Some Comments of Vlasov-Liouville Equation and its Relation with Gravitational Field

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We discuss here the possibility to write the Liouville-Vlasov equation for the Wigner-function of a spinor field coupled to a gauge field with field strength tensor $F^{\mu\nu}$ in a curved space-time versus a local Lorentz manifold (introduction of local Lorentz coordinates) with an appropriate definition of a covariant derivative carried out using a spin connection $B_\mu^{ab}(x)$.

Key-words: Gravitation; Vlasov-Liouville equation.

I Introduction

The general relativity is invariant under general coordinate transformations like $x \rightarrow x' = x + \xi(x)$. One can still define a convenient covariant derivative under a general transformation of coordinate as $\mathcal{D}_\mu\xi_\nu = -\partial_\mu\xi_\nu + \Gamma^\lambda_{\mu\nu}\xi_\lambda$ where $\xi_\mu(x)$ is the infinitesimal parameter and $\Gamma^\lambda_{\mu\nu}$ is the Christoffel symbol or "connection".

The infinitesimal variation of metric under the coordinate transformation can be written as $\delta g_{\mu\nu}(x) = -\mathcal{D}_\mu\xi_\mu(x) - \mathcal{D}_\nu\xi_\mu(x)$.

It's easy to verify that for weak field approximation, $g_{\mu\nu}(x) = \eta_{\mu\nu} + kh_{\mu\nu}$ with a redefinition of the $\xi_\mu(x)$ parameter. We obtain the gauge transformation imposed on the field $h_{\mu\nu}(x)$ such as infinitesimal version in the Minkowsky space under GCT's (general transformation of coordinates).

How do we consider the interaction between gravitation and other fields? The general recipe is to introduce new fields and to define an appropriate covariant derivative that includes such an interaction.

In general there are two ways to follow: Firstly, we can use the coupling between gravitation and other fields. In this case the gravitational field is only a background for other fields. The scenario is geometrical as for gravity which acts as a background field. There is no "interaction" as is the case for the electromagnetic theory, quantum chromodynamics or even in the perturbative quantum gravity. On the other hand we can introduce a flat space time and attack the problem using the perturbative approach. If we wish to interpret the fields as particles associated with that field we need to introduce a local Lorentz manifold to bring in a specific interactions. It is true, in particular if we wish to analyse the interaction between gravitation and spinor fields. The interaction cannot be introduced directly in curved space time for the spinor field. It's necessary to look for a local Lorentz manifold rather than the global manifold as in space time.

This point is emphasized, in particular, for the special case of Liouville-Vlasov equation. We believe that the correct way to introduce the effect of gravitational field in that equation is by defining an appropriate covariant derivative in a local Lorentz manifold against the point of view that one can define the gravitational effect on Vlasov equation directly in curved space time. The latter is written as follows.

We analyse the interaction between gravitation and spinor fields using the vierbein field and the local Lorentz group as our support. As a second example we present the coupling between matter-Maxwell and gravitation fields. The following example is the $U(1)$ gauge group generalized to the case of, $SO(N)$ model coupled to gravity and spinor-vector field (Rarita-Schwinger).

In all cases the interaction is seen on a local Lorentz manifold where it is possible to find a spin connection, and in some cases, to find the "torsion" associated with such a model.

Finally, the Liouville-Vlasov equation is discussed and the effect of gravitational field on that equation is analysed. We argue here that it is impossible to define a covariant derivative for the Vlasov equation if $\psi(q,p)$ means the Wigner function associated with the spinor field. The fundamental reason for it is that the symmetry group of general relativity is incompatible with the presence of spinors [2]; but we can do that only in a local Lorentz manifold where both the Vierbein and spin connection are defined. We start by remembering that the concept of spin 1/2 field makes sense only in a tangent flat space [4,5]. One can write the action for interaction between Dirac's field and gravitation as

$$S = \int d^4x \; e \left[e^{\mu a} \bar{\psi} i \gamma_a \; D_\mu \psi - m \bar{\psi} \psi \right] \tag{1}$$

where

$$\gamma^\mu = e^\mu_a(x) \gamma^a \tag{2}$$

and

$$D_\mu \psi_\alpha = \partial_\mu \psi_\alpha + \frac{i}{8} B^{ab}_\mu [\gamma_a \gamma_b]_{\alpha\beta} \psi_\beta \tag{3}$$

with e^μ_a are the Vierbein field with two indices; one space time index a, μ, varying from zero to three and a local Lorentz group index, a, of a local manifold. The γ^μ are the Dirac's matrices, γ^a meaning the local Dirac matrices and B^{ab}_μ being the spin connection.

