^{(M)}_{\mu\nu}$, which is the projection of the matter-energy momentum tensor in the divergence-free domain, as we have already described in the previous section. The resulting equation of motion becomes

$$G^{(L)}_{\mu\nu} = -k\hat{T}^{(M)}_{\mu\nu}. \tag{10}$$

Let us make here a comment on this formulation. As we know, from Feynman and others, the coherence of the field theory with the law of conservation of energy is obtained, in Einstein's terms, by an infinite series. Here, such an iterative procedure is substituted by non-locality. One can say, in a few words, that non-linearity is replaced by non-locality. Nevertheless, one should emphasize that these theories are not equivalent. They have an overlapping of certain properties but certainly not all, as we shall see in the next section.

Astonishingly enough, such a simple construction is able to describe the four classical standard tests of gravity. The main drawback of DL formulation appears when gravitational waves are present. The reason for this is precisely related to the linearity of DL model, that treats gravity as transparent to gravitons.

It is a pity that one should abandon such a simple and worth scheme to deal with gravitational phenomena. This led us to circunvent the wave problem by looking for a non-linear extension of this theory. Before going into the details of such extension let us point out that we will not analyse the general case but a specific model that indeed does circunvent such difficulty.

We shall comment in the final session of this paper how a series of alternative theories can be generated by the application of the same technique of projection introduced by DL.

Thus let us limit ourselves in our analysis to a specific simple extended theory fron now on.

3 NON-LINEAR EXTENDED MODEL

Using the above considerations we write the field equation representing a spin two field with self interaction in DL scheme by

$$G^{(L)}_{\mu\nu} = -k\hat{\mathcal{T}}_{\mu\nu} \tag{11}$$

where

$$G^{(L)}_{\mu\nu} = \Box h_{\mu\nu} - h^{\alpha}{}_{\mu|\alpha\nu} - h^{\alpha}{}_{\nu|\alpha\mu} + \eta_{\mu\nu} h^{\alpha\beta}{}_{|\alpha\beta} \tag{12}$$

and

$$\hat{\mathcal{T}}_{\mu\nu} \equiv Q_{\mu\nu}{}^{\alpha\beta}\mathcal{T}_{\alpha\beta}, \tag{13}$$

in which we have redefined the field variable by

$$h_{\alpha\beta} \equiv \phi_{\alpha\beta} - \frac{1}{2}\eta_{\alpha\beta}\phi^{\lambda}{}_{\lambda}. \tag{14}$$

The total energy momentum tensor that appears in the field equations is defined as the sum of the matter and the gravitational energy momentum tensor

$$\mathcal{T}_{\mu\nu} \equiv T_{\mu\nu} + t_{\mu\nu}. \tag{15}$$

The natural choice for the object that represents the gravitational contribution to the energy is the *Gupta* tensor:

$$t_{\alpha\beta} = \frac{1}{2k}\left[h_{\rho\nu|\beta}h^{\rho\nu}{}_{|\alpha} - \frac{1}{2}h^{\lambda}{}_{\lambda|\beta}h^{\rho}{}_{\rho|\alpha} + \right.$$

$$\left. -\frac{1}{2}\eta_{\beta\alpha}\left(h_{\rho\nu|\sigma}h^{\rho\nu|\sigma} - \frac{1}{2}h^{\lambda}{}_{\lambda|\sigma}h^{\rho}{}_{\rho}{}^{|\sigma}\right)\right]. \tag{16}$$

Note that contrary to the old (Feynman) procedure we are not led into any sort of trouble with the divergenceless identity, since we are not dealing here with the full Gupta tensor, but only to its restriction in the divergenceless space.

It seems worth at this point to stop for a while and make some comments on the formal aspect of the present model. The recursive approach that yields from the Fierz linear theory to the infinite series allows the summation that provides us with the geometrization of gravity. Here such a procedure is not followed, but instead we look for compatibility of the theory by allowing nonlocality of gravitational interaction. Putting aside the matter of taste, one should count only on observation to decide which one of these ways should indeed be retained.

