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Non Extensive Physics: a Possible Connection Between Generalized Statistical Mechanics and Quantum Groups

by

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Abstract

Two different formalisms have been recently developed for nonextensive Physics, namely the Generalized Statistical Mechanics and Thermodynamics (characterized by $q \neq 1$) and the Quantum Groups (characterized by $q_G \neq 1$). Through the discussion of the mean values of observables, we propose a (temperature dependent) connection between q and q_G, and illustrate with bosonic oscillators.

Key-words: Generalized entropy; Generalized statistical mechanics; Quantum groups; q-deformations.

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An extremely interesting tendency towards nonextensive Physics keeps growing along recent years. This fact appears in two different areas, namely in Statistical Mechanics and Quantum Groups, through apparently independent paths.

In what Statistical Mechanics is concerned, a generalized entropy has been proposed as follows [1]

$$S_q = k \frac{1 - \sum_i p_i^q}{q-1} \qquad (q \in \Re) \tag{1}$$

where p_i is the probability associated with the i-th microscopic state of the system and k a conventionally chosen positive constant. The $q \to 1$ limit of S_q yields the well known Shannon expression $-k_B \Sigma_i p_i \ln p_i$ (where we have used $p_i^{q-1} \sim 1 + (q-1)\ln p_i$). S_q satisfies, for all $q > 0$, the standard properties of positivity, equiprobability, expansibility, concavity (which guarantees thermodynamic stability for the system), H-theorem [2–4], among others. However, if we have two *independent* systems Σ and Σ' (i.e., $\hat{\rho}_{\Sigma U \Sigma'} = \hat{\rho}_{\Sigma}\hat{\rho}_{\Sigma'}$, where $\hat{\rho}$ denotes the density operator), we immediately verify *pseudo-additivity*, more precisely

$$S_q^{\Sigma U \Sigma'} = S_q^{\Sigma} + S_q^{\Sigma'} + (1-q) S_q^{\Sigma} S_q^{\Sigma'} \tag{2}$$

In other words, S_q is generically extensive if and only if $q = 1$; otherwise, it is *nonextensive*. The connection to a consistently generalized equilibrium Thermodynamics is established by extremizing S_q with the constraints $Tr\hat{\rho} = 1$ and (for the canonical ensemble) [5]

$$< \hat{\mathcal{H}} >_q \equiv Tr \hat{\rho}^q \hat{\mathcal{H}} = U_q \tag{3}$$

where $\hat{\mathcal{H}}$ is the Hamiltonian and U_q the generalized internal energy. This optimization yields the distribution [1, 5]

$$\hat{\rho} = \frac{[1 - \beta(1-q)\hat{\mathcal{H}}]^{\frac{1}{1-q}}}{Z_q} \tag{4}$$

where

$$Z_q \equiv Tr[1 - \beta(1-q)\hat{\mathcal{H}}]^{\frac{1}{1-q}} \tag{5}$$

with $\beta \equiv 1/kT$ being a Lagrange parameter. It can be shown [5] that $1/T = \partial S_q / \partial U_q$, $F_q \equiv U_q - TS_q = -\frac{1}{\beta}\frac{Z_q^{1-q}-1}{1-q}$ and $U_q = -\frac{\partial}{\partial\beta}\frac{Z_q^{1-q}-1}{1-q}$. In the $q \to 1$ limit, Eq. (4) recovers the well known Boltzmann-Gibbs distribution $\hat{\rho} \propto \exp(-\beta\hat{\mathcal{H}})$. Furthermore, this generalized statistics has been shown to satisfy appropriate forms of the Ehrenfest theorem [6], von Neumann equation [7], Jaynes Information Theory duality relations [6], fluctuation-dissipation theorem [8, 9], Bogolyubov inequality [10], Langevin and Fokker-Planck equation [11], Callen's identity [12], quantum statistics [13], among others. Moreover, this generalized scheme has enabled [14] the overcome of the Boltzmann-Gibbs inability to provide *finite* mass for astrophysical systems within the polytropic model as studied by

Chandrasekhar and others (Balian and many others [15] had already pointed the need for a nonextensive entropy in Astrophysics; this is a very natural thing to happen whenever the long-range gravitational forces are essentially involved in the problem). Finally, this generalization has recently enabled [16] to derive d-dimensional Lévy flights from an entropic optimization using physically acceptable a priori constraints, q being directly related to the fractal dimension of the random motion. Again, this overcomes a well known inability of $q = 1$ statistics [17].

