CBPF-NF-046/99
JINR E2-99-302
solv-int/yymmnnn

$N = 4$ Sugawara construction on $\widehat{sl(2|1)}$, $\widehat{sl(3)}$ and mKdV-type superhierarchies

E. Ivanov[a,1], S. Krivonos[a,2] and F. Toppan[b,3]

[a] JINR-Bogoliubov Laboratory of Theoretical Physics,
141980 Dubna, Moscow Region, Russia

[a] Centro Brasileiro de Pesquisas Físicas – CBPF
Departamento de Campos e Partículas – DCP
Rua Dr.Xavier Sigaud 150,
Rio de Janeiro, RJ, CEP 22290-180 Brazil

Abstract

The local Sugawara constructions of the "small" $N = 4$ SCA in terms of supercurrents of $N = 2$ extensions of the affine $\widehat{sl(2|1)}$ and $\widehat{sl(3)}$ algebras are investigated. The associated super mKdV type hierarchies induced by $N = 4$ SKdV ones are defined. In the $\widehat{sl(3)}$ case the existence of two inequivalent Sugawara constructions is found. The "long" one involves all the affine $\widehat{sl(3)}$ currents, while the "short" one deals only with those from the subalgebra $\widehat{sl(2) \oplus u(1)}$. As a consequence, the $\widehat{sl(3)}$-valued affine superfields carry two inequivalent mKdV type super hierarchies induced by the correspondence between "small" $N = 4$ SCA and $N = 4$ SKdV hierarchy. However, only the first hierarchy posseses genuine global $N = 4$ supersymmetry. We discuss peculiarities of the realization of this $N = 4$ supersymmetry on the affine supercurrents.

Key-Words: Integrable hierarchies; Superconformal algebras.

E-Mail:
1) eivanov@thsun1.jinr.ru
2) krivonos@thsun1.jinr.ru
3) toppan@cbpf.br

1 Introduction

In the last several years integrable hierarchies of non-linear differential equations have been intensely explored, mainly in connection with the discretized two-dimensional gravity theories (matrix models) [1] and, more recently, with the 4-dimensional super Yang-Mills theories in the Seiberg-Witten approach [2].

A vast literature is by now available on the construction and classification of the hierarchies. In the bosonic case the understanding of integrable hierarchies in $1+1$ dimensions is to a large extent complete. Indeed, a generalized Drinfeld-Sokolov scheme [3] is presumably capable to accomodate all known bosonic hierarchies.

On the other hand, due to the presence of even and odd fields, the situation for supersymmetric extensions remains in many respects unclear. Since a fully general supersymmetric Drinfeld-Sokolov approach to the superhierarchies is still lacking, up to now they were constructed using all sorts of the available tools. These include, e.g., direct methods, Lax operators of both scalar and matrix type, bosonic as well as fermionic, coset construction, etc. [4]-[10].

In [11] a general Lie-algebraic framework for the $N=4$ super KdV hierarchy [12, 13] and, hopefully, for its hypothetical higher conformal spin counterparts (like $N=4$ Boussinesq) has been proposed. It is based upon a generalized Sugawara construction on the $N=2$ superextended affine (super)algebras which possess a hidden (nonlinearly realized) $N=4$ supersymmetry. This subclass seemingly consists of $N=2$ affine superextensions of both the bosonic algebras with the quaternionic structure listed in [14] and proper superalgebras having such a structure. In its simplest version [11], the $N=4$ Sugawara construction relates affine supercurrents taking values in the $sl(2)\oplus u(1)$ algebra to the "minimal" (or "small") $N=4$ superconformal algebra ($N=4$ SCA) which provides the second Poisson structure for the $N=4$ super KdV hierarchy. The Sugawara-type transformations are Poisson maps, i.e. they preserve the Poisson-brackets structure of the affine (super)fields. Therefore for any Sugawara transformation which maps affine superfields, say, onto the minimal $N=4$ SCA, the affine supercurrents themselves inherit an integrable hierarchy which is constructed using the tower of the $N=4$ SKdV hamiltonians in involution. Such $N=4$ hierarchies realized on the affine supercurrents can be interpreted as generalized mKdV-type superhierarchies. The simplest example, the combined $N=4$ mKdV-NLS hierarchy associated with the affine $N=2$ $\widehat{sl(2)\oplus u}(1)$ superalgebra, was explicitly constructed in [11].

In the case of higher-dimensional $N=4$ affine superalgebras this sort of Sugawara construction is expected to yield additional $N=4$ multiplets of currents which would form, together with those of $N=4$ SCA (both "minimal" and "large"), more general nonlinear $N=4$ superalgebras of the W algebra type. Respectively, new SKdV (or super Boussinesq) type hierarchies with these conformal superalgebras as the second Poisson structures can exist, as well as their mKdV type counterparts associated with the initial $N=4$ affine superalgebras. Besides, the linear $N=4$ SCAs can be embedded into a given affine superalgebra in different ways, giving rise to a few inequivalent mKdV-type superhierarchies associated with the same KdV-type superhierarchy.

In this paper we describe inequivalent $N=4$ Sugawara constructions for the eight-dimensional affine (super)algebras $N=2$ $\widehat{sl(2|1)}$ and $N=2$ $\widehat{sl(3)}$. These algebras are natural candidates for the higher-rank affine superalgebras with hidden $N=4$ supersymmetry, next in complexity to the simplest $\widehat{sl(2)\oplus u(1)}$ case treated in ref. [11].

The results can be summarized as follows.

