Some Comments on Wheeler De Witt Equation for Gravitational Collapse and the Problem of Time

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We write the Hamiltonain for a gravitational spherically symmetric scalar field collapse with massive scalar field source, and we discuss the application of Wheeler De Witt equation as well as the appearence of time in this context. Using an Ansatz for Wheeler De Witt equation, solutions are discussed including the appearence of time evolution.

Key-words: Gravitation; Time; Collapse.

Introduction

In this letter we discuss the problem of gravitational collapse of a star using the Wheeler-De Witt equation.

In accordance with [2] we assume a scalar field, ϕ, with a mass term and we assume that the super hamiltonian has a constraint [1-5] such that $H \simeq 0$. Ordering of operators is assumed.

A particular ansatz for the functional is chosen to show qualitatively the appearance of the notion or concept of “time” after quantization.

As in the case of the hydrogen atom the discrete index is identified with an “internal time” just as in any relativistic field theory or general relativity but different from the usual quantum mechanics, where “time” appears as a Galilean time.

We apply the Wheeler-De Witt equation for a special collapse condition despite the fact that the question related to the Copenhagen interpretation for product of functional $\psi(\Lambda, R, \phi)$ is not understood.

Let us begin by writing the super Hamiltonian for a gravitational spherically symmetric scalar field collapse with massive scalar field source such as [2].

$$H = \mathcal{H} + \frac{1}{2}\, m^2 R^2 \Lambda \phi^2 \ , \tag{1}$$

where

$$\begin{aligned} \mathcal{H} &= -R^{-1} P_R P_\Lambda + \frac{1}{2}\, R^{-2} \Lambda P_\Lambda^2 + \Lambda^{-1} R R'' - \Lambda^{-2} R R' \Lambda' + \frac{1}{2}\, \Lambda^{-1} R'^2 + \\ &- \frac{1}{2}\, \Lambda + \frac{1}{2}\, R^{-2} \Lambda^{-1} P_\phi^2 + \frac{1}{2}\, R^2 \Lambda^{-1} \phi'^2 \ . \end{aligned} \tag{2}$$

In the expression above P_R, P_Λ, P_ϕ imply respectively conjugate momenta associated with R, Λ and ϕ variables.

Furthermore $R = R(r,t)$, $\Lambda = \Lambda(r,t)$, $\phi = \phi(r,t)$. We define conjugate momentum as

$$\pi_x = -i\,\frac{\partial}{\partial x} \tag{3}$$

where $\underline{x}$ means R, Λ or ϕ variable.

It is a known fact that using the Hamiltonain (2) some operator ordering problems appear [1, 2].

A simple form to represent the ambiguous order of factors $\left(x\,,\ \frac{\partial}{\partial x}\right)$ and $\left(y\,,\ \frac{\partial}{\partial y}\right)$ is given by [1]. Applying such an ordering for operators in (2) we can find the following squared conjugate momenta

$$\begin{aligned} \pi_x^2 &= -\frac{\partial^2}{\partial x^2} - \frac{p}{x}\,\frac{\partial}{\partial x} \\ \pi_y^2 &= -\frac{\partial^2}{\partial y^2} - \frac{q}{y}\,\frac{\partial}{\partial y} \end{aligned} \tag{4}$$

where (p, q) are c-numbers.

It is assumed that the Hamiltonian (2) is a constraint for a classical Hamiltonain with the mass term present for the scalar field ϕ. In other words, the canonical quantization needs the annihilation of the wave function ψ by the corresponding quantum operator

$$\hat{H}\psi = 0 \tag{5}$$

that results in the Wheeler-De Witt equation. Using eq. (2-5) we get

$$\frac{\Lambda}{2R^2}\left(\frac{\partial^2\psi}{\partial\Lambda^2} + \frac{p}{\Lambda}\,\frac{\partial\psi}{\partial\Lambda}\right) + \frac{1}{2R^2\Lambda}\left(\frac{\partial^2\Lambda}{\partial\phi^2} + \frac{q}{\phi}\frac{\partial\psi}{\partial\phi}\right) - \frac{1}{R}\,\frac{\partial^2\psi}{\partial R\partial\Lambda} \equiv V\psi \tag{6}$$

where ψ is a functional of Λ, ϕ and R functions, and V is a potential term written as

$$V = \frac{R}{\Lambda}\,R'' - \frac{R}{\Lambda^2}\,R'\Lambda' + \frac{1}{2\Lambda}\,{R'}^2 - \frac{1}{2}\,\Lambda + \frac{1}{2}\,\frac{R^2}{\Lambda}\,{\phi'}^2 + \frac{1}{2}\,m^2R^2\Lambda\phi^2 \tag{7}$$