Thus, let us now turn to the considerations of some properties of our model that could be checked observationally. In the next section we will analyse the behaviour of material particles acted upon by gravitational forces generated through the proposed non-local scheme. We will follow the same lines as in the original by quoted paper, ref.[2].

3.1 PARTICLE MOTION

Let us consider an incoherent dust cloud of non-interacting matter. The fluid is characterized by a 4-vector velocity U^{μ} and proper density ρ_0. We define the 4-velocity as

$$U^{\mu} = \left(\frac{\partial x^{\mu}}{\partial \tau}\right). \tag{17}$$

The scalar density ρ_0 is measured by an observer comoving with the fluid. The matter and interaction lagrangians are constructed in the standard way, that is

$$\mathcal{L}_M = \frac{1}{2}\rho_0 U_{\mu}U^{\mu} \tag{18}$$

and

$$\mathcal{L}_I = \phi^{\mu\nu}\hat{\mathcal{T}}_{\mu\nu}, \tag{19}$$

The interaction lagrangian (19) can be equivalently rewritten, up to surface terms, in the following way

$$\mathcal{L}_I = \psi^{\mu\nu}\mathcal{T}_{\mu\nu}, \tag{20}$$

with

$$\psi_{\mu\nu} = \left(P_{\mu\alpha}P_{\nu\beta} + p\eta_{\mu\nu}P_{\alpha\beta} + qP_{\mu\nu}P_{\alpha\beta}\right)\phi^{\alpha\beta}. \tag{21}$$

Finally, the resulting total Lagrangian $\mathcal{L}$

$$\mathcal{L} = \mathcal{L}_M + \mathcal{L}_I,$$

is given by

$$\mathcal{L} = \frac{1}{2}\rho_0 U_\mu U^\mu + \psi^{\mu\nu}\mathcal{T}_{\mu\nu}. \tag{22}$$

The resulting equations of motion are obtained by variation of $\mathcal{L}$ with respect to proper time τ. Using the Hamilton Principle we have

$$\left[\eta_{\mu\nu}U^\mu\frac{\partial U^\nu}{\partial\tau} + \frac{1}{\rho_0}\frac{\partial}{\partial\tau}\left(\psi^{\mu\nu}\mathcal{T}_{\mu\nu}\right)\right]\delta\tau = 0.$$

Or, equivalently,

$$\frac{1}{2}\frac{\partial}{\partial\tau}\left(\eta_{\mu\nu}U^\mu U^\nu + \frac{2}{\rho_0}\psi^{\mu\nu}\mathcal{T}_{\mu\nu}\right) = 0.$$

From the above equation we conclude that

$$\eta_{\mu\nu}U^\mu U^\nu + \frac{2}{\rho_0}\psi^{\mu\nu}\mathcal{T}_{\mu\nu} = constant \tag{23}$$

From now on we will normalize this constant.

The matter energy momentum tensor takes the usual form:

$$T_{\mu\nu} \equiv \rho_0 U_\mu U_\nu. \tag{24}$$

Then introducing (15), (17) and (24) in (23), and performing some algebraic manipulations we get

$$\left[1 - \frac{2}{\rho_0}\psi^{\mu\nu}t_{\mu\nu}\right](d\tau)^2 = \left(\eta_{\mu\nu} + 2\psi_{\mu\nu}\right)dx^\mu dx^\nu. \tag{25}$$

Let us define the quantity

$$S(\phi) \equiv \left[1 - \frac{2}{\rho_0}\psi^{\mu\nu}t_{\mu\nu}\right]. \tag{26}$$

We can then write:

$$(d\tau)^2 = S(\phi)^{-1}\left(\eta_{\mu\nu} + 2\psi_{\mu\nu}\right)dx^\mu dx^\nu, \tag{27}$$

with

$$S(\phi)^{-1} = \left\{1 + \frac{2}{\rho_0}\psi^{\mu\nu}t_{\mu\nu} + \frac{4}{\rho_0^2}\psi^{\mu\nu}\psi^{\alpha\beta}t_{\mu\nu}t_{\alpha\beta} + ...\right\}. \tag{28}$$

In this formulation one obtains a perfect equivalence with a geometric theory by identifying an effective metric $g_{\mu\nu}$, given by

$$g_{\mu\nu} = S(\phi)^{-1}\left(\eta_{\mu\nu} + 2\psi_{\mu\nu}\right).$$

Although the $g_{\mu\nu}$ are local functions, we can express them in terms of non local fields as one can see directly through the definition of $\psi_{\mu\nu}$.