In the $\widehat{sl(2|1)}$ case there are no other *local* Sugawara constructions leading to the "small" $N=4$ SCA besides the one which proceeds from the bosonic $\widehat{sl(2)\oplus u(1)}$ subalgebra supercurrents. The $\widehat{sl(2|1)}$ affine supercurrents carry a unique mKdV type hierarchy, the evolution equations for the extra four superfields being induced from their Poisson brackets with the $N=4$ SKdV hamiltonians constructed from the $sl(2)\oplus u(1)$-valued supercurrents. The full hierarchy possesses by construction the manifest $N=2$ supersymmetry and also reveals some extra exotic "$N=2$ supersymmetry". These two yield the standard $N=4$ supersymmetry only on the $\widehat{sl(2)\oplus u(1)}$ subset of currents ("standard" means closing on z translations). Actually, such an extra $N=2$ supersymmetry is present in *any* $N=2$ affine (super)algebra with a $\widehat{sl(2)\oplus u(1)}$ subalgebra. As the result, neither the $N=2$ $\widehat{sl(2|1)}$ superalgebra itself, nor the above-mentioned mKdV hierarchy reveal the genuine $N=4$ supersymmetry.

The $\widehat{sl(3)}$ case is more interesting since it admits such an extended supersymmetry. In this case, besides the "trivial" $N=4$ SCA based on the $\widehat{sl(2)\oplus u(1)}$ subalgebra, one can define an extra $N=4$ SCA containing the full $N=2$ stress-tensor and so involving all affine $\widehat{sl(3)}$ supercurrents [1]. We have explicitly checked that no other inequivalent local $N=4$ Sugawaras exist in this case. The supercurrents of the second $N=4$ SCA generate global $N=4$ supersymmetry closing in the standard way on z-translations. The defining relations of the $N=2$ $\widehat{sl(3)}$ algebra are covariant under this supersymmetry, so it is actually $N=4$ extension of $\widehat{sl(3)}$, similarly to the $\widehat{sl(2)\oplus u(1)}$ example. In the original basis, where the affine currents satisfy nonlinear constraints, the hidden $N=2$ supersymmetry transformations are essentially nonlinear and mix all the currents. After passing, by means of a non-local field redefinition, to the basis where the constraints become the linear chirality conditions, the supercurrents split into some invariant subspace and a complement which transforms through itself and the invariant subspace. In other words, they form a not fully reducible representation of the $N=4$ supersymmetry. This phenomenon was not previously encountered in $N=4$ supersymmetric integrable systems. We expect it to hold also in higher rank $N=2$ affine superalgebras with the hidden $N=4$ structure.

The "long" Sugawara gives rise to a new mKdV type hierarchy associated with the $N=4$ SKdV one. Thus the $\widehat{sl(3)}$ affine supercurrents provide an example of a Poisson structure leading to two inequivalent mKdV-type hierarchies, both associated with $N=4$ SKdV, but recovered from the "short" and, respectively, "long" $N=4$ Sugawara constructions. Only the second hierarchy possesses global $N=4$ supersymmetry.

As a by-product, we notice the existence of another sort of super mKdV hierarchies associated with both affine superalgebras considered. They are related to the so-called "quasi" $N=4$ SKdV hierarchy [15] which still possesses the "small" $N=4$ SCA as the second Poisson structure but lacks global $N=4$ supersymmetry. In the $\widehat{sl(3)}$ case

[1] In what follows we name the corresponding Sugawara construction "long" $N=4$ Sugawara, as opposed to the "short" one based on the $\widehat{sl(2)\oplus u(1)}$ subalgebra.

there also exist two inequivalent "quasi" super mKdV hierarchies generated through the "short" and 'long" Sugawara constructions.

Like in [11], in the present paper we use the $N=2$ superfield approach with the manifest linearly realized $N=2$ supersymmetry. The results are presented in the language of classical OPEs between $N=2$ supercurrents, which is equivalent to the Poisson brackets formalism used in [11]. When evaluating these $N=2$ OPEs, we systematically exploit the Mathematica package of ref. [16].

2 $N=2$ conventions and the minimal $N=4$ SCA

Here we fix our notation and present the $N=2$ superfield Poisson brackets structure of the "minimal" ("small") $N=4$ superconformal algebra (in the OPE language).

The $N=2$ superspace is parametrized by the coordinates $Z \equiv \{z,\theta,\overline{\theta}\}$, with $\{\theta,\overline{\theta}\}$ being Grassmann variables. The (anti)-chiral $N=2$ derivatives $D,\overline{D}$ are defined as

$$D=\frac{\partial}{\partial\theta}-\frac{1}{2}\overline{\theta}\partial_z\ ,\ \ \overline{D}=\frac{\partial}{\partial\overline{\theta}}-\frac{1}{2}\theta\partial_z\ ,\ \ D^2=\overline{D}^2=0\ ,\ \ \{D,\overline{D}\}=-\partial_z\ . \tag{1}$$

In the $N=2$ superfield notation the minimal $N=4$ SCA is represented by the spin 1 general superfield $J(Z)$ and two (anti)-chiral spin 1 superfields W, $\overline{W}$ ($DW=\overline{D}\,\overline{W}=0$), with the following classical OPE's

$$\begin{aligned}
\underline{J(1)J(2)} &= \frac{2}{{Z_{12}}^2}-\frac{\theta_{12}\overline{\theta}_{12}}{{Z_{12}}^2}J-\frac{\overline{\theta}_{12}}{Z_{12}}\overline{D}J+\frac{\theta_{12}}{Z_{12}}DJ-\frac{\theta_{12}\overline{\theta}_{12}}{Z_{12}}J'\ ,\\
\underline{J(1)W(2)} &= -\frac{\theta_{12}\overline{\theta}_{12}}{{Z_{12}}^2}W-\frac{2}{Z_{12}}W-\frac{\overline{\theta}_{12}}{Z_{12}}\overline{D}W-\frac{\theta_{12}\overline{\theta}_{12}}{Z_{12}}W'\ ,\\
\underline{J(1)\overline{W}(2)} &= -\frac{\theta_{12}\overline{\theta}_{12}}{{Z_{12}}^2}\overline{W}+\frac{2}{Z_{12}}\overline{W}+\frac{\theta_{12}}{Z_{12}}D\overline{W}-\frac{\theta_{12}\overline{\theta}_{12}}{Z_{12}}\overline{W}'\ ,\\
\underline{W(1)\overline{W}(2)} &= \frac{\theta_{12}\overline{\theta}_{12}}{{Z_{12}}^3}-\frac{1}{{Z_{12}}^2}-\frac{\frac{1}{2}\theta_{12}\overline{\theta}_{12}}{{Z_{12}}^2}J+\frac{\overline{\theta}_{12}}{Z_{12}}\overline{D}J+\frac{1}{Z_{12}}J\ .
\end{aligned} \tag{2}$$