The interaction between gravitation and fermionic field can be seen immediately from eq. (3) because of the appropriate definition of a covariant derivative D_μ for a local Lorentz group.

The metric can be written as

$$g_{\mu\nu} = e^a_\mu e^b_\nu \eta_{ab} \; . \tag{4}$$

Thus, the "e" in eq. (1) means the determinant of the metric. *The Dirac equation coupled to gravitation* is given by

$$(e^{\mu a} i \gamma_a D_\mu - m) \psi(x) = 0 \tag{5}$$

The spin connection in (3) can be completly fixed by the coefficient of non holomicity as

$$B^{ab}_\mu = -\frac{1}{2} e^c_\mu (\Omega_{cda} + \Omega_{acd} - \Omega_{dac}) \tag{6}$$

where

$$\Omega_{cda} = e^\rho_c e^\sigma_j (\partial_\rho e_{\sigma a} - \partial_\sigma e_{\rho a}) \tag{7}$$

The equation for e^a_μ field, $R_{\mu a} - \frac{1}{2} e_{\mu a} R = 0$ fixes $e^a_\mu(x)$ exactly; then we have that B^{ab}_μ shall be fixed too.

The gravitational degree of freedom is carried by Vierbein. All information about spins from gravitational field will be given by $e^a_\mu(x)$.

One may see clearly that the kinetic term of the fermionic lagrangean contributes to the equation of motion for the spin connection and we can show that the presence of the fermionic field generates torsion [4,5] given as

$$T^a_{\mu\nu} \sim D_\mu (e e^\mu_{[a} e^\nu_{b]}) \sim \bar{\psi} \gamma [\gamma, \gamma] \psi \; . \tag{8}$$

where $[a,b]$ here means symmetrization and the right hand side represents the fermionic density.

The interaction between matter-Maxwell and gravitational fields

The action for interaction between matter-Maxwell and gravitational fields is given as

$$S^{Maxwell}_{Mat-Grav} = \int d^4x \; e [e^{\mu a} \bar{\psi} i \gamma_a \nabla_\mu \psi - m \bar{\psi} \psi] \tag{9}$$

where ∇_μ is a covariant derivative of gravitation and the gauge is given by

$$\nabla_\mu \psi = D_\mu + igqA_\mu \tag{10}$$

with D_μ being the covariant derivative as in eq. (3), $A_\mu(x)$ is the vector potential, q is a coupling constant associated with the $U(1)$ symmetry and g is another coupling constant linking the local Lorentz group. The action is invariant under local Lorentz group, GCT's (transformation coordinates group) and $U(1)$ symmetry simultaneously.

A term is needed that gives the dynamics for the gauge degree of freedom (Maxwell term); so the complete action which couples Maxwell-Dirac-gravitation is written as

$$S^{Maxwell}_{Dirac-Grav} = \int d^4x \; e [-\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + e^{\mu a} \bar{\psi} i \gamma_a \nabla_\mu \psi - m \bar{\psi} \psi] \tag{11}$$

The $U(1)$ case generalized

Consider the scalar and spinor field matter fields coupled to gravitation and Maxwell field. The lagrangean is given as

$$\mathcal{L} = (\partial_\mu \varphi^* \partial^\mu \varphi - m^2 \varphi^* \varphi) + \bar{\psi}(i\gamma^\mu \partial_\mu - M)\psi + \frac{\lambda}{4}(\varphi^* \varphi)^2 \ . \tag{12}$$

To obtain a covariant lagrangean the convenient covariant derivative is introduced as $\partial_\mu \to \nabla_\mu$, where

$$\nabla_\mu \psi = \left(\partial_\mu + \frac{i}{8} B_\mu^{ab}[\gamma_a, \gamma_b] + igQA_\mu(x)\right)\psi \tag{13}$$

Thus, the complete action for Maxwell, matter and gravitational field is

$$S_{mat-grav}^{Maxwell} = \int d^4x \ e\left[(\nabla_\mu \varphi^*)(\nabla^\mu \varphi) - V(\varphi^*, \varphi)\right] + e \ e^{\mu a} \bar{\psi}(i\gamma_a \nabla_\mu \psi - M)\psi \tag{14}$$

where $V(\varphi^* \varphi) = m^2 \varphi^* \varphi$ is the potential term and Q means the generalized charge.