3.2 THE STATIC SPHERICALLY SYMMETRICAL SOLUTION

Let us consider a static spherically symmetric configuration having a point-like source. In this case the d'Alembertian operator $\Box$ reduces to Laplacian Δ

$$\Delta \equiv \eta^{\alpha\beta}\partial_\alpha\partial_\beta \equiv -\nabla^2,$$

and the corresponding inverse obeys the same rule.

The equations of motion reduce then to

$$\Delta h_{\mu\nu} = -k\hat{\mathcal{T}}_{\mu\nu}. \tag{29}$$

in which we are using the particular gauge [3]

$$h^{\alpha\beta}{}_{|\alpha} = 0. \tag{30}$$

Since the projection operation is associative, it follows that

$$\hat{\mathcal{T}}_{\mu\nu} = \hat{T}_{\mu\nu} + \hat{t}_{\mu\nu}. \tag{31}$$

In our case, these quantities are given by

$$\hat{T}_{\mu\nu} \equiv (P_{\mu\alpha}P_{\nu\beta} + pP_{\mu\nu}\eta_{\alpha\beta} + qP_{\mu\nu}P_{\alpha\beta})\,T^{\alpha\beta}, \tag{32}$$

and

$$\hat{t}_{\mu\nu} \equiv (P_{\mu\alpha}P_{\nu\beta} + pP_{\mu\nu}\eta_{\alpha\beta} + qP_{\mu\nu}P_{\alpha\beta})\,t^{\alpha\beta}. \tag{33}$$

Then, using the above considerations, equation (29) becomes:

$$\Delta h_{\mu\nu} = -k\left(\hat{T}_{\mu\nu} + \hat{t}_{\mu\nu}\right). \tag{34}$$

In such a static spherically symmetric distribution the only non null component of $T_{\mu\nu}$ is T_{00} and the components of $\hat{T}_{\mu\nu}$ are consequently:

$$\hat{T}_{00} = (1 + p + q)\,T_{00}, \tag{35}$$

and

$$\hat{T}_{ij} = (p+q)\left(\eta_{ij} - \Delta^{-1}\partial_i\partial_j\right)T_{00}. \tag{36}$$

Using equations (33) and (16) we obtain the components of $\hat{t}_{\mu\nu}$:

$$\begin{aligned}\hat{t}_{00} = -\frac{1}{2k}\Big[\frac{1}{2}(1+2p+2q)\left(h_{\rho\nu|\sigma}h^{\rho\nu|\sigma} - \frac{1}{2}h^{\alpha}{}_{\alpha|\sigma}h^{\beta}{}_{\beta}{}^{|\sigma}\right) + \\ +q\Delta^{-1}\Big(\Delta h_{\rho\nu}\Delta h^{\rho\nu} + \Delta h_{\rho\nu|\sigma}h^{\rho\nu|\sigma} + \\ -\frac{1}{2}\Delta h^{\alpha}{}_{\alpha}\Delta h^{\beta}{}_{\beta} - \frac{1}{2}\Delta h^{\alpha}{}_{\alpha|\sigma}h^{\beta}{}_{\beta}{}^{|\sigma}\Big)\Big];\end{aligned} \tag{37}$$

[3] This choice is equivalent to using harmonic coordenates in the linear approximation of General Relativity.