Here $Z_{12}=z_1-z_2+\frac{1}{2}\left(\theta_1\overline{\theta}_2-\theta_2\overline{\theta}_1\right)$, $\theta_{12}=\theta_1-\theta_2$, $\overline{\theta}_{12}=\overline{\theta}_1-\overline{\theta}_2$, and the superfields in the r.h.s. are evaluated at the point $(2)\equiv(\,z_2,\theta_2,\overline{\theta}_2\,)$.

3 The superaffinization of the $sl(2|1)$ superalgebra

In this and next Sections we follow the general $N=2$ superfield setting for $N=2$ extensions of affine (super)algebras [17, 18].

The $N=2$ $\widehat{sl(2|1)}$ superalgebra is generated by four fermionic and four bosonic superfields, respectively $(H,\overline{H},F,\overline{F})$ and $(S,\overline{S},R,\overline{R})$.

The superfields $H,\overline{H}$ are associated with the Cartan generators of $sl(2|1)$ and satisfy the (anti)chiral constraints

$$\overline{D}\,\overline{H}=DH=0 \tag{3}$$

while the remaining superfields are associated with the root generators of $sl(2|1)$. In particular $F, \overline{F}$ are related to the bosonic $(\pm)$-simple roots and, together with $H, \overline{H}$, close on the superaffine $\widehat{sl(2)\oplus u(1)}$ subalgebra. The extra superfields satisfy the non-linear chiral constraints

$$\begin{aligned} &\overline{D}\,\overline{R}=0\ , \quad \overline{D}\,\overline{F}=\overline{H}\,\overline{F}\ , \quad \overline{D}\,\overline{S}=-\overline{F}\,\overline{R}+\overline{H}\,\overline{S}\ , \\ &DR=HR\ , \quad DF=-HF\ , \quad DS=FR\ . \end{aligned} \tag{4}$$

The full set of OPEs defining the classical $N=2$ superaffine $\widehat{sl(2|1)}$ algebra is given by

$$\begin{aligned}
&\underline{H(1)\overline{H}(2)} = \frac{\frac{1}{2}\theta_{12}\overline{\theta}_{12}}{{Z_{12}}^2} - \frac{1}{Z_{12}}\ ,\ \underline{H(1)F(2)} = \frac{\overline{\theta}_{12}}{Z_{12}}F\ ,\ \underline{H(1)\overline{F}(2)} = -\frac{\overline{\theta}_{12}}{Z_{12}}\overline{F}\ ,\\
&\underline{H(1)S(2)} = \frac{\overline{\theta}_{12}}{Z_{12}}S\ ,\ \underline{H(1)\overline{S}(2)} = -\frac{\overline{\theta}_{12}}{Z_{12}}\overline{S}\ ,\ \underline{\overline{H}(1)F(2)} = \frac{\theta_{12}}{Z_{12}}F\ ,\ \underline{\overline{H}(1)\overline{F}(2)} = -\frac{\theta_{12}}{Z_{12}}\overline{F}\ ,\\
&\underline{\overline{H}(1)R(2)} = -\frac{\theta_{12}}{Z_{12}}R\ ,\ \underline{\overline{H}(1)\overline{R}(2)} = \frac{\theta_{12}}{Z_{12}}\overline{R}\ ,\\
&\underline{F(1)\overline{F}(2)} = \frac{\frac{1}{2}\theta_{12}\overline{\theta}_{12}}{{Z_{12}}^2} - \frac{1-\overline{\theta}_{12}\overline{H}-\theta_{12}H-\theta_{12}\overline{\theta}_{12}(F\overline{F}+H\overline{H}+\overline{D}H)}{Z_{12}}\ ,\\
&\underline{F(1)S(2)} = -\frac{\theta_{12}\overline{\theta}_{12}}{Z_{12}}FS\ ,\ \underline{F(1)\overline{S}(2)} = \frac{\overline{\theta}_{12}\overline{R}+\theta_{12}\overline{\theta}_{12}(F\overline{S}+H\overline{R})}{Z_{12}}\ ,\\
&\underline{F(1)R(2)} = -\frac{\overline{\theta}_{12}S+\theta_{12}\overline{\theta}_{12}HS}{Z_{12}}\ ,\ \underline{\overline{F}(1)S(2)} = -\frac{\theta_{12}R+\theta_{12}\overline{\theta}_{12}\overline{H}R}{Z_{12}}\ ,\\
&\underline{\overline{F}(1)R(2)} = \frac{\theta_{12}\overline{\theta}_{12}}{Z_{12}}R\overline{F}\ ,\ \underline{\overline{F}(1)\overline{R}(2)} = \frac{\theta_{12}\overline{S}-\theta_{12}\overline{\theta}_{12}(\overline{F}\,\overline{R}-\overline{H}\,\overline{S})}{Z_{12}}\ ,\\
&\underline{S(1)\overline{S}(2)} = -\frac{\frac{1}{2}\theta_{12}\overline{\theta}_{12}}{{Z_{12}}^2} + \frac{1-\overline{\theta}_{12}\overline{H}-\theta_{12}\overline{\theta}_{12}(F\overline{F}-R\overline{R})}{Z_{12}}\ ,\ \underline{S(1)R(2)} = -\frac{\theta_{12}\overline{\theta}_{12}}{Z_{12}}SR\ ,\\
&\underline{S(1)\overline{R}(2)} = \frac{\theta_{12}F+\theta_{12}\overline{\theta}_{12}\overline{D}F}{Z_{12}}\ ,\ \underline{\overline{S}(1)R(2)} = \frac{\overline{\theta}_{12}\overline{F}+\theta_{12}\overline{\theta}_{12}(R\overline{S}+H\overline{F}-D\overline{F})}{Z_{12}}\ ,\\
&\underline{R(1)\overline{R}(2)} = -\frac{\frac{1}{2}\theta_{12}\overline{\theta}_{12}}{{Z_{12}}^2} + \frac{1+\theta_{12}H+\theta_{12}\overline{\theta}_{12}\overline{D}H}{Z_{12}}\ .
\end{aligned} \tag{5}$$