Let us now focus, on the other hand, Quantum Groups (q_G-deformations, q_G-oscillators, q_G-calculus, where we use q_G, instead of the traditional notation q, in order to avoid confusion with the present entropy parameter q). These are generalizations of Lie groups and algebras, which are recovered for $q_G \to 1$. They have provided applications in as varied areas as (see [18–23] and references therein) inverse scattering method, vertex models, anisotropic spin chains Hamiltonians, knot theory, conformal field theory, heuristic phenomenology of deformed molecules and nuclei, non-commutative approach to quantum gravity and anyon physics. They have enabled, in particular, a formulation of quantum mechanics [19] in a *discontinuous space time* (where $(q_G - 1)$ plays the role of minimal lattice step, and $(q_G - 1)^2$ that of minimal time step). To illustrate the *nonextensivity* associated with quantum groups let us consider a bosonic $q_G - oscillator$. Its Hamiltonian is given (see [23] and references therein) by

$$\hat{\mathcal{H}} = \hbar\omega \hat{A}^+ \hat{A} = \hbar\omega [\hat{N}]_A \tag{6}$$

where $[\hat{N}]_A \equiv (q_G^{2\hat{N}} - 1)/(q_G^2 - 1)$, $\hat{N}$ being the q_G-generalized number operator, $\omega > 0$ is a characteristic frequency, and $\hat{A}^+$ and $\hat{A}$ respectively are the creation and annihilation operators satisfying

$$\hat{A}\hat{A}^+ - q_G^2 \hat{A}^+ \hat{A} = \hat{1} \tag{7}$$

as well as

$$[\hat{N}, \hat{A}^+] = \hat{A}^+ \tag{8}$$

and

$$[\hat{N}, \hat{A}] = -\hat{A} \tag{9}$$

The eigenvalues of $[\hat{N}]_A$ are given by [23]

$$[n]_A = \frac{q_G^{2n} - 1}{q_G^2 - 1} \qquad (n = 0, 1, 2, \cdots) \tag{10}$$

In the $q_G \to 1$ limit, Eq. (10) recovers the well known extensive expression $[n]_A = n$.

Let us now address the main aim of the present paper, namely the basic question of a possible connection between q and q_G. We shall use as a guideline the following trivial observation: within the generalized statistics, the mean value of any observable $\hat{O}$ is given by [5-9,12]

$$< \hat{O} >_q \equiv Tr \hat{\rho}^q \hat{O} = Tr \hat{\rho} (\hat{\rho}^{q-1} \hat{O}) \equiv < \hat{\rho}^{q-1} \hat{O} > \tag{11}$$

i,e., it can be thought as *nonextensive statistics* ($\hat{\rho}^q$) *on an extensive operator* ($\hat{O}$), *or as extensive statistics* ($\hat{\rho}$) *on a nonextensive operator* ($\hat{\rho}^{q-1}\hat{O}$).

We consider now a q_G-deformed arbitrary extensive observable $\hat{O}$ (e.g., number of particles, energy, or any other observable which, were it not for the q_G-deformation, would be extensive) associated with two *independent* systems Σ and Σ'($\hat{\rho}_{\Sigma U \Sigma'} = \hat{\rho}_\Sigma \hat{\rho}_{\Sigma'}$ with $Tr\hat{\rho}_{\Sigma U\Sigma'} = Tr\hat{\rho}_\Sigma = Tr\hat{\rho}_{\Sigma'} = 1$). We have

$$\hat{O}(\Sigma U \Sigma') = \hat{O}(\Sigma) + \hat{O}(\Sigma') + (q_G - 1)\hat{\eta}_{q_G}^{\Sigma\Sigma'} \tag{12}$$

where $\hat{\eta}_{q_G}^{\Sigma\Sigma'}$ is, by definition, the nonextensive correction associated with q_G (although q_G is a complex number, we restrict our discussion to $q_G \in \Re$, as in [19]). Eq. (12) implies