$$\hat{t}_{ij} = \frac{1}{2k}\left\{-\frac{1}{2}(1+2p+2q)\left(h_{\rho\nu|\sigma}h^{\rho\nu|\sigma} - \frac{1}{2}h^{\alpha}{}_{\alpha|\sigma}h^{\beta}{}_{\beta}{}^{|\sigma}\right)\eta_{ij} + \right.$$

$$-\Delta^{-1}\left[h_{\rho\nu|i}\Delta h^{\rho\nu}{}_{|j} + h_{\rho\nu|j}\Delta h^{\rho\nu}{}_{|i} + 2h_{\rho\nu|ij}\Delta h^{\rho\nu} + \right.$$

$$\left. -\frac{1}{2}h^{\alpha}{}_{\alpha|i}\Delta h^{\beta}{}_{\beta|j} - \frac{1}{2}h^{\alpha}{}_{\alpha|j}\Delta h^{\beta}{}_{\beta|i} - h^{\alpha}{}_{\alpha|ij}\Delta h^{\beta}{}_{\beta}\right] +$$

$$+2(p+q)\Delta^{-1}\left[h_{\rho\nu|\alpha ij}h^{\rho\nu|\alpha} + h_{\rho\nu|\alpha i}h^{\rho\nu|\alpha}{}_{j} + \right.$$

$$\left. -\frac{1}{2}h^{\alpha}{}_{\alpha|\gamma ij}h^{\beta}{}_{\beta}{}^{|\gamma} - \frac{1}{2}h^{\alpha}{}_{\alpha|\gamma i}h^{\beta}{}_{\beta}{}^{|\gamma}{}_{j}\right] + h_{\rho\nu|i}h^{\rho\nu}{}_{|j} - \frac{1}{2}h^{\alpha}{}_{\alpha|i}h^{\beta}{}_{\beta|j} +$$

$$-q\eta_{ij}\Delta^{-1}\left[\Delta h_{\rho\nu}\Delta h^{\rho\nu} + \Delta h_{\rho\nu|\beta}h^{\rho\nu|\beta} - \frac{1}{2}\Delta h^{\alpha}{}_{\alpha}\Delta h^{\beta}{}_{\beta} + \right.$$

$$\left. -\frac{1}{2}\Delta h^{\alpha}{}_{\alpha|\gamma}h^{\beta}{}_{\beta}{}^{|\gamma}\right] + (1+q)\Delta^{-2}\left[2\Delta h_{\rho\nu|ij}\Delta h^{\rho\nu} + \right.$$

$$+2\Delta h_{\rho\nu|i}\Delta h^{\rho\nu}{}_{|j} + \Delta h_{\rho\nu|\beta ij}h^{\rho\nu|\beta} + \Delta h_{\rho\nu|\beta}h^{\rho\nu|\beta}{}_{ij} +$$

$$+\Delta h_{\rho\nu|\beta i}h^{\rho\nu|\beta}{}_{j} + \Delta h_{\rho\nu|\beta j}h^{\rho\nu|\beta}{}_{i} - \Delta h^{\alpha}{}_{\alpha|ij}\Delta h^{\beta}{}_{\beta} +$$

$$-\Delta h^{\alpha}{}_{\alpha|i}\Delta h^{\beta}{}_{\beta|j} - \frac{1}{2}\Delta h^{\alpha}{}_{\alpha|\gamma ij}h^{\beta}{}_{\beta}{}^{|\gamma} - \frac{1}{2}\Delta h^{\alpha}{}_{\alpha|\gamma}h^{\beta}{}_{\beta}{}^{|\gamma}{}_{ij} +$$

$$\left.\left. -\frac{1}{2}\Delta h^{\alpha}{}_{\alpha|\gamma i}h^{\beta}{}_{\beta}{}^{|\gamma}{}_{j} - \frac{1}{2}\Delta h^{\alpha}{}_{\alpha|\gamma j}h^{\beta}{}_{\beta}{}^{|\gamma}{}_{i}\right]\right\}. \qquad (38)$$