All other OPEs are vanishing. The superfields in the r.h.s. are evaluated at the point (2).

There is only one local Sugawara realization of $N=4$ SCA associated with this affine $sl(2|1)$ superalgebra. It is explicitly given by the relations

$$J=\overline{D}H+D\overline{H}+H\overline{H}+F\overline{F}\ ,\ W=D\overline{F}\ ,\ \overline{W}=\overline{D}F\ . \tag{6}$$

It involves only the superfields $(H, \overline{H}, F, \overline{F})$ which generate just the $\widehat{sl(2)\oplus u(1)}$-superaffine subalgebra. It can be checked that no Sugawara construction involving all the $sl(2|1)$ superfields exists in this case. The $N=4$ SKdV hamiltonians constructed from the superfields (6) produce an mKdV type hierarchy of the evolution equations for the $\widehat{sl(2|1)}$ supercurrents through the OPE relations (5).

Note that the supercurrents (6) generate global non-linear automorphisms of $N=2$ $\widehat{sl(2|1)}$ (preserving both the OPEs (5) and the constraints (4)), such that their algebra formally coincide with the $N = 4$ supersymmetry algebra. However, these fermionic transformations close in a standard way on z-translations only on the $\widehat{sl(2) \oplus u(1)}$ subset. On the rest of affine supercurrents they yield complicated composite objects in the closure. It is of course a consequence of the fact that the true z-translations of all supercurrents are generated by the full $N = 2$ stress-tensor on the affine superalgebra, while $N = 4$ SCA (6) contains the stress-tensor on a subalgebra. So this fermionic automorphisms symmetry cannot be viewed as promoting the manifest $N = 2$ supersymmetry to $N = 4$ one [2]. Thus the $N = 2$ superaffine $\widehat{sl(2|1)}$ algebra as a whole possesses no hidden $N = 4$ structure, as distinct from its $\widehat{sl(2) \oplus u(1)}$ subalgebra. This obviously implies that the super mKdV hierarchy induced on the full set of the $\widehat{sl(2|1)}$ supercurrents through the Sugawara construction (6) is not $N = 4$ supersymmetric as well.

4 The superaffine $\widehat{sl(3)}$ algebra

The superaffinization of the $sl(3)$ algebra is spanned by eight fermionic $N = 2$ superfields subjected to non-linear (anti)chiral constraints. We denote these superfields H, F, R, S (their antichiral counterparts are $\overline{H}, \overline{F}, \overline{R}, \overline{S}$). The $\widehat{sl(2) \oplus u(1)}$ subalgebra is represented by $H, \overline{H}, S, \overline{S}$. As before the Cartan subalgebra is represented by the standard (anti)chiral $N = 2$ superfields $H, \overline{H}$

$$DH = \overline{D}\,\overline{H} = 0 \ . \tag{7}$$

The remaining supercurrents are subject to the non-linear constraints:

$$\begin{aligned} &DS = -HS \ , \quad DF = -\overline{\alpha}HF + SR \ , \quad DR = \alpha HR \ , \\ &\overline{D}\,\overline{S} = \overline{H}\,\overline{S} \ , \quad \overline{D}\,\overline{F} = \alpha\overline{H}\,\overline{F} - \overline{S}\,\overline{R} \ , \quad \overline{D}\,\overline{R} = -\overline{\alpha}\overline{H}\,\overline{R} \ , \end{aligned} \tag{8}$$

where

$$\alpha = \frac{1 + i\sqrt{3}}{2} \ , \quad \overline{\alpha} = \frac{1 - i\sqrt{3}}{2} \ . \tag{9}$$

The non-vanishining OPEs of the classical $N = 2$ superaffine $\widehat{sl(3)}$ algebra read:

$$\begin{aligned}
&\underline{H(1)\overline{H}(2)} = \frac{\frac{1}{2}\theta_{12}\overline{\theta}_{12}}{Z_{12}{}^2} - \frac{1}{Z_{12}} \ , \ \underline{H(1)F(2)} = \frac{\alpha\overline{\theta}_{12}}{Z_{12}}F \ , \ \underline{H(1)\overline{F}(2)} = -\frac{\alpha\overline{\theta}_{12}}{Z_{12}}\overline{F} \ , \\
&\underline{H(1)S(2)} = \frac{\overline{\theta}_{12}}{Z_{12}}S \ , \ \underline{H(1)\overline{S}(2)} = -\frac{\overline{\theta}_{12}}{Z_{12}}\overline{S} \ , \ \underline{H(1)R(2)} = -\frac{\overline{\alpha}\overline{\theta}_{12}}{Z_{12}}R \ , \ \underline{H(1)\overline{R}(2)} = \frac{\overline{\alpha}\overline{\theta}_{12}}{Z_{12}}\overline{R} \ , \\
&\underline{\overline{H}(1)F(2)} = \frac{\overline{\alpha}\theta_{12}}{Z_{12}}F \ , \ \underline{\overline{H}(1)\overline{F}(2)} = -\frac{\overline{\alpha}\theta_{12}}{Z_{12}}\overline{F} \ , \ \underline{\overline{H}(1)S(2)} = \frac{\theta_{12}}{Z_{12}}S \ , \ \underline{\overline{H}(1)\overline{S}(2)} = -\frac{\theta_{12}}{Z_{12}}\overline{S} \ , \\
&\underline{\overline{H}(1)R(2)} = -\frac{\alpha\theta_{12}}{Z_{12}}R \ , \ \underline{\overline{H}(1)\overline{R}(2)} = \frac{\alpha\theta_{12}}{Z_{12}}\overline{R} \ , \\
&\underline{F(1)\overline{F}(2)} = \frac{\frac{1}{2}\theta_{12}\overline{\theta}_{12}}{z_{12}{}^2} - \frac{1 - \alpha\overline{\theta}_{12}\overline{H} - \overline{\alpha}\theta_{12}H - \theta_{12}\overline{\theta}_{12}(F\overline{F} + H\overline{H} + R\overline{R} + S\overline{S} + \overline{\alpha}\overline{D}H)}{Z_{12}} \ ,
\end{aligned}$$

[2]This kind of odd automorphisms is inherent to any $N = 2$ affine algebra or superalgebra containing $\widehat{sl(2) \oplus u(1)}$ subalgebra.

$$
\begin{aligned}
&\underline{F(1)S(2)} = \frac{\alpha\theta_{12}\overline{\theta}_{12}}{Z_{12}}FS\,,\ \underline{F(1)\overline{S}(2)} = \frac{\theta_{12}R + \theta_{12}\overline{\theta}_{12}(\overline{D}R + \overline{\alpha}F\overline{S} - \overline{H}R)}{Z_{12}}\,,\\
&\underline{F(1)R(2)} = \frac{\overline{\alpha}\theta_{12}\overline{\theta}_{12}}{Z_{12}}FR\,,\ \underline{F(1)\overline{R}(2)} = -\frac{\theta_{12}S + \theta_{12}\overline{\theta}_{12}(\overline{D}S - \alpha F\overline{R} + \overline{\alpha}\overline{H}S)}{Z_{12}}\,,\\
&\underline{\overline{F}(1)S(2)} = -\frac{\overline{\theta}_{12}\overline{R} - \theta_{12}\overline{\theta}_{12}(H\overline{R} - \alpha\overline{F}S + D\overline{R})}{Z_{12}}\,,\ \underline{\overline{F}(1)\overline{S}(2)} = -\frac{\overline{\alpha}\theta_{12}\overline{\theta}_{12}}{Z_{12}}\overline{F}\,\overline{S}\,,\\
&\underline{\overline{F}(1)R(2)} = \frac{\overline{\theta}_{12}\overline{S} - \theta_{12}\overline{\theta}_{12}(D\overline{S} - \alpha H\overline{S} + \overline{\alpha}\overline{F}R)}{Z_{12}}\,,\ \underline{\overline{F}(1)\overline{R}(2)} = -\frac{\alpha\theta_{12}\overline{\theta}_{12}}{Z_{12}}\overline{F}\,\overline{R}\,,\\
&\underline{S(1)\overline{S}(2)} = \frac{\frac{1}{2}\theta_{12}\overline{\theta}_{12}}{{Z_{12}}^2} - \frac{1 - \overline{\theta}_{12}\overline{H} - \theta_{12}H - \theta_{12}\overline{\theta}_{12}(S\overline{S} + H\overline{H} + \overline{D}H)}{Z_{12}}\,,\\
&\underline{S(1)R(2)} = -\frac{\overline{\theta}_{12}F + \theta_{12}\overline{\theta}_{12}(HF - \overline{\alpha}SR)}{Z_{12}}\,,\ \underline{S(1)\overline{R}(2)} = -\frac{\overline{\alpha}\theta_{12}\overline{\theta}_{12}}{Z_{12}}S\overline{R}\,,\\
&\underline{\overline{S}(1)R(2)} = \frac{\alpha\theta_{12}\overline{\theta}_{12}}{Z_{12}}\overline{S}R\,,\ \underline{\overline{S}(1)\overline{R}(2)} = \frac{\theta_{12}\overline{F} + \theta_{12}\overline{\theta}_{12}(\overline{H}\,\overline{F} - \alpha\overline{S}\,\overline{R})}{Z_{12}}\,,\\
&\underline{R(1)\overline{R}(2)} = \frac{\frac{1}{2}\theta_{12}\overline{\theta}_{12}}{{Z_{12}}^2} - \frac{1 + \overline{\alpha}\overline{\theta}_{12}\overline{H} + \alpha\theta_{12}H - \theta_{12}\overline{\theta}_{12}(H\overline{H} + R\overline{R} - \alpha\overline{D}H)}{Z_{12}}\,.
\end{aligned}
\tag{10}
$$

There exist two inequivalent ways to embed the affine supercurrents into the minimal $N = 4$ SCA via a local Sugawara construction. One realization, like in the $\widehat{sl(2|1)}$ case, corresponds to the "short" Sugawara construction based solely upon the $\widehat{sl(2)\oplus u(1)}$ subalgebra. The second one, which in what follows is referred to as the "long" Sugawara construction, involves *all* the $sl(3)$-valued affine supercurrents. This realization corresponds to a new globally $N = 4$ supersymmetric hierarchy realized on the full set of superaffine $\widehat{sl(3)}$ supercurrents. Thus the set of superfields generating the superaffine $\widehat{sl(3)}$ algebra supplies the first known example of a Poisson-brackets structure carrying two inequivalent hierarchies of the super mKdV type associated with $N = 4$ SKdV hierarchy.

