CBPF-NF-049-81

CURVATURE TENSOR COPIES IN AFFINE GEOMETRY

by

Prem P. Srivastava

CENTRO BRASILEIRO DE PESQUISAS FÍSICAS - CBPF/CNPq
AV. WENCESLAU BRAZ, 71, FUNDOS
22290 - Rio de Janeiro - RJ - BRASIL

ABSTRACT

The sets of space-time and spin-connections which give rise to the same curvature tensor are constructed. The corresponding geometries are compared. We illustrate our results by an explicit calculation and comment on the copies in Einstein-Cartan and Weyl-Cartan geometries.

- December, 1981.

Gauge equivalent potentials (gauge copies) are very useful in certain problems. In Wu and Yang formulation [1] of Dirac monopole theory each monopole divides the space in two overlapping sections. Regular potentials in each of the sections can be defined such that in the intersection region the two are gauge equivalent. Another example is the calculation [2] of topological number in a simple way of the multi-instanton solution in Yang-Mills theory using 't Hooft's solution [3] together with another gauge equivalent compact form [2]. The problem of the existence, in non--abelian gauge theory, of two or more potentials not related by any gauge transformations associated to the same field strength has been discussed recently in several papers [1] [4]. Here we will discuss the analogous 'field strength copies' or rather curvature tensor copies and gauge copies in the context of an affine manifold. The case of Weyl geometry can also be treated in our general frame work.

A local frame is specified by a set of tetrad fields $e^{\ell}_{\mu}(x)$ where Greek indices label the components of world vector while Latin indices indicate those in local tangent space. The tangent space group is taken to be the Lorentz group. The covariant derivative $e^{\ell}_{\mu;\lambda}$ is written as

$$e^{\ell}_{\mu;\lambda} = \partial_{\lambda} e^{\ell}_{\mu} - \Gamma^{\alpha}_{\mu\lambda} e^{\ell}_{\alpha} + \omega^{\ell}_{\lambda\ m} e^{m}_{\mu} \tag{1}$$

Here $\Gamma^{\alpha}_{\mu\lambda}$ are space-time connections while $\omega_{\lambda} = (\omega^{\ell}_{\lambda m})$ are spin connections. The constant metric in tangent space is $\eta_{\ell m}$ while $g_{\mu\nu}(x)$ indicates the metric in space time. Under the local Lorentz rotations $\Lambda = (\Lambda^{\ell}_{m})$ of the tetrad frame

$$\omega_\mu \to \Lambda\omega_\mu\Lambda^{-1} - (\partial_\mu\Lambda)\Lambda^{-1} \tag{2}$$

The corresponding field strength defined by

$$P_{\lambda\rho}(\omega) = \partial_\lambda\omega_\rho - \partial_\rho\omega_\lambda + [\omega_\lambda, \omega_\rho] \tag{3}$$

is gauge covariant, e.g.,

$$P_{\lambda\rho} \to \Lambda P_{\lambda\rho}\Lambda^{-1}.$$

The space-time curvature tensor is given by

$$R^\mu{}_{\nu\lambda\rho}(\Gamma) = \partial_\lambda\Gamma^\mu{}_{\nu\rho} + \Gamma^\mu{}_{\beta\lambda}\,\Gamma^\beta{}_{\nu\rho} - (\lambda \leftrightarrow \rho) \tag{4}$$

while $(P_{\lambda\rho}(\omega))^\ell{}_m = R^\ell{}_{m\lambda\rho}(\omega)$ defines the spin curvature tensor. They are connected by the following equation

$$\begin{aligned} &e^\ell_{\mu;\lambda\rho}(\Gamma,\omega) - e^\ell_{\mu;\rho\lambda}(\Gamma,\omega) + (\Gamma^\alpha_{\lambda\rho} - \Gamma^\alpha_{\rho\lambda})e^\ell_{\mu;\alpha}(\Gamma,\omega) \\ &= R^\alpha{}_{\mu\lambda\rho}(\Gamma)e^\ell_\alpha - R^\ell{}_{m\lambda\rho}(\omega)e^m_\mu \end{aligned} \tag{5}$$