The field equations take then the form:

$$\Delta h_{00} = -\frac{1+p+3q}{1+3q}kT_{00} + \frac{q}{1+3q}\Delta h^{\alpha}{}_{\alpha} +$$

$$+\frac{1+2p+3q}{4(1+3q)}\left(h_{\rho\nu|\alpha}h^{\rho\nu|\alpha} - \frac{1}{2}h^{\alpha}{}_{\alpha|\gamma}h^{\beta}{}_{\beta}{}^{|\gamma}\right); \qquad (39)$$

$$\Delta h_{ij} = \left(\eta_{ij} + \Delta^{-1}\partial_i\partial_j\right)kT_{00} + \left(\eta_{ij} + 2\Delta^{-1}\partial_i\partial_j\right)\Delta h_{00} +$$

$$-\Delta^{-1}\partial_i\partial_j\Delta h^{\alpha}{}_{\alpha} + \Delta^{-1}\left(h_{\rho\nu|ij}\Delta h^{\rho\nu} - h_{\rho\nu|\alpha i}h^{\rho\nu|\alpha}{}_{j} + \right.$$

$$\left. -\frac{1}{2}h^{\alpha}{}_{\alpha|ij}\Delta h^{\beta}{}_{\beta} + \frac{1}{2}h^{\alpha}{}_{\alpha|\gamma i}h^{\beta}{}_{\beta}{}^{|\gamma}{}_{j}\right). \qquad (40)$$

The above set of equations, for the case of the static gravitational field configuration, is too intrincate. To simplify its manipulation and to allow for a future interpretation of the solutions it seems worth to examine a particular class these equation, by looking for the case in which the solution takes the form:

$$h_{\alpha\beta} = \omega\rho_{\alpha\beta} + \omega^2\sigma_{\alpha\beta}, \tag{41}$$

where ω is an arbitrary function (that could be constant) and that will be fixed later on. To solve this intrincate system of differential equation we will employ the following method. We start by assuming a hypothesis (to be checked for consistency later on) that the linear and the nonlinear parts are decoupled.

A remarkable property of this set of equations then appears. The functions $\rho_{\alpha\beta}$ and $\sigma_{\alpha\beta}$ are determined from the orders ω and ω^2 up to an equal number of free functions $\chi_{\alpha\beta}$ and $\pi_{\alpha\beta}$ that can in turn be used to satisfy the coherence, from the obtained solution, of the remaining orders ω^3 and ω^4. Then the equations that should be explicitly solved reduce to the following set:

$$\Delta\rho_{00} = -\frac{1+p+3q}{1+3q}kT_{00} + \frac{q}{1+3q}\Delta\rho^{\alpha}{}_{\alpha}; \tag{42}$$

$$\Delta\sigma_{00} = \frac{1+2p+3q}{4\left(1+3q\right)}\left(\rho_{\alpha\beta|\gamma}\rho^{\alpha\beta|\gamma} - \frac{1}{2}\rho^{\alpha}{}_{\alpha|\gamma}\rho^{\beta}{}_{\beta}{}^{|\gamma}\right) + $$
$$+\frac{q}{1+3q}\Delta\sigma^{\alpha}{}_{\alpha}; \tag{43}$$

$$\Delta\rho_{ij} = \left(\eta_{ij} + \Delta^{-1}\partial_i\partial_j\right)kT_{00} - \Delta^{-1}\partial_i\partial_j\Delta\rho^{\alpha}{}_{\alpha} +$$
$$+\left(\eta_{ij} + 2\Delta^{-1}\partial_i\partial_j\right)\Delta\rho_{00}; \tag{44}$$

$$\Delta\sigma_{ij} = \left(\eta_{ij} + 2\Delta^{-1}\partial_i\partial_j\right)\Delta\sigma_{00} - \Delta^{-1}\partial_i\partial_j\Delta\sigma^{\alpha}{}_{\alpha} +$$
$$+\Delta^{-1}\left(\rho_{\rho\nu|ij}\Delta\rho^{\rho\nu} - \rho_{\rho\nu|\alpha i}\rho^{\rho\nu|\alpha}{}_{j} + \right.$$
$$\left. -\frac{1}{2}\rho^{\alpha}{}_{\alpha|ij}\Delta\rho^{\beta}{}_{\beta} + \frac{1}{2}\rho^{\alpha}{}_{\alpha|\gamma i}\rho^{\beta}{}_{\beta}{}^{|\gamma}{}_{j}\right). \tag{45}$$