The prime means derivative with respect to the coordinate $\underline{r}$. Observe that in equation (6) we don't have any derivative with respect to time. This means that the equation (6) could be describing a spherically symmetric gravitational collapse but without any explicit time dependence for functional ψ. The concept of "time" in this case may appear only after quantization in accordance with [3].

This suggests that eq. (6) is like the usual Schrödinger equation of quantum mechanics applied to gravitational collapse but with a difference depending on the operator ordering [1-5].

The usual Schrödinger equation is written as

$$H\psi = i\frac{\partial\psi}{\partial t} \tag{8}$$

where H means the Hamiltonian of the system. It means that the wave function of the system has an important difference with equation (6) besides the fact that ψ in (8) to be a function while ψ in (6) being a functional $\psi(\Lambda, \phi, R)$. The parameter "time" $\underline{t}$ in (8) is a universal time-"external time" in the sence of Galili-Newton time, while in equation (6) "time" is an internal parameter. In some sense there is no "time" with which we could describe the evolution of gravitational collapse of the star for exemplo. Thus, in principle we might apply the equation for a static case such as Schwarszchild solution but not for a dynamic case where the functions R, Λ, ϕ might be time dependent. In other words, one can apply Wheeler-De Witt equation (6) for static Schwarszchild case where $R = R(r)$, $\Lambda = \Lambda(r)$ and $\phi = \phi(r)$ but shall we apply the same equation for the general case, with $R = R(r,t)$, $\Lambda = \Lambda(r,t)$ and $\phi = \phi(r,t)$?

How does the conception of "time" appear in this case?

How can we get the notion of evolution in time of a collapsing star using equation (6) without explicit time dependence of the functional ψ?

The equation (8) can be applied for steady systems such as hydrogen atom where the right side is zero and we have

$$\hat{H}\psi = E\psi = 0 \tag{9}$$

where E is the energy. In the particular case of $E = 0$ this equation has a strong resemblance to the Wheeler-De Witt equation.

It is a well known fact that stationary solution can be find from equation (9) in terms of $R(r)$, $\Theta(\theta)$, $\phi(\varphi)$ with R, the radial solution and $\Theta(\theta)\phi(\varphi) = Y(\theta, \varphi)$ being the spherical harmonics. The obvious similarity of eq. (9) and eq. (5) leads us to think that eq. (6) can be solved in the general case, with an "internal time" and the idea of "evolution" being identified with some discrete index $i = 1, 2, 3 \cdots$. after solving eq. (6).

We know that there are many different $\psi_{k\ell m}(r, \theta, \varphi)$ for different values of k, ℓ, m for the hydrogen atom and in some sense "the evolution of the system" can be seen as a changing of wave function for a stationary situation. There is no "external time" in eq. (8) for the hydrogen atom.

In the same way we can think of applying in eq. (6) with an "internal time" or without an external time any way and to obtain the functional $\psi(\Lambda, \phi, R)$.

We may take an appropriate ansatz for the eq. (6) and to verify if it really does satisfy eq. (6). But immediately two questions can be raised.

First, which ansatz? There are an inifinite number of possibilities.

Second, the introduction of a mass term in (1) for scalar field ϕ can break the "constraint" character for H and eq. (5) may not be valid anymore. We must remind that we are assuming the presence of mass of the scalar field and it does not break the constraint of super Hamiltonian as in [2].