If we impose $e^\ell_{\mu;\lambda}(\Gamma,\omega)=0$ we obtain

$$R^\alpha{}_{\mu\lambda\rho}(\Gamma) = R^\ell{}_{m\lambda\rho}(\omega)e^m_\mu e^\alpha_\ell \tag{6}$$

Consider now, for simplicity, the following tetrad condition

$$e^\ell_{\mu;\lambda}(\bar{\Gamma},\omega) = -K^\ell_{\lambda m}e^m_\mu \tag{7}$$

which is equivalent to $e^\ell_{\mu;\lambda}(\bar{\Gamma},\bar{\omega}) = 0$ where $\bar{\omega}_\lambda = (\omega+K)_\lambda$. Applying eq.(5) to this case we establish that if K_λ satisfies the equation

$$P_{\lambda\rho}(K)+[\omega_\lambda,K_\rho]-[\omega_\rho,K_\lambda]=0 \tag{8}$$

the following relation is satisfied

$$R^\alpha{}_{\mu\lambda\rho}(\bar{\Gamma})=R^\ell{}_{m\lambda\rho}(\omega)e^m_\mu e^\alpha_\ell \tag{9}$$

We note that eq.(8) is equivalent to the relation $P_{\lambda\rho}(\bar{\omega})=P_{\lambda\rho}(\omega)$. and does not depend explicitly on $\bar{\Gamma}$. The expression for $\bar{\Gamma}$ is easily shown to be

$$\bar{\Gamma}^\alpha{}_{\mu\lambda}= \Gamma^\alpha_{\mu\lambda}+L^\alpha_{\mu\lambda} \;;\; L^\alpha_{\mu\lambda}= K^\ell_{\lambda m}e^m_\mu e^\alpha_\ell \tag{10}$$

The sets of connections $(\bar{\Gamma},\bar{\omega})$ and (Γ,ω) where K_λ satisfies eq. (8) give rise to the same curvature tensors and correspond to 'field strength copies' if $\bar{\omega}_\lambda$ and ω_λ are not connected by a Lorentz gauge transformation. The condition for $\bar{\omega}_\lambda$ to be a gauge transform of ω_λ requires the existence of a lorentz transformation Λ such that

$$K_\lambda= \Lambda\omega_\lambda\Lambda^{-1}-\omega_\lambda-(\partial_\lambda\Lambda)\Lambda^{-1} \tag{11}$$

The eq. (8) then gives expected result, $\Lambda P_{\lambda\rho}(\omega)=P_{\lambda\rho}(\omega)\Lambda$. It is worth remarking that the pure gauge term in eq. (11) is antisymmetric in local indices and we note

$$\eta_{\ell m;\lambda}(\bar{\omega})= -2[\omega_\lambda+K_\lambda]_{(\ell m)}$$

$$g_{\mu\nu;\lambda}(\bar{\Gamma}) = -2\left[\omega_\lambda + K_\lambda\right]_{(\ell m)} e^{\ell}_{\mu} e^{\mu}_{\nu} \tag{12}$$

It is obvious that $K_\lambda = -I(\partial_\lambda \chi(x))$ is a solution of eq. (8). It corresponds to $L^{\alpha}_{\mu\lambda} = -\delta^{\alpha}_{\mu}(\partial_\lambda \chi)$ and

$$g_{\mu\nu;\lambda}(\bar{\Gamma}) = 2g_{\mu\nu}\partial_\lambda \chi - 2\omega_{\lambda(\ell m)} e^{\ell}_{\mu} e^{m}_{\nu}.$$

The Einstein-Cartan geometry is characterized by the metric condition $g_{\mu\nu;\lambda}=0$ implying $\omega_{\lambda(\ell m)}=0$. Thus in this case a copy is obtained in E.C. geometry only if $K_{\lambda(\ell m)}=0$ and the corresponding $\bar{\omega}_\lambda$ is not a gauge transform of ω_λ.