$$\begin{aligned} < \hat{O}(\Sigma U \Sigma') >_q &= Tr\hat{\rho}^q_\Sigma \hat{\rho}^q_{\Sigma'}[\hat{O}(\Sigma) + \hat{O}(\Sigma') \\ &+ (q_G - 1)\hat{\eta}_{q_G}^{\Sigma\Sigma'}] \\ &= [Tr\hat{\rho}^q_\Sigma \hat{O}(\Sigma)][Tr\hat{\rho}^q_{\Sigma'}] \\ &+ [Tr\hat{\rho}^q_{\Sigma'} \hat{O}(\Sigma')][Tr\hat{\rho}^q_\Sigma] \\ &+ (q_G - 1) < \hat{\eta}_{q_G}^{\Sigma\Sigma'} >_q \end{aligned} \tag{13}$$

hence, by using (1) (i.e., $Tr\hat{\rho}^q = 1 + (1-q)S_q$),

$$\begin{aligned} < \hat{O}(\Sigma U \Sigma') >_q &= < \hat{O}(\Sigma) >_q + < \hat{O}(\Sigma') >_q \\ &+ (1-q)\left[< \hat{O}(\Sigma) >_q S_q^{\Sigma'} + < \hat{O}(\Sigma') >_q S_q^{\Sigma}\right]/k \\ &+ (q_G - 1) < \hat{\eta}_{q_G}^{\Sigma\Sigma'} >_q \end{aligned} \tag{14}$$

If we impose now that the *mean values of the observable must be extensive whenever these are measurable quantities* (i.e., that the $q \neq 1$ effect is exactly compensated by the $q_G \neq 1$ effect) we obtain

$$q - 1 = k(q_G - 1)\frac{< \hat{\eta}_{q_G}^{\Sigma\Sigma'} >_q}{< \hat{O}(\Sigma) >_q S_q^{\Sigma'} + < \hat{O}(\Sigma') >_q S_q^{\Sigma}} \tag{15}$$

which yields the connection we were looking for. Generically, $q = 1$ if and only if $q_G = 1$. In the $q_G \to 1$ limit, Eq. (15) becomes

$$q - 1 \sim k_B(q_G - 1)\frac{<\hat{\eta}_1^{\Sigma\Sigma'}>}{<\hat{O}(\Sigma)>S_1^{\Sigma'} + <\hat{O}(\Sigma')>S_1^{\Sigma}} \tag{16}$$

$$\propto q_G - 1 \tag{16'}$$

The situation is schematically indicated in Fig. 1.

Before going on let us remark that the *entropy operator* $\hat{S}_q \equiv k(\hat{1} - \hat{\rho}^{1-q})/(1-q)$ (so denominated because it satisfies $< \hat{S}_q >_q = S_q$) is not included among the operators $\hat{O}$ which Eq. (15) refers to. Indeed, it satisfies $\hat{S}_q^{\Sigma U \Sigma'} = \hat{S}_q^{\Sigma} + \hat{S}_q^{\Sigma'} + (q-1)\hat{S}_q^{\Sigma}\hat{S}_q^{\Sigma'}$, consequently it is not extensive (unless $q=1$). The application of $Tr\hat{\rho}_{\Sigma}^{q}\hat{\rho}_{\Sigma'}^{q}$ on both sides of this equality naturally recovers Eq. (2).

Let us now illustrate the present calculation with *two* bosonic oscillators of the type described by Hamiltonian (6). Let the observable $\hat{O}$ be $[\hat{N}]_A$. According to Eq. (10) we have

$$[n_{\Sigma}]_A = \frac{q_G^{2n_{\Sigma}}-1}{q_G^2-1} \qquad (n_{\Sigma}=0,1,2,\cdots) \tag{17}$$

and

$$[n_{\Sigma'}]_A = \frac{q_G^{2n_{\Sigma'}}-1}{q_G^2-1} \qquad (n_{\Sigma'}=0,1,2,\cdots) \tag{18}$$

hence

$$[n_{\Sigma U \Sigma'}]_A = \frac{q^{2(n_{\Sigma}+n_{\Sigma'})}-1}{q_G^2-1} \tag{19}$$

Consequently, using Eq. (12), we have

$$\begin{aligned} (q_G-1)\hat{\eta}_{q_G}^{\Sigma\Sigma'} &= [\hat{N}_{\Sigma U\Sigma'}]_A-[\hat{N}_{\Sigma}]_A-[\hat{N}_{\Sigma'}]_A \\ &= \frac{q_G^{2(\hat{N}_{\Sigma}+\hat{N}_{\Sigma'})}-q_G^{2\hat{N}_{\Sigma}}-q_G^{2\hat{N}_{\Sigma'}}+1}{q_G^2-1} \\ &= \frac{(q_G^{2\hat{N}_{\Sigma}}-1)(q_G^{2\hat{N}_{\Sigma'}}-1)}{q_G^2-1} \end{aligned} \tag{20}$$

