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Some Properties of Charge-Conjugated Spinors in D dimensions

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Abstract

Spinors for an arbitrary Minkowski space with signature (t, s) are reassessed in connection with D-dimensional free Dirac action. The possibility of writing down kinetic and mass terms for charge-conjugated spinors is discussed in terms of the number of time-like directions of the space-time. The results found out here are commented in the light of early works on the subject.

Key-words: Spinors; Arbitrary space time dimensions.

Spinors in arbitrary Minkowski-like spaces with signature $(t,\ s)$, $M^{t,s}$, have been carefully investigated in the papers of Refs. [1, 2, 3]. There are several ways of imposing suitable conditions in order to restrict the number of degrees of freedom carried by the spinors, according to the particular space-time one is considering.

Usually, one discusses the kinematical aspects of spinors by dealing with the Dirac equation in the space-time $M^{t,s}$. A collection of interesting results can be found in a very complete paper by Kugo and Townsend [2]. With the requirements that the Dirac equation be derivable from a Lagrangian and that the spinors be described by anticommuting components, we adopt here the viewpoint of discussing the kinematical properties of spinors in D dimensions through a free Dirac-like action, rather than in terms of the free Dirac equation itself. In particular, we find a number of peculiar results regarding the propagation of charge-conjugated spinors, so that Majorana and Majorana-Weyl spinors are directly concerned by our discussion.

We shall now present our analysis and the main results shall be discussed in our general conclusions. The considered metric for the space-time $M^{t,s}$ is the generalized Minkowski one:

$$\eta^{\mu\nu} = \mathrm{diagonal}(\underbrace{+,+,\cdots,+}_{t-\mathrm{times}},\ \underbrace{-,-,\cdots,-}_{s-\mathrm{times}}), \quad \mu,\ \nu = (1,\cdots,D)\ . \tag{1}$$

The Dirac γ-matrices for a space-time of dimension $D = t{+}s$ can be represented by $2^{[D/2]} \times 2^{[D/2]}$ complex matrices (the bracket $[D/2]$ denotes the integral part of $D/2$) that satisfy the Clifford algebra:

$$\{\gamma^\mu, \gamma^\nu\} = 2\eta^{\mu\nu}\mathbb{1}\ , \tag{2}$$

where $\mathbb{1}$ is the $2^{[D/2]} \times 2^{[D/2]}$ identity matrix. The γ's which satisfy $\gamma^{\mu 2} = \mathbb{1}$ and $\gamma^{\mu 2} = -\mathbb{1}$ will be respectively called t-like and s-like. The matrices γ^μ, $\gamma^{\mu\dagger}$, $\gamma^{\mu *}$ and $\gamma^{\mu T}$, up to constant factors, yield equivalent representations of the Clifford algebra; thus, for a given t:

$$\gamma^{\mu\dagger} = -(-1)^t A\gamma^\mu A^{-1}\ , \tag{3.a}$$
$$\gamma^{\mu *} = \eta B\gamma^\mu B^{-1}\ , \tag{3.b}$$
$$\gamma^{\mu T} = -\eta(-1)^t C\gamma^\mu C^{-1}\ ; \tag{3.c}$$

the t-like and s-like γ-matrices can always be chosen Hermitean and anti-Hermitean, respectively. It can be seen, by using the Schur's lemma, that such a choice amounts to taking A, B and C as unitary matrices. The *unitarity* of A, B and C also plays a central role in the construction of a real action for the spinors and their corresponding charge-conjugated partners. The unitary matrix that satisfies (3.a) is given by

$$A = \gamma^1 \cdots \gamma^t\ . \tag{4}$$

As we shall see, the η-factor in (3.b) will have its value (± 1) fixed, for a given t, through the existence, or not, of the action for the charge-conjugated spinor. The factor in (3.c) follows from the consistency with the previous one. So, the matrices A, B and C exhibit the properties [2]:

