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Toroidal Magnetic Field in Closed Robertson-Walker Cosmologies

*A.L. Oliveira** and *A.F.F. Teixeira*

Centro Brasileiro de Pesquisas Físicas - CBPF
Rua Dr. Xavier Sigaud, 150
22290-180 Rio de Janeiro/RJ, Brasil

*Universidade Federal do Rio de Janeiro,
Departamento de Astronomia,
Observatório do Valongo,
20080-090 Rio de Janeiro/RJ, Brasil

Abstract

In the background of a Robertson-Walker universe model with positive spatial curvature and arbitrary time evolution, the exact solution of a sourceless magnetic field with finite range is presented. The field seems an inhomogeneous solid torus spanning the entire space, and is non-singular. Both topologies that permit global isotropy are considered: the three-sphere $\mathbf{S}^3$ and the multiply connected projective three-space $\mathbf{P}^3$.

Key-words: Magnetic field; Cosmology; Topology.

1 Introduction

In the near future a lot of information is expected from both land based and space orbiting telescopes, much more precise and reliable than those available at present. These new data, it is hoped, will eventually lead to a decision concerning what are perhaps the two major mathematical questions in cosmology, namely what are the geometry and the topology of our universe [1].

Measurements in cosmology are associated most commonly with electromagnetic radiation; it then follows indisputably the necessity of studying the electromagnetic field in the background of a given geometry. On the grounds that the universe looks spatially homogeneous and locally isotropic, special emphasis has been given in the literature to study of Robertson-Walker (RW) geometries.

As concerns the electromagnetism, the non-quantized electromagnetic field is correctly described by Maxwell's equations. However, from the experimental point of view the non-quantized Proca equations prove to be equally good, provided the range λ of the Proca field is sufficiently large (when $\lambda \to \infty$ we recover the Maxwell field); further, from the theoretical viewpoint the Proca field seems more suitable to represent electromagnetism in compact three-spaces, where infinite distances are nonsensical [2].

Extensive magnetic fields are commonplace in the universe, so they have deserved the attention of a fairly large number of authors. Nevertheless a quite interesting physical possibility, mathematically very simple, seems to have eluded all investigation so far: that of a non-uniform magnetic field shaped like a solid torus and pervading the entire universe. The field obeys the sourceless Proca equations and is non-singular. The $t = const$ sections of the spatially homogeneous and globally isotropic universe models are endowed with positive curvature (with unit radius for simplicity). The inquiry into such a field is the purpose of this report.

2 Equations

We concern ourselves with the RW line element

$$ds^2 = dt^2 - R^2(t)dl^2 \; , \tag{2.1}$$
$$dl^2 = d\rho^2 + \sin^2\rho \; d\phi^2 + \cos^2\rho \; d\zeta^2 \; , \tag{2.2}$$

where $R(t)$ is arbitrary. In the background of this four-geometry we assume the vector potential

$$A_\mu = a[0; 0, 0, \cos^2\rho] \;\; , \;\; a = const \; ; \tag{2.3}$$

it is trivial to show that A_μ satisfies the Lorentz gauge $\partial_\mu[(-g)^{1/2}A^\mu] = 0$.

The skew-symmetric field $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ corresponding to A_μ has only one independent component, which is

$$F_{13} = -F_{31} = -a\sin 2\rho \; . \tag{2.4}$$

Finally, we remark that the finite range, source-free equations for $F^{\mu\nu}$, namely

$$\partial_\mu[(-g)^{1/2}F^{\mu\nu}] - \lambda^{-2}(-g)^{1/2}A^\nu = 0 \; , \tag{2.5}$$

are obeyed provided the range equals one-half the (unit) radius of the universe, $\lambda = 1/2$ [3].

3 Discussion

As pointed out by Schrödinger [4], the line element (2.2) corresponds to the three-dimensional spherical geometry with unit radius. The spatial coordinates (ρ, ϕ, ζ) have been termed cylindrical, because when $\rho \to 0$ one finds that $dl^2 \to d\rho^2 + \rho^2\ d\phi^2 + d\zeta^2$. The azimuthal angular coordinate $-\pi < \phi \leq +\pi$ is familiar, while the radial coordinate ρ ranges from 0 to $\pi/2$. The altitude ζ is a cyclical coordinate whose range depends on the topology of the three-manifold: for the three-sphere $\mathbf{S}^3$ the altitude goes from $-\pi$ to $+\pi$, while for the real projective three-space $\mathbf{P}^3$ the range is $[-\pi/2, +\pi/2]$ only.

The vector potential A_μ in (2.3) seems to be novel in the literature; it has a maximum intensity along the circle $\rho = 0$ and weakens monotonically to nought on the circle $\rho = \pi/2$. In both topologies $\mathbf{S}^3$ and $\mathbf{P}^3$ these two circles are geodetic lines, indeed they are polar to each other.

The magnetic field is purely azimuthal, with one physical component

$$H_\phi = -(g_{\rho\rho} g_{\zeta\zeta})^{-1/2} F_{13} = -2aR^{-2}(t) \sin\rho\ . \tag{3.1}$$

We notice that for a fixed time t the field intensity monotonically increases from the zero value along $\rho = 0$ to the maximum value along $\rho = \pi/2$; each line of the field is a circle with ρ and ζ constant, the intensity of flux being also constant along each such circle. The overall aspect of the field for a given instant t is that of a spatially inhomogeneous solid torus who permeates the whole space of $\mathbf{S}^3$ or $\mathbf{P}^3$.

A result which has interesting physical significance seems worth mentioning. We first notice that the physical magnetic field given in Eq. (3.1) is time-depending, due to the factor $R^{-2}(t)$: the lines of the field behave as if they were frozen in the background three-space, and this space expands or contracts at the rate described by the factor $R(t)$. However, we have seen that no electric field is produced when the magnetic lines expand or contract in the curved space of Eq. (2.1); this is probably an unexpected result, since it is in striking contrast to the predictions of the ordinary Maxwell theory in flat space-time in regions free from electromagnetic source.

It should be noticed that the toroidal magnetic field described here is compatible with a fairly large number of universe models. Indeed, not only the field does not imply any particular material content or state of motion of the universe (since the time-depending function $R(t)$ has been left arbitrary), nor is it peculiar to any specific geometrodynamical theory (since the origins of the line element (2.1) were neither declared).

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