This is a good point for commenting the generic form we expect for $\hat{\eta}_{q_G}^{\Sigma\Sigma'}$. As illustrated in Eq. (20) (and in what follows from it), we expect to be $\hat{\eta}_{q_G}^{\Sigma\Sigma'}=f(\hat{O}(\Sigma);\, q_G)f(\hat{O}(\Sigma');\, q_G)$, where $f(x;q_G)$ is analytic in q_G at $q_G=1$, being generically $f(x;1)\neq 0$.

By denoting now $\eta_{q_G}^{\Sigma\Sigma'}$ the eigenvalues of $\hat{\eta}_{q_G}^{\Sigma\Sigma'}$ we have that $\eta_{q_G}^{\Sigma\Sigma'}$ vanishes if $n_{\Sigma}=0$ or $n_{\Sigma'}=0$; otherwise (i.e., if $n_{\Sigma}\geq 1$ and $n_{\Sigma'}\geq 1$), Eq. (20) implies

$$\begin{aligned} &(q_G-1)\eta_{q_G}^{\Sigma\Sigma'} \\ &= \frac{\left[\sum_{i=1}^{2n_{\Sigma}}\binom{2n_{\Sigma}}{i}(q_G-1)^i\right]\left[\sum_{i'=1}^{2n_{\Sigma'}}\binom{2n_{\Sigma'}}{i'}(q_G-1)^{i'}\right]}{q_G^2-1} \\ &= \frac{q_G-1}{q_G+1}\left[\sum_{j=0}^{2n_{\Sigma}-1}\binom{2n_{\Sigma}}{j+1}(q_G-1)^j\right]\left[\sum_{j'=0}^{2n_{\Sigma}-1}\binom{2n_{\Sigma}}{j'+1}(q_G-1)^j\right] \end{aligned} \tag{21}$$

Consequently, in the $q_G\to 1$ limit, we have

$$\eta_1^{\Sigma\Sigma'} \sim 2n_{\Sigma}n_{\Sigma'} \qquad (n_{\Sigma},n_{\Sigma'}=0,1,2,\cdots) \tag{22}$$

hence, using Eq. (16),

$$q-1 \sim (q_G-1) < \hat{N} > k_B/S_1 \tag{23}$$

where we have used $< \hat{N}_\Sigma \hat{N}_{\Sigma'} >=< \hat{N}_\Sigma >< \hat{N}_{\Sigma'} >$ and $< \hat{N}_\Sigma >=< \hat{N}_{\Sigma'} >\equiv< \hat{N} >$.

By using the well known quantum harmonic oscillator results $< \hat{N} >= (e^{\beta\hbar\omega}-1)^{-1}$ and $S_1/k_B = \beta\hbar\omega e^{\beta\hbar\omega}/(e^{\beta\hbar\omega}-1) - \ln(e^{\beta\hbar\omega}-1)$, we finally obtain

$$q-1 \sim \frac{q_G-1}{\beta\hbar\omega e^{\beta\hbar\omega} - (e^{\beta\hbar\omega}-1)\ln(e^{\beta\hbar\omega}-1)} \tag{24}$$

which is represented in Fig. 2 (where the $k_BT/\hbar\omega \to 0$ and $k_BT/\hbar\omega \to \infty$ asymptotic behaviors are indicated). We remark: (i) A temperature exists ($k_BT^*/\hbar\omega \simeq 2.31$) for which $q-1 \sim q_G-1$, hence q can (asymptotically) equal q_G !; (ii) At very low temperatures (where the system is practically not excited) q_G can vary a lot without making the thermodynamics appreciably nonextensive; (iii) At very high temperatures (where the system is highly excited), the slightest departure of q_G from unity yields a highly nonextensive thermodynamics; (iv) It seems plausible that, for any finite temperature, q monotonically increases from zero to infinity when q_G increases from zero to infinity. The Plastino and Plastino's discussion [14] of the $q=1$ paradox of the polytropic model for stellar systems was done in the classical limit ($\hbar \to 0$), where it seems now reasonable to expect that a value of q_G *slightly different from unity* would imply in a value of q *quite different from unity*, as they indeed found. Since it seems possible to interpret $q_G > 1$ as a discontinuous space-time [19], we should certainly not exclude the possibility for astrophysical systems being the right candidates for exploring the deepest effects of gravitation in Nature, *including nonextensivity of the entropy.*

