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# "$q$-Deformation and Energy Deficit in Liquid Helium Phonon Spectrum"

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## Abstract

The instability of the phonon spectrum in liquid Helium for $T < 1\ K$ is a well established experimental fact. We discuss the role of $q$-deformation as a possible mechanism to supply the energy deficit that forbiddens one-phonon decay into two phonons when the constant $\gamma$ in the phonon anomalous dispersion relation ($\omega^{ph} = c_0 p(1 - \gamma p^2)$) is positive, through the analysis of three-phonon processes in a $q$-phonons gas.

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The superfluid properties of $^4He$, first discussed by Kapitza [1], appear below the lambda-point, $T_\lambda = 2.18\ K$ [2]. In Landau theory [3], those properties follow from phonon and roton excitations [4]. Even for very low temperatures, $T \leq 0.6\ K$, where phonons dominate, they present an anomalous dispersion given

$$\omega^{ph} = c_0 p(1 - \gamma p^2)\ ; \tag{1}$$

$c_0$ is the sound velocity.

The non-linearity of (1) makes impossible the simultaneous conservation of energy and momentum in three-phonon processes. The sign of the constant $\gamma$ is crucial: if $\gamma > 0$, then one phonon of energy $\omega_1$ and momentum $p_1$,

$$\omega_1 = c_0 p_1(1 - \gamma p_1^2)\ , \tag{2}$$

cannot decay into two phonons of energies $\omega_2$, $\omega_3$ and momenta $p_2, p_3$, where

$$\vec{p}_1 = \vec{p}_2 + \vec{p}_3 \tag{3}$$

because

$$\omega_1(p_1) < \omega_2(p_2) + \omega_3(p_3)\ , \tag{4}$$

with an energy deficit

$$\omega_2(p_2) + \omega_3(p_3) - w_1(p_1) = 3\gamma p_2^2 p_3 + 3\gamma p_2 p_3^2\ . \tag{5}$$

Therefore, the positivity of $\gamma$ implies that the phonon spectrum is stable [2]. Nevertheless, the unstable character of the phonon spectrum in $^4He$ is a well established experimental fact for $T < 1\ K$. The occurence of one-phonon decay two into phonons is supported by experimental data of phonon lifetime in scattering of neutrons [5]; also, a negative $\gamma$ is obtained (for most values of the pressure) in $^4He$ specific heat measurements [6, 7]. In spite of that, there are unsolved discrepancies concerning the unstable character of phonon spectrum in $^4He$. From high-resolution neutron scattering experiments [8], $\gamma$ was deduced to be positive for the observed excitation spectrum for liquid $^4He$ below $T_\lambda$. This result agrees with the indirect determination of $\gamma$ from $X$-ray scatering experiments [9]. As the temperature at which the experiments were carried on are higher than $1\ K$, $T = 1.1\ K$, this discrepancy suggests that the sign of $\gamma$ may change according to the temperature [2].

There is also disagreement between theory and experiment. The dispersion relation (1) was theoretically derived from a hydrodynamic Hamiltonian and through a self-energy calculation using the lowest order perturbation theory, the constant $\gamma$ was estimated to be positive [10].

The facts above can be taken as an indication that for very low temperatures ($T \leq 0.6\ K$), where phonon excitations are dominant, there must appear some mechanism that makes the phonon spectrum unstable, even if $\gamma > 0$, as theoretically predicted [10] and experimentally determined for $T > 1\ K$ [8, 9].

In recent paper it was shown that these discrepancies concerning the stability of the phonon spectrum can be solved when the phonon excitations are treated as an ideal deformed bosonic gas [11]. In this note we discuss the mechanism behind the role of $q$-deformation in solving the question of the sign of $\gamma$ and its relation to the existence of three phonon processes in liquid Helium at very low temperatures.

$q$-Deformed Heisenberg algebras are non-trivial generalizations of the Heisenberg algebra through the introduction of deformation parameters [12]. $q$-Oscillators [13, 14] are objects that satisfy the deformed Heisenberg algebras; with them one constructs a deformed ideal bosonic $q$-gas that generalizes the usual ideal boson gas. Their statistical properties have been studied both in the $q \simeq 1$ approximation [15] and in the highly deformed region [16], in the so called "fundamental representation" [17]. More recently, it was shown that there are more general representations where a second parameter, $\nu_0$, is introduced [18]. Some consequences of using these inequivalent representations and their possible connection with superfluidity have been discussed [19].

A bosonic $q$-oscillator [13, 14] is the associated algebra generated by elements $a, a^+$ and $N$ satisfying the relations

$$\begin{aligned} &[N, a^+] = a^+ \quad , \quad [N, a] = -a \\ &aa^+ - qa^+a = q^{-N} \ , \end{aligned} \tag{6}$$

where the deformation parameter, $q \in I\!R^+$. For $q > 1$, denoting the normalized basis vector by $|n>$, the inequivalent $\nu_0$ representations [17] are given by:

$$\begin{aligned} a^+|n\rangle &= q^{\nu_0/2}[n+1]^{1/2}|n+1\rangle \\ a|n\rangle &= q^{\nu_0/2}[n]|n-1\rangle \\ N|n\rangle &= (\nu_0 + n)|n\rangle \ , \end{aligned} \tag{7}$$

where $[n] = \dfrac{q^n - q^{-n}}{q - q^{-1}}$. $\nu_0$ is a real free parameter that goes to zero when $q \to 1$ and is interpreted as a "background effect" [17].