In general the Wheeler-De Witt equation can be separated depending on the potential term (7). The role of $V\,(R, R', R'', \Lambda, \Lambda', \phi, \phi', m)$ is similar to the coordinates system for decoupling of the Schrödinger equation. It is a known fact that the Schrödinger Equation can be separated in several coordinates systems. In the same manner eq. (6) may decouple for $\psi(R, \Lambda, \phi)$ depending on the potential term and the particular choice of the *ansatz* for the ψ functional. But eq. (6-9) is too complicated and again there is no derivative in "time".

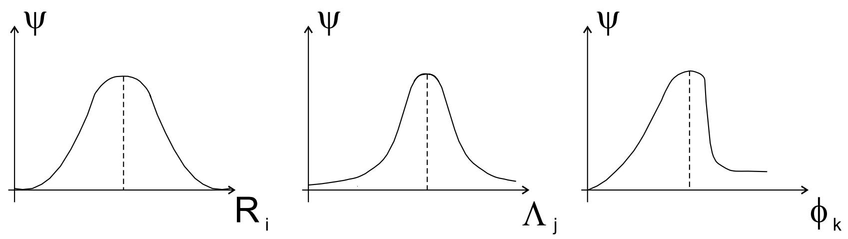
Qualitatively the problem can be solved in the following way. Suppose that ψ functional reads as

$$\psi\,(\Lambda, \phi, R) = \Lambda(r + c)\sqrt{\phi(r + c)}\ R(r + c) \qquad (10)$$

where Λ, R, ϕ are functions of $\underline{r}$ only since there is no"external time" as in eq. (8) or an "internal time" as in general Relativity theory or in the relativistic Klein Gordon equation.

In eq. (9) $\underline{c}$ is a constant that can be identified with "time" after quantization.

A class of solutions such as is shown below may be found



In reality we can find a sequence of $R_i(r)$, $\Lambda_j(r)$ and $\phi_k(r)$ where $i, j, k = 1, 2, 3 \cdots$ the concept of "time" being identified with $i, j, k \sim t$ (Time).

In the Schrödinger equation for the hydrogen atom the wave function $\psi_{k\ell m}(r, t)$ can be written as a product of $R_{k\ell}(r), \Theta_{\ell m}(\theta)$ and $\phi_m(\varphi)$ for stationary states and one may see a notion of "evolution" through the different configurations is possible given by different values of k, ℓ, m.

In our case the same idea can be utilised by identifying with a discrete index $(i = 1, 2, 3 \cdots)$ as the "time" where i, j, k are the different functions that contribute to our functional ψ.

Finally, we need to be clear that eq. (6) has a infinite number of solutions with the proposal given by eq. (10) being one of them.

The Wheeler-De Witt equation itself has many different possibilities depending of the operator ordering [1, 2, 3, 4, 5]. Then, in principle one can write different mathematics (different Wheeler-De Witt equations) and each one of them with infinite number of ansatz. Each possibility is given us a notion of "Time" after quantization.

The natural question that we can put is:

Shall we find the same "physics" for different Wheeler-De Witt equations?

Can we find the same notion of "time" from different Wheeler-De Witt equation with infinite possibilities of the ansatz ?

The physical "time" is the same for each possibility or do we have many times in physics as in [6]?

Admitting that our equation (6) has some meaning and that the ansatz eq. (10) can provide us with a notion of "time" arises from the discretization of the index $i, j, k \sim t$. The next question we need to resolve is: if eq. (6) implies the Schrödinger equation for a global Universe in general and in our particular case it is a Schrödinger equation for a gravitational collpse of a body like a star how can we improve the Copenhagen interpretation for the functional $\psi(\Lambda, \phi, R)$?

Maybe the answer can be found as in eq. (6) and the ansatz given by eq. (10) describing the possibility of finding the star between $\underline{m}$ and $\underline{m + dm}$ mass states.

But if so, can it be supported by the condition $m \neq 0$ for the scalar field ϕ in (1)?

Should the superhamiltonian be a real constraint $H \sim 0$ on that condition?

In any case we need to understand the real meaning of operator ordering in quantum mechanics as well as the meaning of time in all of physics. While we don't know the final answer for these open questions there have been uncertain consequences for a complete understanding of physics and our interpretation for the world.

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