$$B^T = \xi C A^{-1}\ ,\ \ \xi \text{ being a phase factor,} \tag{5.a}$$

$$
\begin{aligned}
A^{-1} &= (-1)^{t(t-1)/2} A\,, &(5.b)\\
B^T &= \varepsilon B\,, \quad \varepsilon = \pm 1\,, &(5.c)\\
C^T &= \varepsilon \eta^t (-1)^{t(t-1)/2} C\,, &(5.d)
\end{aligned}
$$

$$
\begin{aligned}
A^* &= \eta^t B A B^{-1}\,, &(5.e)\\
A^T &= \eta^t C A^{-1} C^{-1}\,, &(5.f)\\
\xi C &= \eta^t B^T (\xi C)^* B\,. &(5.g)
\end{aligned}
$$

The ε-factor appearing in (5.c), for dimensions $D = (t{+}s)$ and $(D{+}1)$ (with D even), is shown to be given by [2, 4, 5]

$$
\varepsilon = \cos\frac{\pi}{4}(s-t) - \eta \sin\frac{\pi}{4}(s-t) = \pm 1. \tag{6}
$$

Spinors are objects that, under the action of the group $SO_0(t,s)$, transform as

$$
\Psi \to \Psi' = e^{\frac{1}{2}\omega_{\kappa\lambda}\Sigma^{\kappa\lambda}}\Psi\,, \quad \Sigma^{\kappa\lambda} = \tfrac{1}{4}[\gamma^\kappa, \gamma^\lambda]. \tag{7}
$$

A conjugated spinor, $\overline{\Psi}$, can be defined in such a way that the bilinear $\overline{\Psi}\Psi$ be a scalar under the $SO_0(t,s)$ group transformation. Using (3.a), it can be readily seen that, under such a transformation,

$$
\Psi^\dagger A \to \Psi'^\dagger A = \Psi^\dagger A e^{-\frac{1}{2}\omega_{\kappa\lambda}\Sigma^{\kappa\lambda}}. \tag{8}
$$

Therefore, $\overline{\Psi}$ can be identified with $\overline{\Psi} = \Psi^\dagger A$. Using the relation

$$
e^{-\frac{1}{2}\omega_{\kappa\lambda}\Sigma^{\kappa\lambda}}\gamma^\mu e^{\frac{1}{2}\omega_{\rho\sigma}\Sigma^{\rho\sigma}} = \Lambda^\mu{}_\nu \gamma^\nu\,, \tag{9}
$$

where $\Lambda^\mu{}_\nu$ is the transformation matrix of the $SO_0(t,s)$ group for vectors, it can be found that $\overline{\Psi}\gamma^\mu\Psi$ transforms like a vector under $SO_0(t,s)$ transformations.

The free Dirac action, $\mathcal{A}$, is a real functional and a scalar under $SO_0(t,s)$. It must carry information on the propagation of Ψ:

$$
\mathcal{A} = \int d^D x \mathcal{L}(\Psi, \partial_\mu\Psi). \tag{10}
$$

The Dirac Lagrangian, $\mathcal{L}$, up to a "surface term" (ST), has the form

$$
\mathcal{L}(\Psi, \partial_\mu\Psi) = \alpha\overline{\Psi}\gamma^\mu\partial_\mu\Psi + \beta m\overline{\Psi}\Psi\,, \tag{11}
$$

where the parameter m is real and α and β are factors that may assume the values (± 1), $(\pm i)$, depending on the number of time-like directions. The determination of α and β, for a given t, can be achieved through the Hermiticity of $\mathcal{L}$. Using (3.a), and enforcing the Grassmannian character of the spinors, it can be found, up to a ST, that

$$
\mathcal{L}^\dagger = \alpha^*(-1)^{t(t+1)/2}\,\overline{\Psi}\gamma^\mu\partial_\mu\Psi + \beta^*(-1)^{t(t-1)/2}\,m\overline{\Psi}\Psi\,. \tag{12}
$$

In order that $\mathcal{A}$ be real, one must have:

$$\alpha^*(-1)^{t(t+1)/2} = \alpha \; , \quad (13.a)$$
$$\beta^*(-1)^{t(t-1)/2} = \beta \; . \quad (13.b)$$

With the aid of these equations, we can build up Table 1, where the values of α and β are displayed for different numbers of time-like directions.

t	0 mod 4	1 mod 4	2 mod 4	3 mod 4
α	1	i	i	1
β	1	1	i	i

Table 1: Conditions for the reality of the Dirac action $\mathcal{A}$.