The two Sugawara realizations are respectively given by:

i) in the "short" case,

$$
J = D\overline{H} + \overline{D}H + H\overline{H} + S\overline{S}\,, \quad W = D\overline{S}\,, \quad \overline{W} = \overline{D}S\,, \tag{11}
$$

ii) in the "long" case

$$
J = H\overline{H} + F\overline{F} + R\overline{R} + S\overline{S} + \overline{\alpha}\overline{D}H + \alpha D\overline{H}\,, \quad W = D\overline{F}\,, \quad \overline{W} = \overline{D}F\,. \tag{12}
$$

Their Poisson brackets (OPEs) are given by the relations (2).

5 $N = 4$ supersymmetry

Like in the $\widehat{sl(2|1)}$ case, the "short" Sugawara $N = 4$ supercurrents (11) do not produce the true global $N = 4$ supersymmetry for the entire set of the affine supercurrents, yielding it only for the $\widehat{sl(2)\oplus u(1)}$ subset. At the same time, the "long" Sugawara (12) generates such a supersymmetry. In the $z, \theta, \bar{\theta}$ expansion of the supercurrents $J, W, \overline{W}$ the global supersymmetry generators are present as the coefficients of the monomials $\sim \theta/z$. From J there come out the generators of the manifest linearly realized $N=2$ supersymmetry, while those of the hidden $N=2$ supersymmetry appear from $W, \overline{W}$. The precise form of the hidden supersymmetry transformations can then be easily read off from the OPEs (10):

$$
\begin{aligned}
\delta H &= \overline{\epsilon}(HF - \alpha\, SR) + \epsilon\alpha\, D\overline{F}\,, \qquad \delta\overline{H} = \overline{\epsilon}\,\overline{\alpha}\,\overline{D}F - \epsilon\left(\overline{H}\,\overline{F} - \overline{\alpha}\,\overline{S}\,\overline{R}\right)\,, \\
\delta F &= -\epsilon\left(\alpha D\overline{H} + F\overline{F} + H\overline{H} + R\overline{R} + S\overline{S}\right)\,, \delta\overline{F} = -\overline{\epsilon}\left(\overline{\alpha}\,\overline{D}H + F\overline{F} + H\overline{H} + R\overline{R} + S\overline{S}\right)\,, \\
\delta S &= -\overline{\epsilon}\alpha\, FS - \epsilon\left(D\overline{R} - \alpha\,\overline{F}S + H\overline{R}\right)\,, \ \delta\overline{S} = -\overline{\epsilon}\left(\overline{D}R + \overline{\alpha}\,F\overline{S} - \overline{H}R\right) + \epsilon\overline{\alpha}\,\overline{F}\,\overline{S}\,, \\
\delta R &= -\overline{\epsilon}\,\overline{\alpha}\,FR + \epsilon\left(D\overline{S} + \overline{\alpha}\,\overline{F}R - \alpha\, H\overline{S}\right)\,, \ \delta\overline{R} = \overline{\epsilon}\left(\overline{D}S - \alpha\, F\overline{R} + \overline{\alpha}\,\overline{H}S\right) + \epsilon\alpha\,\overline{F}\,\overline{R}\,.
\end{aligned} \tag{13}
$$

Here $\epsilon, \overline{\epsilon}$ are the corresponding odd transformation parameters. One can check that these transformations have just the same standard closure in terms of ∂_z as the manifest $N=2$ supersymmetry transformations, despite the presence of nonlinear terms. Also it is straightforward to verify that the constraints (8) and the OPEs (10) are covariant under these transformations.

Let us examine the issue of reducibility of the set of the $N=2$ $\widehat{sl(3)}$ currents with respect to the full $N=4$ supersymmetry. In the $\widehat{sl(2)\oplus u(1)}$ case the involved currents form an irreducible $N=4$ multiplet which is a nonlinear version of the multiplet consisting of two chiral (and anti-chiral) $N=2$ superfields [13]. In the given case one can expect that eight $N=2$ $\widehat{sl(3)}$ currents form a reducible multiplet which can be divided into a sum of two irreducible ones, each involving four superfields (a pair of chiral and anti-chiral superfields together with its conjugate). However, looking at the r.h.s. of (13), it is difficult to imagine how this could be done in a purely algebraic and local way. Nevertheless, there is a non-local redefinition of the supercurrents which partly makes this job. As the first step one introduces a prepotential for the chiral superfields $H, \overline{H}$

$$
H = DV, \quad \overline{H} = -\overline{D}\,\overline{V} \tag{14}
$$

and chooses a gauge for V in which it is expressed through $H, \overline{H}$ [19]

$$
\begin{aligned}
V &= -\partial^{-1}(\overline{D}H + \overline{\alpha}\, D\overline{H})\,, \quad \overline{V} = \partial^{-1}(D\overline{H} + \alpha\,\overline{D}H)\,, \quad V = -\overline{\alpha}\overline{V}\,, & (15) \\
\delta V &= \alpha(\overline{\epsilon}F - \epsilon\overline{F})\,, \qquad \delta\overline{V} = \overline{\alpha}(\overline{\epsilon}F - \epsilon\overline{F})\,. & (16)
\end{aligned}
$$