As we said, from a direct analysis of the equations (42), (44), (43) and (45) there still remains a number of free functions. This can be seen by taking the transformations

$$\rho_{00} \rightarrow \rho'_{00} = \rho_{00} + \chi_{00},$$

$$\rho_{ij} \rightarrow \rho'_{ij} = \rho_{ij} + \chi_{ij},$$

with

$$\Delta\chi_{00} = 0,$$
$$\Delta\chi_{ij} = 0;$$

and

$$\sigma_{00} \to \sigma'_{00} = \sigma_{00} + \pi_{00},$$

$$\sigma_{ij} \to \sigma'_{ij} = \sigma_{ij} + \pi_{ij},$$

with

$$\Delta\pi_{00} = 0,$$
$$\Delta\pi_{ij} = 0.$$

The complete solution of the previous set of equations is then

$$\rho_{00} = -2\frac{1+p+4q+3pq+3q^2}{1+3q}\frac{m}{r}, \tag{46}$$

$$\rho_{ij} = -\frac{p+q+3pq+3q^2}{1+3q}\frac{m}{r}\left(\eta_{ij} - \frac{x_i x_j}{r^2}\right). \tag{47}$$

$$\sigma_{00} = c'_2\frac{m^2}{r^2} \tag{48}$$

and

$$\sigma_{ij} = c'_3\frac{m^2}{r^2}\frac{x_i x_j}{r^2}, \tag{49}$$

with

$$c'_2 = -\frac{5}{4}q^3 - p^3 - \frac{7}{2}pq^2 - \frac{13}{4}p^2 q - \frac{3}{2}q^2 - \frac{3}{2}p^2 - 3pq + \frac{1}{4} \tag{50}$$

and

$$c'_3 = \frac{5}{3}q^3 + 2p^3 + 7pq^2 + \frac{13}{2}p^2 q + \frac{11}{4}q^2 + \frac{11}{4}p^2 + \frac{11}{2}pq - \frac{1}{2}q - \frac{1}{2}p - \frac{1}{4}. \tag{51}$$

In terms of the field variables $\phi_{\mu\nu}$ we have

$$\phi_{00} = \frac{1}{2}\left(a' - 4b'\right)\frac{km}{r} + \frac{1}{2}\left(d' + f'\right)\left(\frac{km}{r}\right)^2, \tag{52}$$

$$\phi_{ij} = \left[\left(-b' - \frac{a'}{2}\right)\frac{km}{r} + \left(-\frac{d'}{2} + \frac{f'}{2}\right)\left(\frac{km}{r}\right)^2\right]\eta_{ij} +$$

$$+\left[-b'\frac{km}{r} + f'\left(\frac{km}{r}\right)^2\right]\frac{x^i x^j}{r^2}, \tag{53}$$

where the coeficients a', b', d' and f' are parameter combinations and will be stated later.

3.3 STANDARD CLASSICAL TESTS

We are now prepared to come back to the modified expression of the proper time. Using the solutions (46), (47), (48) and (49) and assuming the arbitary parameter ω to be identified to the coupling constant k, we obtain the corresponding prescription for the non-local variable $\psi_{\mu\nu}$ [4],

$$\psi_{00} = \frac{A'}{r} + \frac{B'}{r^2}, \tag{54}$$

and

$$\psi_{ij} = \left(\frac{C'}{r} + \frac{D'}{r^2}\right)\eta_{ij} + \left(\frac{F'}{r} + \frac{G'}{r^2}\right)\frac{x^i x^j}{r^2}. \tag{55}$$