The equation of motion is immediately obtained as

$$\nabla^\mu \nabla_\mu \varphi - m^2 \varphi + \cdots = 0 \tag{15}$$

and the dynamics is verified on local Lorentz manifold again.

Model $SO(N)$ coupled to gravity

The interest in this model is formal. There appear magnetic monopoles of t'Hooft-Polyakov coupling to gravitational field.

Suppose that φ_a are scalars fields in the representation of $SO(N)$ and $a = 1, 2, 3 \cdots N$.

Thus,

$$\varphi_a{}' = R_{ab} \varphi_b \tag{16}$$

where R_{ab} is the transformation matrix and it satisfies the condition $R^+ R = 1$.

The lagrangean satisfying the invariance under global $SO(N)$ is given by

$$\mathcal{L} = \frac{1}{2} \partial_\mu \varphi_a \partial^\mu \varphi_a - \frac{1}{2} m^2 \varphi_a \varphi_a \ . \tag{17}$$

the $SO(N)$ will be calibrated by equation (16) and we need to change again the derivative ∂_μ to a new covariant derivative ∇_μ written as

$$\nabla_\mu \varphi_a = (\partial_\mu \delta_{ab} + igA_\mu^I (G_I)_{ab})\varphi_b \tag{18}$$

where G_I are generators of $SO(N)$ group and $A_\mu^I(x)$ are the non abellian vector potentials. The number of generators being written as $I = 1 \cdots \dfrac{N(N-1)}{2}$.

The complete action for this case can be written as

$$S_{mat-grav}^{Y.M} = \int d^4x \ e\left[-\frac{1}{4} F_{\mu\nu I} F^{\mu\nu I} + \frac{1}{2} \nabla_\mu \varphi_a \nabla^\mu \varphi_a - \frac{1}{2} m^2 \varphi_a \varphi_a + \right.$$
$$\left. -\frac{\lambda}{4!}(\varphi_a \varphi_b)^2 + fR\varphi_a \varphi_a\right] \tag{19}$$

where f means the dimensionless coupling constant, R in the last term is the scalar curvature and $F_{\mu\nu I}$ represents the field strength given by

$$F_{\mu\nu I} = \partial_\mu A_{\nu I} - \partial_\nu A_{\mu I} + gf_{IJK} A_{\mu J} A_{\nu K} \tag{20}$$

Spinor-vector fields

We define now $\psi_{a\alpha}$ as a spinor-vector field with $\alpha = 1, 2, 3, 4$ and $a = 0, 1, 2, 3$. The transformation law for Rarita-Schwinger field can be written as

$$\delta\psi_{a\alpha} = w_a^b \psi_{b\alpha} + \frac{1}{2} w^{mn} [\gamma_m, \gamma_n]_{\alpha\beta} \psi_{\alpha\beta} \qquad (21)$$

where w_a^b are continuously varying parameters. The first part on right hand side being responsible for spin-1 and the second part being associated with spin field 1/2 and 3/2.

The spin 1/2 can be eliminated consistently by a symmetry of the free lagrangean [5]. Thus, $\psi_{a\alpha}$ shall describe a pure spin 3/2.

It appears in supergravity [5] as a fermionic mediator of the gravitational interaction.

The lagrangean which describes a free spinor-vector field can be given as

$$\mathcal{L}_{R.S.} = \frac{1}{2} \varepsilon^{\mu\nu\rho\sigma} \bar{\psi}_\mu \gamma_5 \gamma_\nu \partial_\rho \psi_\sigma \qquad (22)$$

with $\varepsilon^{\mu\nu\rho\sigma}$ being the Levi-Civita tensor and $\gamma_5 = \gamma_0\gamma_1\gamma_2\gamma_3$ is a Dirac's matrix. We can write eq. (22) in the component form as

$$\mathcal{L}_{R.S.} = \frac{1}{2} \varepsilon^{\mu\nu\rho\sigma} \bar{\psi}_{\mu\alpha} (\gamma_5 \gamma_\nu)_{\alpha\beta} \partial_\rho \psi_{\sigma\beta} \qquad (23)$$

The local invariance $U(1)$ that permits the elimination of spin 1/2 components is given as

$$\delta\psi_{\mu\alpha} = \partial_\mu \chi \qquad (24)$$

where χ is a spinor field. As in the case of the fermionic field the Rarita-Schwinger field generates torsion [5].