Let us now synthetize the present work. Although the connection between Generalized Statistical Mechanics and Quantum Groups was done on effective grounds (i.e., the relation between q and q_G depends on temperature for a system in thermal equilibrium), it was established through a remarkably simple and generic assumption, namely that *nonextensive statistics can* exactly compensate *nonextensive* mechanics in such a way as to provide *extensive mean values of the observables.* This fact might help for the understanding of one among the most puzzling problems of contemporary science, namely the deep nature of space-time.

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Caption for figures

Fig. 1 - Typical (finite temperature) relation between q and q_G. By "Extensive Physics" we mean Shannon entropy, Boltzmann-Gibbs statistics, continuous space-time, differential-equations physics, etc.

Fig. 2 - Temperature dependent relation between q and q_G in the region $q \simeq q_G \simeq 1$ for bosonic oscillators ($k_B T^*/\hbar\omega \simeq 2.31$). The asymptotic behaviors for $T << T^*$ and $T >> T^*$ are respectively given by $k_B T/\hbar\omega$ and $(k_B T/\hbar\omega)/\ln(k_B T/\hbar\omega)$.

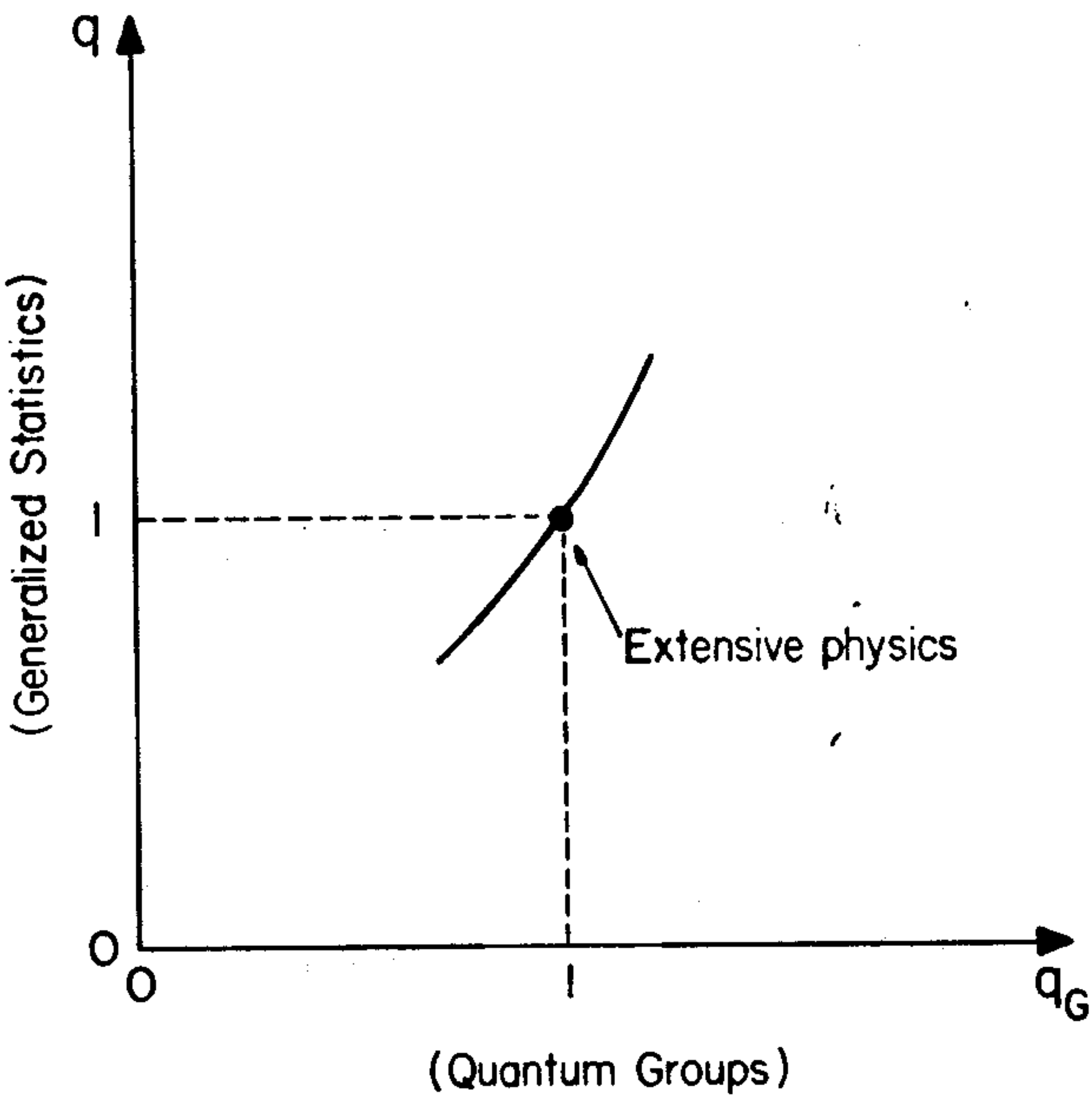
q
(Generalized Statistics)
1
0
0
1
q_G
Extensive physics
(Quantum Groups)