Then, the expression of $(d\tau)^2$, Eq. (27), reduces to:

$$(d\tau)^2 = \left(1 - \frac{a}{r} - \frac{A}{r^2}\right)dt^2 - \left(1 + \frac{b}{r} + \frac{B}{r^2}\right)dr^2 - \left(1 + \frac{d}{r} + \frac{D}{r^2}\right)r^2 d\Omega, \tag{56}$$

in which[5]

$$d\Omega \equiv d\theta^2 + sin^2\theta\, d\phi^2.$$

The coefficients that appear in this expression are combinations of the constants p and q given by:

$$a = -3(p+q)^2 - 2(p+q) + 1; \tag{57}$$

$$b = 3(p+q)^2 + 2(p+q) + 1; \tag{58}$$

$$d = \frac{3}{2}(p+q)^2 + \frac{5}{2}(p+q) + 1; \tag{59}$$

$$A = -\frac{15}{16}q^4 - \frac{3}{4}p^4 - \frac{57}{16}pq^3 - \frac{51}{16}p^3q - \frac{81}{16}p^2q^2 - \frac{11}{8}q^3 - \frac{21}{16}p^3 + -\frac{65}{16}pq^2 - 4p^2q - \frac{3}{16}(p+q)^2 + \frac{1}{4}(p+q); \tag{60}$$

$$B = \frac{15}{8}q^4 + \frac{3}{2}p^4 + \frac{57}{8}pq^3 + \frac{51}{8}p^3q + \frac{81}{8}p^2q^2 + \frac{29}{16}q^3 + \frac{15}{8}p^3 + +\frac{11}{2}pq^2 + \frac{89}{16}p^2q - \frac{9}{16}(p+q)^2 - \frac{1}{8}(p+q); \tag{61}$$

$$D = \frac{15}{16}q^3 + \frac{3}{4}p^3 + \frac{21}{8}pq^2 + \frac{39}{16}p^2q + \frac{17}{16}(p+q)^2 - \frac{1}{8}. \tag{62}$$

By the same procedure as in the case of General Relativity, we obtain the following constants of motion E and L:

$$E \equiv \left(1 - \frac{a}{r} - \frac{A}{r^2}\right)\frac{dt}{d\tau} \tag{63}$$

[4]From now on and until convenient we will use $2km$ units. The above equations will be accordingly rewritten.

[5]Note that Eq. (56) reduces to DL form in the case $A = B = D = 0$.

and

$$L \equiv \left(1 + \frac{d}{r} + \frac{D}{r^2}\right) r^2 \frac{d\varphi}{d\tau}. \tag{64}$$

Defining the a new variable $u \equiv \frac{1}{r}$, and using the above relations, the equation governing the shape of the orbit is given by

$$\begin{aligned} \frac{d^2u}{d\varphi^2} + u &= \frac{a}{2}\frac{E^2}{L^2} + \frac{1}{2}(2d-b)\frac{E^2-1}{L^2} + \\ &+ \left[\left(A + a^2 - ab + 2ad\right)\frac{E^2}{L^2} + (-B + 2D + \right. \\ &\left. + b^2 - 2bd + d^2\right)\frac{E^2-1}{L^2}\Big] u + \frac{3}{2}(b-d)\,u^2. \end{aligned} \tag{65}$$

Using Eq. (65) we find the corresponding equation for a planetary orbit. Taking into account the approximation $E^2 \approx 1$, it follows that

$$\frac{d^2u}{d\varphi^2} + u = \frac{a}{2}\frac{E^2}{L^2} + \left(A + a^2 - ab + 2ad\right)\frac{E^2}{L^2}u + \frac{3}{2}(b-d)\,u^2. \tag{66}$$

Using the approximation, valid for the solar system,[6]

$$u \approx \frac{a}{2}\frac{E^2}{L^2}, \tag{67}$$

it follows[7]

$$\frac{d^2u}{d\varphi^2} + u = \frac{GM}{L^2}a + 2\left(\frac{2A}{a} + 2a - \frac{b}{2} + \frac{5}{2}d\right) GMu^2. \tag{68}$$

The last term gives the perihelion precession of our present theory. This should be compared with the analogous result in General Relativity. Before this, however, it is more convenient to analyse the consequences of the remaining equations.