The interaction between gravitation and the spinor-vector field is shown following the same recipe as in earlier cases so that

$$\mathcal{L}_{R.S.}^{gravity} = \frac{e}{2} \varepsilon^{\mu\nu\rho\sigma} \bar{\psi}_{\mu\alpha} (\gamma^5 \gamma_\nu)_{\alpha\beta} D_\rho \psi_{\sigma\beta} \qquad (25)$$

where

$$D_\rho \psi_{\sigma\beta} = \partial_\rho \psi_{\sigma\beta} + \frac{i}{8} B_\rho^{ab} [\gamma_a, \gamma_b]_{\beta\gamma} \psi_{\sigma\gamma} \qquad (26)$$

It is clear that the only difference between eq. (25) and the other cases is that here ψ has two indices. One can show again that the spinor-vector field generates torsion again as $T_{\mu\nu}^a \sim \bar{\psi}\gamma_5\gamma[\gamma, \gamma]\psi$ forming a spinor condensity exactly the same way as given in (8).

The Liouville-Vlasov equation with the gravitational interaction

Finally, we obtain the Liouville-Vlasov equation [3] for the Wigner function of a "spinor field" coupled to a gauge field with the field strength tensor $F^{\mu\nu}$ as

$$(-p_\mu F^{\mu\nu} D_\nu + P^\mu D_\mu)\psi(q, p) = 0 \qquad (27)$$

where $\psi(q, p)$ is the Wigner function. Here D_μ cannot be a covariant derivative including the curved space-time spin connection unlike the case discussed in [1,3].

The reason for this is that the coordinate tranformation group of general relativity is incompatible with the presence of spinor field and so, incompatible with the appearance of spin connection B_μ^{ab} in curved space-time [2,4].

We can introduce an appropriate covariant derivative in eq. (27) the same way as in preceding cases. So, considering the tangent space or local Lorentz group we do the same use of eq. (3) and in all analyzed cases, obtained for D_μ in eq. (27) a covariant derivative like

$$D_\mu \psi = \partial_\mu \psi + \frac{i}{8} B_\mu^{ab} [\gamma_a, \gamma_b] \psi \qquad (28)$$

where ψ is the Wigner function associated with a spinor field coupled to a gauge field in a local Lorentz manifold (tangent space). It is well-known that [3] $\psi^*(q, p)\psi(q, p)$ can be interpreted as the classical distribution function in the relativistic phase space.

We observe that if we wish to include the real effects from gravitational field in an approach of general relativity together with the kinetic theory some troubles will appear. For example: it is clear [3] "that the treatment of transport theory in a curved space-time background is hidered by the fact that a definition of the Wigner function $\psi(q, p)$ depends on the use of the Fourier transform of a space-time correlation function with a translated argument".

Fourier transformation (nor translation) are globally available in general curved spacetime. On the other hand the symmetry group of general relativity is not compatible with the presence of spinors or spin connection field [2] if the Wigner function of a spinor field in a curved space time has to be found.

However, as discussed in [3], the construction of Wigner function $\psi(q, p)$ is still meaningful if carried out in the tangent space as in local Lorentz manifold as applied in our case. The problem is that the correlation between two points on that manifold is by an exponential map [3]. Only this way, we can introduce the fermionic field interacting with the gravitation and to define an appropriate covariant derivative containing the spin connection.

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Referências

[1] F.C. Khanna, A.E. Santana, A. Matos Neto, J.D.M. Vianna, Preprint, submitted for publications, 1998.

[2] S. Weimberg, "Gravitation and Cosmology".

[3] A.E. Santana, A. Matos Neto, J.D.M. Vianna and F.C. Khanna, T. Kopf, Preprint submitted, 1998.

[4] Hagen Kleinert, "Gauge Fields in Condensed Matter", vol. II, World Scientific, 1998.

[5] J. Helayël-Neto, C. Pinheiro. G.O. Pires, Physics Lectures, on "Quantum Gravity", CBPF, Rio de Janeiro, 1994.