Fig. 1

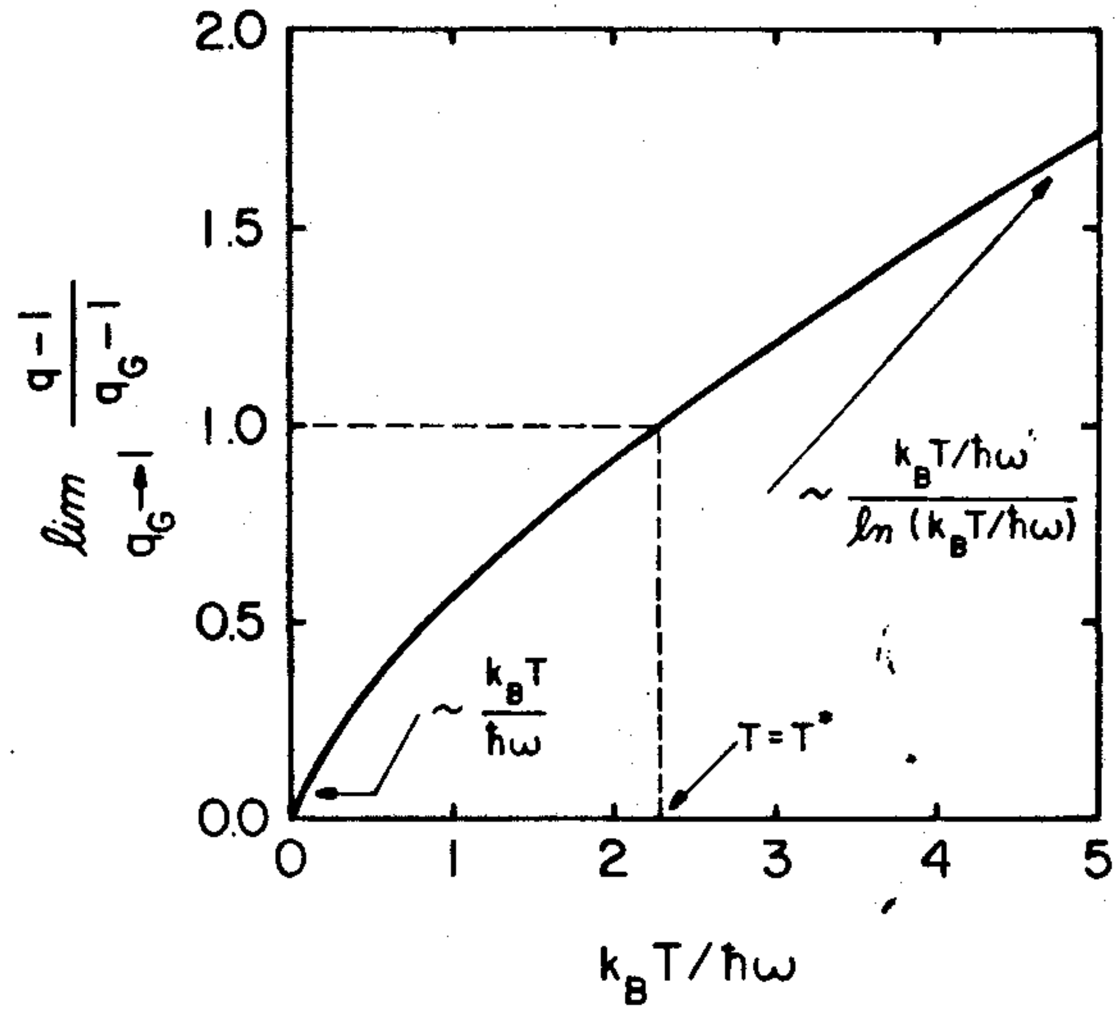
2.0
1.5
1.0
0.5
0.0
0
1
2
3
4
5
$\lim_{q_G \to 1} \frac{q-1}{q_G-1}$
$k_B T/\hbar\omega$
$\sim \frac{k_B T}{\hbar\omega}$
$T = T^*$
$\sim \frac{k_B T/\hbar\omega}{\ln(k_B T/\hbar\omega)}$