Taking now the complex conjugate of $\mathcal{L}$, and using (3.b), it can be found that

$$\mathcal{L}^* = -\alpha^*\eta^{t+1}\,\overline{\Psi^c}\gamma^\mu\partial_\mu\Psi^c - \beta^*\eta^t\, m\overline{\Psi^c}\Psi^c \; , \quad (14)$$

where

$$\Psi^c \equiv B^{-1}\Psi^* \quad \text{and} \quad \overline{\Psi^c} = (\Psi^c)^\dagger A \; . \quad (15)$$

The transformation $\Psi \to \Psi^c$ is the so-called charge conjugation operation. Next, assuming the Hermiticity of $\mathcal{L}$, and upon use of the equations (13.a) and (13.b), it results that

$$\mathcal{L}^* = \mathcal{L}^T = \mathcal{L}'(\Psi^c, \partial_\mu\Psi^c) = -\alpha\eta^{t+1}(-1)^{t(t+1)/2}\,\overline{\Psi^c}\gamma^\mu\partial_\mu\Psi^c - \beta\eta^t(-1)^{t(t-1)/2}\, m\overline{\Psi^c}\Psi^c \; . \quad (16)$$

On the other hand, taking the transposition of $\mathcal{L}$, and using (3.c), it can be found, up to a ST, that

$$\mathcal{L}^T = -\alpha\eta^{t+1}(-1)^{t(t+1)/2}\,\overline{\Psi'}\gamma^\mu\partial_\mu\Psi' - \beta\eta^t(-1)^{t(t-1)/2}\, m\overline{\Psi'}\Psi' \; , \quad (17)$$

where

$$\Psi' = \eta^t(-1)^{t(t-1)/2}\, C^{-1}\overline{\Psi}^T \quad \text{and} \quad \overline{\Psi'} = \Psi^T C \; . \quad (18)$$

From this result, one may conclude, by comparing with (16), that $\Psi' \propto \Psi^c$. It can be quickly computed that $\Psi^c = \xi^*\varepsilon\Psi'$ (ξ is the phase factor in (5.a)), so that

$$\Psi^c = \xi^* C^* \overline{\Psi}^T \; . \quad (19)$$

A Majorana spinor is defined by the condition $\Psi^c = \Psi$, or equivalently

$$\Psi = B^{-1}\Psi^* = \xi^* C^* \overline{\Psi}^T \; . \quad (20)$$

Note that, in the so-called Majorana representation ($B = \mathbb{1}$), a Majorana spinor is always *real*. It is worthwhile to mention that Majorana spinors can only be defined for $\mathrm{M}^{t,s}$ such that $\varepsilon = 1$, as it will be verified in the sequel. Inverting (20) for Ψ^*, it can be obtained that

$$\Psi^* = B\Psi \ . \tag{21}$$

Complex conjugation of the previous equation yields

$$\Psi = B^*\Psi^* \ . \tag{22}$$

Next, plugging (21) into (22), it can be found that $B^*B = \mathbb{1}$, so that, from (5.c), it results $\varepsilon = 1$.[1]

The free Lagrangian for the Majorana spinor Ψ takes the form

$$\mathcal{L} = \alpha\Psi^T\xi C\gamma^\mu\partial_\mu\Psi + \beta\, m\Psi^T\xi C\Psi \ . \tag{23}$$