We consider an ideal $q$-gas in the representation (7), described by the Hamiltonian

$$H = \sum_i \omega_i a_i^+ a_i = \sum_i \omega_i([N_i] - q^{N_i} C_i) \ , \tag{8}$$

where $a_i$ and $a_i^+$ are interpreted as annihilation and creation operators of particles in levels $i$ with energy $\omega_i$ and $N_i$ is an operator that can be interpreted as the number operator of particles in levels $i$ when $\nu_0 = 0$. $a_i$, $a_i^+$ and $N_i$ satisfy algebra (6) and commute for different levels. $C_i$ is a Casimir operator,

$$C_i = q^{N_i}([N_i] - a_i^+ a_i) \tag{9}$$

and in representation (7) one has

$$C_i|n_i\rangle = q^{\nu_0^i}[\nu_0^i]|n\rangle \ . \tag{10}$$

Let us now suppose that the phonons in $^4He$ are described by relations (8-10). Using (7), we have

$$H|n\rangle = \sum_i \omega_i q^{\nu_0^i}[n]|n\rangle \tag{11}$$

and the $q$-phonon energy in level $i$ is

$$\omega_i^{ph} = \omega_i q^{\nu_0^i}[n_i] \ . \tag{12}$$

We take $\nu_0^i$ different for each level and in the continuum limit $\nu_0^i \to \nu_0(p)$. $\omega_i$ is the usual phonon energy such that in the continuum limit we have the anomalous dispersion relation (1). Then, the energy-momentum relation

$$\omega^{ph}(p) = q^{\nu_0(p)} c_0 p(1 - \gamma p^2) \tag{13}$$

can be interpreted as the effective energy of one $q$-phonon in $^4He$. Therefore deformed phonon presents a modified anomalous dispersion relation and it is important to remark that the background parameter $\nu_0$ is the origin of this modification [11].

Let us now discuss a three-phonon process. We take two phonons with energies $\omega_1(p_1)$ and $\omega_2(p_2)$, respectively. In order to have simultaneous conservation of energy and momentum, we must have

$$\omega_1(p_1) + \omega_2(p_2) = \omega_3(p_1 + p_2) \ , \tag{14}$$

that is,

$$q^{\nu_0(p_1)}c_0p_1(1-\gamma p_1^2)+q^{\nu_0(p_2)}c_0p_2(1-\gamma p_2^2)=$$
$$q^{\nu_0(p_1+p_2)}c_0p_1p_2(1-\gamma(p_1+p_2)^2)\ . \tag{15}$$

First, relation (15) can only be satisfied if $f(p)=q^{\nu_0(p)}c_0p(1-\gamma p^2)$ is a linear function of the momentum. This only happens in two cases: a) $q^{\nu_0(p)}$ is a constant and $\gamma=0$; b) $\nu(p)\propto\ln(1-\gamma p^2)^{-1}$. In the first case, we have the usual linear phonon dispersion relation, which is ruled out by experiment [6–9]. The second possibility means that $q$-deformation can lead to simultaneous energy and momentum conservation by rendering *the q-phonon spectrum non-anomalous*, even if $\gamma\neq 0$.

On the other side, an appropriate choice of the factor $q^{\nu_0(p)}$ in (13) can make the coefficient of the $p^3$ term negative even if $\gamma$ is kept positive. For example, if

$$\nu_0(p)=\frac{\ln(1+\lambda p^2)}{\ln q}\ , \tag{16}$$

the dispersion relation (13) will become:

$$\omega^{ph}(p)=\frac{c_0p}{\ln q}(1-(\gamma-\lambda)p^2-\gamma\lambda p^4)\ . \tag{17}$$

Inspection of relation (17) shows that if $\gamma-\lambda<0$, *q-deformation has the consequence of giving the phonon the necessary energy for its spectrum to be unstable*, independently of the sign of $\gamma$.

Of course, relation (16) is not the only possible choice. Indeed, it has recently been shown that choosing $q^{\nu_0(p)}=e^{\delta^2p^2}$ with $\delta^2=\dfrac{\ln q}{2m_{^4He}k_BT_\lambda}$, it is possible to reproduce, within less than 5% of discrepancy, the experimental values of $^4He$ molar specific heat [7] in the temperature range $0.14\leq T\leq 0.86$, obtaining positive values for $\gamma$ and negative coefficients of $p^3$ in the $q$-deformed dispersion relation in all cases [11].

Summing up, the modified dispersion relation that appears when we treat the phonon as an ideal bosonic $q$-gas in a $\nu_0$ representation can solve the energy deficit that forbiddens one-phonon decay into two phonons.

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