Limiting our analysis here to the path of light rays we have

$$\frac{d^2u}{d\varphi^2} + u = \frac{3}{2}GM\,(a+b)\,u^2. \tag{69}$$

Using the values for a and b taken from Eqs.(57) and (58), it follows identically that $a + b \equiv 2$. Substituting this value into the above Eq. (69) we obtain the same value predicted by GR for the bending of light. Indeed,

$$\frac{d^2u}{d\varphi^2} + u = 3GMu^2.$$

The fourth test (radar delay time) is automatically satisfied. To proceed we need to specify the values of a and b. Let us examine a simple possible case by choosing

$$a = 1, \tag{70}$$

$$b = 1. \tag{71}$$

[6]The reader may consult the quoted article of Deser et al for more details.

[7]From here on we will write explicitly the constants GM of our problem instead of working in a system of units in which they are made equal to 1. This is done here just for comparison with observations.

Coming back to the original equation for the perihelion precession results, in this case, in

$$\frac{d^2u}{d\varphi^2} + u = \frac{GM}{L^2} + 3GMu^2 + (4A + 5d)\,GMu^2.$$

By comparison with the analogous result in GR we are led to impose

$$4A + 5d = 0, \tag{72}$$

which provides the second[8] equation for the unknown parameters p and q. Solving the algebraic equation we obtain

$$p = -1, \tag{73}$$

$$q = \frac{1}{3}. \tag{74}$$

Coming back to Eq. (63) and using these values of p and q we obtain

$$E = \left(1 - \frac{2GM}{r}\right)\frac{dt}{d\tau}, \tag{75}$$

which is precisely the formula of GR for the redshift.

From these results we conclude that the choice $p = -1$ and $q = \frac{1}{3}$ makes the present theory, as far as the four tests are concerned, indistinguishable from General Relativity. Thus the next step should be the examination of the strong field structure, i.e. pulsar binary system. This issue is presently under investigation.

4 GENERAL FORMULATION OF NON-LINEAR NON-LOCAL GRAVITY THEORY

The scheme that we presented in this paper can be generalized to incorporate higher orders of the basic contribution of the gravitational field. In each order a self consistent theory is obtained. Thus a whole class of consistent gravity theories that circunvent the tradicional problems of combining the conservation of the energy momentum tensor of matter and gravity is obtained.

Although we have limited our analysis here to describe the gravitational field by just a quadratic non linear non local theory, this kind of natural extension can be immediately formulated.

The formal aspect of the equations of motion remains the same, that is,

$$G^{(L)}_{\mu\nu} = -k\,\hat{\mathcal{T}}_{\mu\nu}.$$

In the general case the right hand side contains the non linear gravity contribution to the energy momentum tensor, whose terms are written as

$$\mathcal{T}_{\mu\nu} = T^{(M)}_{\mu\nu} + \sum_{i=1}^{n} c_i t^{(i)}_{\mu\nu}.$$

[8]The other equation is provided by Eq.(70) or Eq.(71). Note that any one of these yields the same restriction on the parameters p and q.

The constant c_i can assume the values 0 or 1 and establishes the diferent theories to be constructed in this formulation. Deser and Laurent theory is the case when all constants c_i are equal to zero. The choice $c_1 = 1$ and all other constants $c_i, i \neq 1$ vanishing constitutes the non linear non local theory presented in this paper. The summation of the complete series became identical to Einstein's. In this case the application of the non local projection operator is not justified anymore.

5 CONCLUSION

In this paper we have shown that it is possible to obtain a coherent scheme of massless spin two field theory that describes gravitational interaction beyond the standard geometrical way (Einstein, Feynman, Deser, etc.). We do this by appealing to a generalization of a previous method set out by Deser and Laurent. This scheme uses the mathematical properties of projection operators in the divergenceless space of arbitrary tensors. Although Deser and Laurent approach deals with a linear theory (which is precisely the main drawback of their model), we use here the same technique but in a non linear framework. We can thus combine the non local and the non linear properties in a unified scheme that provides a self-consistent model for gravitational interactions. We have shown that our model satisfies all the observational tests. We have exibit this by showing a static spherically symmetrical solution for the gravitational field. The next step should then be to undertake the task of solving our equations for strong gravitational fields mainly to the respect of the consequences of the behaviou in the neighbourhood of neutrons stars; and, beyond the static case, to generate a cosmological framework within our theory. We are now presently examining these cases.

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