Before our final considerations on the η-factor, let us discuss also the case of Majorana-Weyl spinors. For even D, we can decompose $\Sigma^{\kappa\lambda}$ in generators $\Sigma_+^{\kappa\lambda}$ and $\Sigma_-^{\kappa\lambda}$ of independent transformations, which determine complementary sectors in the Ψ space (Ψ_+ and Ψ_- chiral subspaces respectively). In the case $(s-t) = 0 \bmod 4$, $\Sigma_\pm^{\kappa\lambda}$ are real in the Majorana representation [2]. A Majorana spinor in the Majorana representation is real, such that its sector Ψ_+, when transformed via $\Sigma_+^{\kappa\lambda}$, will always be *real*. Analogously, Ψ_-, when transformed via $\Sigma_-^{\kappa\lambda}$, will always be *real*: that is, the fundamental property of Majorana spinors in the Majorana representation (the reality) is preserved in each chirality sector. This guarantees that we can expect that the property $\Psi_\pm^* = B\Psi_\pm$, under transformation via the generators $\Sigma_\pm^{\kappa\lambda}$, will be preserved in each corresponding sector, for any chosen representation. Ψ_+ and Ψ_-, with the properties shown above are known as Majorana-Weyl spinors. We are now ready to state our general results on the η-factor, and we shall quote some interesting properties on the dynamics of the charge-conjugated spinors.

From the invariance of the Lagrangian under charge conjugation operation, it follows, by virtue of (11) and (16), that

$$\eta^{t+1}(-1)^{t(t+1)/2} = -1 \ , \tag{24.a}$$
$$\eta^{t}(-1)^{t(t-1)/2} = -1 \ . \tag{24.b}$$

Relations (24.a) and (24.b) do *not* show up from an analysis based directly on the Dirac equation; they govern respectively the existence of the *kinetic term* and *mass term* of the action for the charge-conjugated spinor (and consequently for the Majorana spinor).

We present now some general conclusions concerning the existence of the kinetic and mass terms for the charge-conjugated spinors:

a) For even t, from (24.a) it results that kinetic terms can always be written down, since $\eta = (-1)^{\frac{t}{2}+1}$. In this case, (24.b) turns out to be a consistency condition, from which it follows that a mass term is only possible for $t = 2 \bmod 4$, such that $\eta = 1$.

b) On the other hand, for odd t, (24.a) becomes a consistency condition, and then it rules out the existence of the kinetic action for the charge-conjugated spinor in the case

[1] In the case $\varepsilon{=}-1$, one can impose, for example, the so-called SU(2)-reality condition and define SU(2)-Majorana spinors[2].

$t = 3 \bmod 4$. So, a kinetic term is possible only for $t = 1 \bmod 4$, so that the existence of mass for these spinors determines $\eta = -1$.

c) We therefore conclude that for $t = 1, 2 \bmod 4$, we can always write down actions with both kinetic and mass terms for the charge-conjugated spinor. For $t = 0 \bmod 4$, the charge-conjugated spinor cannot be massive; whereas for $t = 3 \bmod 4$, it cannot be dynamical (no kinetic term).

d) When the value of η is such that it prohibits mass terms, the condition $\Psi^c = \Psi$ defines the so-called pseudo-Majorana spinor. According to the paper of [2], this type of spinor appears indeed to be massless.

The results mentioned above are all summarized in Table 2.

η \ t	0 mod 4	1 mod 4	2 mod 4	3 mod 4
1	—	pseudo-Majorana	Majorana	—
−1	pseudo-Majorana	Majorana	—	—

Table 2: Types of self-conjugated spinors (ε=1) as a function of η and the number, t, of time-like directions. — means that no kinetic term can be written down for charged-conjugated spinors with the pair of values (t,η).

To conclude this letter, we should stress that our results on the possibility of writing down kinetic and mass Lagrangians for charge-conjugated spinors rely on our assumption that spinors be always taken to be Grassmann-valued, no matter what the space-time signature is. For non-usual signatures with $t \geq 2$, instead of giving up a kinetic term for charge-conjugated spinors in the cases indicated in Table 2, one could perhaps decide to work with commuting spinors, for which a kinetic term becomes non-trivial. Nevertheless, this is not the viewpoint we take here: we assume spinors to be anticommuting in any case, for we have in mind to formulate supersymmetric field theories in space-times with arbitrary signatures, and so these spinors may well be identified with supersymmetry charge generators and fermionic matter that, as it is well-known, should always exhibit an anticommuting character.

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