Fig. 2

References

[1] C. Tsallis, J. Stat. Phys. A **52**, 479 (1988).

[2] A.M. Mariz, Phys. Lett. A **165**, 409 (1992).

[3] J.D. Ramshaw, Phys. Lett. A **175**, 169 (1993).

[4] J.D. Ramshaw, Phys. Lett. A **175**, 171 (1993).

[5] E.M.F. Curado and C. Tsallis, J. Phys. A **24**, L 69 (1991); Corrigenda: **24**, 3187 (1991) and **25**, 1019 (1992).

[6] A.R. Plastino and A. Plastino, Phys. Lett. A **177**, 177 (1993).

[7] A.R. Plastino and A. Plastino, "Dynamical Aspects of Tsallis Entropy", Physica A (1993), in press.

[8] E.P. da Silva, C. Tsallis and E.M.F. Curado, Physica A **199**, 137 (1993).

[9] A. Chame and E.V.L. de Mello, "The fluctuation-dissipation theorem in the framework of the Tsallis statistics", preprint (1993).

[10] A. Plastino and C. Tsallis, J. Phys. A **26**, L893 (1993).

[11] D.A. Stariolo, "The Langevin and Fokker-Planck Equations in the Framework of a Generalized Statistical Mechanics", preprint (1993).

[12] E.F. Sarmento, "Generalization of Single-site Callen's Identity Within Tsallis Statistics", preprint (1993).

[13] F. Buyukkiliç and D. Demirhan, Phys. Lett. A **181**, 24 (1993).

[14] A.R. Plastino and A. Plastino, Phys. Lett. A **174**, 384 (1993).

[15] R. Balian, "From Microphysics to Macrophysics", Vol. 1, page 134 (Springer-Verlag, Berlin, 1991); A.M. Salzberg, J. Math. Phys. **6**, 158 (1965); H.E. Kandrup, Phys. Rev. A **40**, 7265 (1989); L. G. Taff, "Celestial Mechanics" (J. Wiley, New York, 1985) p. 437; W.C. Saslaw, "Gravitational Physics of Stellar and Galactic Systems" (Cambridge University Press, Cambridge, 1985) p. 217; J. Binney and S. Tremaine, "Galactic dynamics" (Princeton University Press, Princeton, 1987) p. 268; P.T. Landsberg, J. Stat. Phys. **35**, 159 (1984).

[16] P. Alemany and D.H. Zanette, "Fractal random walks from a variational formalism for Tsallis entropies", Phys. Rev. E/Rapid Comm. (1993), in press.

[17] E.W. Montroll and M.F. Shlesinger, J. Stat. Phys. **32**, 209 (1983); E.W. Montroll and B.J. West, "Fluctuation Phenomena" (Elsevier, Amsterdam, 1987), p. 186.

[18] E. Witten, Nucl. Phys. B **330**, 285 (1990).

[19] A. Dimakis and F. Müller-Hoissen, Phys. Lett. B **295**, 242 (1992).

[20] P. Aschieri and L. Castellani, Int. J. Mod. Phys. A **8**, 1667 1993).

[21] R. Caracciolo and M.R-Monteiro, Phys. Lett. B **308**, 58 (1993).

[22] A Lerda and S. Sciuto, Nucl. Phys. B (1993), in press.

[23] M.R-Monteiro and I. Roditi, "Thermo Field Dynamics of Deformed Systems", (1993) Mod. Phys. Lett. B, in press.