Using this newly introduced quantity, one can pass to the supercurrents which satisfy the standard chirality conditons following from the original constraints (7), (8) and equivalent to them

$$
\begin{aligned}
S &= \exp\{-V\}\tilde{S}\,, \quad \overline{S} = \exp\{\alpha V\}\overline{\tilde{S}}\,, \quad R = \exp\{\alpha V\}\tilde{R}\,, \quad \overline{R} = \exp\{-V\}\overline{\tilde{R}}\,, \\
F &= \exp\{-\overline{\alpha}V\}[\tilde{F} - \partial^{-1}\overline{D}(\tilde{S}\tilde{R}) + \partial^{-1}D(\overline{\tilde{S}}\,\overline{\tilde{R}})]\,, \\
\overline{F} &= \exp\{-\overline{\alpha}V\}[\overline{\tilde{F}} - \partial^{-1}\overline{D}(\tilde{S}\tilde{R}) + \partial^{-1}D(\overline{\tilde{S}}\,\overline{\tilde{R}})]\,,
\end{aligned} \tag{17}
$$

$$
D\tilde{S} = D\tilde{R} = D\tilde{F} = 0\,, \qquad \overline{D}\overline{\tilde{S}} = \overline{D}\overline{\tilde{R}} = \overline{D}\overline{\tilde{F}} = 0\,. \tag{18}
$$

The $N=4$ transformation rules (13) are radically simplified in the new basis

$$\begin{aligned}
&\delta\tilde{S} = -\epsilon D\overline{\tilde{R}}\,, \quad \delta\overline{\tilde{S}} = -\bar{\epsilon}\overline{D}\tilde{R}\,, \quad \delta\tilde{R} = \epsilon D\overline{\tilde{S}}\,, \quad \delta\overline{\tilde{R}} = \bar{\epsilon}\overline{D}\tilde{S}\,, \\
&\delta\tilde{F} = \epsilon D\overline{D}(\exp\{\overline{\alpha}V\})\,, \qquad \delta\overline{\tilde{F}} = -\bar{\epsilon}\overline{D}D(\exp\{\overline{\alpha}V\})\,, \\
&\delta(\exp\{\overline{\alpha}V\}) = \bar{\epsilon}\tilde{F} - \epsilon\overline{\tilde{F}} - (\bar{\epsilon}-\epsilon)\partial^{-1}[\overline{D}(\tilde{S}\tilde{R}) - D(\overline{\tilde{S}}\,\overline{\tilde{R}})]\,.
\end{aligned} \tag{19}$$

We see that the supercurrents $\tilde{S}\,, \overline{\tilde{S}}\,, \tilde{R}\,, \overline{\tilde{R}}$ form an irreducible $N=4$ supermultiplet, just of the kind found in [13]. At the same time, the superfields $V, \tilde{F}\,, \overline{\tilde{F}}$ do not form a closed set: they transform through the former multiplet. We did not succeed in finding the basis where these two sets of transformatons entirely decouple from each other. So in the present case we are facing a new phenomenon consisting in that the $N=2$ $\widehat{sl(3)}$ supercurrents form a not fully reducible representation of $N=4$ supersymmetry. The same can be anticipated for higher rank affine supergroups with a hidden $N=4$ structure. One observes that putting the supercurrents $\tilde{S}\,, \overline{\tilde{S}}\,, \tilde{R}\,, \overline{\tilde{R}}$ (or their counterparts in the original basis) equal to zero is the truncation consistent with $N=4$ supersymmetry. After this truncaton the remaining supercurrents $H, F, \overline{H}, \overline{F}$ form just the same irreducible multiplet as in the $\widehat{sl(2)\oplus u(1)}$ case [11].

Note that the above peculiariaty does not show up at the level of the composite supermultiplets like (12). Indeed, it is straightforward to see that the supercurrents in (12) form the same irreducible representation as in the $\widehat{sl(2)\oplus u(1)}$ case [11]

$$\delta J = -\epsilon\overline{D}W - \bar{\epsilon}D\overline{W}\,, \qquad \delta W = \bar{\epsilon}DJ\,, \qquad \delta\overline{W} = \epsilon\overline{D}J\,. \tag{20}$$

Another irreducible multiplet is comprised by the following composite supercurrents

$$\begin{aligned}
\hat{J} &= H\overline{H} + F\overline{F} + S\overline{S} + R\overline{R}\,, \\
\hat{W} &= DF = -\overline{\alpha}HF + SR\,,\ \overline{\hat{W}} = \overline{D}\,\overline{F} = \alpha\overline{H}\,\overline{F} - \overline{S}\,\overline{R}\,.
\end{aligned} \tag{21}$$

Under (13) they transform as

$$\delta\hat{J} = -\epsilon D\overline{\hat{W}} - \bar{\epsilon}\overline{D}\hat{W}\,, \qquad \delta\hat{W} = \epsilon D\hat{J}\,, \qquad \delta\overline{\hat{W}} = \bar{\epsilon}\overline{D}\hat{J}\,. \tag{22}$$

The OPEs of these supercurrents can be checked to generate another "small" $N=4$ SCA with zero central charge, i.e. a topological "small" $N=4$ SCA. The same SCA was found in the $\widehat{sl(2)\oplus u(1)}$ case [11]. This SCA and the first one together close on the "large" $N=4$ SCA in some particular realization [20, 11]. Thus the $N=2$ $\widehat{sl(3)}$ affine superalgebra provides a Sugawara type construction for this extended SCA as well. It would be of interest to inquire whether this superalgebra conceals in its enveloping algebra any other SCA containing $N=4$ SCA as a subalgebra, e.g., possible $N=4$ extensions of nonlinear W_n algebras.

6 $N=4$ mKdV-type hierarchies

Both two inequivalent $N=4$ Sugawara constructions, eqs. (11) and (12), define Poisson maps. As a consequence, the superaffine $sl(3)$-valued supercurrents inherit all the integrable hierarchies associated with $N=4$ SCA.