We will illustrate the discusssion above by considering the manifold with the line element given by [5]

$$ds^2 = dt^2 - 2A(t)dzdt - C^2(t)\left[dx^2 + dy^2\right] \tag{13}$$

For definiteness sake we choose $\omega_\lambda = \overset{0}{\omega}_\lambda$, $\Gamma = {}^{g}\Gamma$ (Christoffel affinity) and $\overset{0}{\omega}_\lambda$ is defined from $e^{\ell}_{\mu;\lambda}({}^{g}\Gamma, \overset{0}{\omega})=0$. We verify that $\overset{0}{\omega}_{\lambda(\ell m)}=0$. A simple antisymmetric solution is found to be $K_\lambda = A(\partial_\lambda \chi) \dfrac{\overset{0}{\omega}_1}{\dot{C}}$ and $L^{\mu}_{\nu\lambda} = \left[\frac{A}{C}\,\delta^{\mu}_{1}\delta^{0}_{\nu} - C\delta^{\mu}_{3}\delta^{1}_{\nu}\right](\partial_\lambda \chi)$. The relation $R^{\alpha}_{\mu\lambda\rho}(\bar{\Gamma}) = R^{\alpha}{}_{\mu\lambda\rho}(\Gamma)$ may also be checked directly. This case, however, corresponds to a 'gauge copy' with the corresponding Lorentz matrix given by the non-vanishing elements:

$$\Lambda^{1}{}_{1} = \Lambda^{2}{}_{2} = 1\ ,\quad \Lambda^{0}{}_{0} = (1-\Lambda^{0}{}_{3}),$$

$$\Lambda^{3}{}_{3} = (1+\Lambda^{0}{}_{3}),\quad \Lambda^{1}{}_{0} = \Lambda^{0}{}_{1} = -A\chi$$

and $2\Lambda^3{}_0 = -2\Lambda^0{}_3 = (\Lambda^1{}_0)^2$. A symmetric solution is also easily constructed with $K_\lambda = A^2(\partial_\lambda \chi)M$ where M is a constant matrix with nonvanishing elements given by $M^0{}_0 = M^3{}_0 = -M^0{}_3 = -M^3{}_3 = -1$. In view of eq.(11) and the antisymmetric of $\overset{0}{\omega}$ the spin connection $\bar{\omega}$ is not a gauge transform of $\overset{0}{\omega}$. We also find $L^\mu_{\nu\lambda} = A(\partial_\lambda \chi)\delta^\mu{}_3\delta^0{}_\nu$ and $g_{\mu\nu;\lambda}(\bar{\Gamma}) = 2A^2(\partial_\lambda\chi)\delta^0_\mu\delta^0_\nu$ and verify directly that $R^\alpha{}_{\mu\lambda\rho}(\bar{\Gamma}) = R^\alpha{}_{\mu\lambda\rho}(\Gamma)$.

The case of Weyl (-Cartan) geometry may be discusses along similar lines. It is characterized by $g_{\mu\nu;\lambda} = 2g_{\mu\nu}\phi_\lambda$ where ϕ_λ is Weyl field and $\omega_{\lambda(\ell m)} \neq 0$. We may, for example, take $\omega_\lambda = \overset{0}{\omega}_\lambda - I\phi_\lambda$ and $\Gamma^\lambda_{\mu\nu} = {}^g\Gamma^\lambda_{\mu\nu} - \delta^\lambda_\mu\phi_\nu$ and repeat the procedure discussed above.

ACKNOWLEGMENTS

The author acknowledges with thanks conversations with Professors J.Tiomno, M.Novello, C.G. de Oliveira and I.Damião Soares.

REFERENCES

[1] T.T. Wu and C.N.Yang, Phys. Rev. D13(1976) 437; Phys. Rev. D12(1975) 3845.

[2] P.P.Srivastava, Phys. Rev. D17(1978) 1613.

[3] G.'t Hooft, unpublished. See also R. Jackiw, C.Nohl and C.Rebbi, Phys. Rev. D15(1977) 1642.

[4] S.Deser and F.Wilezek, Phys. Lett. 65B(1976) 391;
S.Roskies, Phys. Rev. D15(1977) 1731;
M.Calvo, Phys. Rev. D15(1977) 1733;
M.B.Halpern, Phys. Rev. D16(1977) 1798;
D19(1979) 517;
C.G.Bollini, J.J.Giambiagi and J.Tiomno, Phys. Lett 83B(1979) 185.

[5] M.Novello and I.Damião Soares, Phys. Lett. 56A(1976) 431.