The first known example of hierarchy with $N = 4$ SCA as the Poisson structure is $N = 4$ SKdV hierarchy (see [11]). The densities of the lowest hamiltonians from an infinite sequence of the corresponding superfield hamiltonians in involution, up to an overall normalization factor, read

$$\begin{aligned}
\mathcal{H}_1 &= J \\
\mathcal{H}_2 &= -\frac{1}{2}(J^2 - 2W\overline{W}) \\
\mathcal{H}_3 &= \frac{1}{2}(J[D,\overline{D}]J + 2W\overline{W}' + \frac{2}{3}J^3 - 4JW\overline{W}) \,.
\end{aligned} \tag{23}$$

Here the $N = 2$ superfields J, W, $\overline{W}$ satisfy the Poisson brackets (2).

Let us condensely denote by Φ_a, $a = 1, 2, ..., 8$, the $\widehat{sl(3)}$-valued superfields H, F, R, S together with the barred ones. Their evolution equations which, by construction, are compatible with the $N = 4$ SKdV flows, for the k-th flow ($k = 1, 2, ...$) are written as

$$\frac{\partial}{\partial t_k}\Phi_a(X, t_k) = \{\int dY \mathcal{H}_k(Y, t_k), \Phi_a(X, t_k)\} \,. \tag{24}$$

The Poisson bracket here is given by the superaffine $\widehat{sl(3)}$ structure (10), with X, Y being two different "points" of $N = 2$ superspace.

The identification of the superfields J, W, $\overline{W}$ in terms of the affine supercurrents can be made either via eqs. (11), i.e. the "short" Sugawara, or via eqs. (12), that is the "long" Sugawara. Thus the same $N = 4$ SKdV hierarchy proves to produce two inequivalent mKdV type hierarchies for the affine supercurrents, depending on the choice of the underlying Sugawara construction. The first hierarchy is $N = 2$ supersymmetric, while the other one gives a new example of globally $N = 4$ supersymmetric hierarchy.

Let us briefly outline the characteristic features of these two hierarchies.

It is easy to see that for the superfields $H, \overline{H}, S, \overline{S}$ corresponding to the superaffine algebra $\widehat{sl(2) \oplus u(1)}$ as a subalgebra in $\widehat{sl(3)}$, the "short" hierarchy coincides with $N = 4$ NLS-mKdV hierarchy of ref. [11]. For the remaining $\widehat{sl(3)}$ supercurrents one gets the evolution equations in the "background" of the basic superfields just mentioned.

New features are revealed while examining the "long", i.e. $N = 4$ supersymmetric $\widehat{sl(3)}$ mKdV hierarchy. It can be easily checked that for all non-trivial flows ($k \geq 2$) the evolution equations for any given superfield Φ_a necessarily contain in the r.h.s. the whole set of eight $\widehat{sl(3)}$ supercurrents. In this case the previous $N = 4$ NLS-mKdV hierarchy can also be recovered. However, it is obtained in a less trivial way. Namely, it is produced only after coseting out the superfields R, S and $\overline{R}, \overline{S}$, i.e. those associated with the simple roots of $sl(3)$ (as usual, the passing to the Dirac brackets is required in this case). As was mentioned in the preceding Section, this truncation preserves the global $N = 4$ supersymmetry.

Let us also remark that, besides the two mKdV hierarchies carried by the superaffine $\widehat{sl(3)}$ algebra and discussed so far, this Poisson bracket structure also carries at least one extra pair of inequivalent hierarchies of the mKdV type possessing only global $N = 2$ supersymmetry. It was shown in [15] that the enveloping algebra of $N = 4$ SCA contains, apart from an infinite abelian subalgebra corresponding to the genuine $N = 4$ SKdV hierarchy, also an infinite abelian subalgebra formed by the hamiltonians in involution associated with a different hierarchy referred to as the "quasi" $N=4$ SKdV one. This hierarchy admits only a global $N=2$ supersymmetry and can be thought of as an integrable extension of the $a=-2$, $N=2$ SKdV hierarchy. In the same paper [15] there was found a non-polynomial Miura-type transformation which in a surprising way relates $N=4$ SCA to the non-linear $N=2$ super-W_3 algebra. This transformation maps the "quasi" $N=4$ SKdV hierarchy onto the $\alpha=-2$, $N=2$ Boussinesq hierarchy. Since the results of [15] can be rephrased in terms of the Poisson brackets structure alone, and the same is true both for our "short" (11) and "long" (12) Sugawara constructions, it immediately follows that the super-affine $\widehat{sl(3)}$ superfields also carry two inequivalent "quasi" $N=4$ SKdV structures and can be mapped in two inequivalent ways onto the $\alpha=-2$, $N=2$ Boussinesq hierarchy.

7 Conclusions

In this work we have investigated the local Sugawara constructions leading to the $N=4$ SCA expressed in terms of the superfields corresponding to the $N=2$ superaffinization of the $sl(2|1)$ and the $sl(3)$ algebras. We have shown that the $\widehat{sl(3)}$ case admits a non-trivial $N=4$ Sugawara construction involving all eight affine supercurrents and generating the hidden $N=4$ supersymmetry of $N=2$ $\widehat{sl(3)}$ algebra. This property has been used to construct a new $N=4$ supersymmetric mKdV hierarchy associated with $N=4$ SKdV. Another mKdV hierarchy is obtained using the $N=4$ Sugawara construction on the subalgebra $\widehat{sl(2)\oplus u(1)}$. Thus the $N=2$ $\widehat{sl(3)}$ algebra was shown to provide the first example of a Poisson brackets structure carrying two inequivalent integrable mKdV type hierarchies associated with the $N=4$ SKdV one. Also, the existence of two non-trivial $N=2$ supersymmetric mKdV-type hierarchies associated with the same superaffine Poisson structure and "squaring" to the quasi $N=4$ SKdV hierarchy of ref. [15] was noticed.

An interesting problem is to generalize the two Sugawara constructions to the full quantum case and to find out (if existing) an $N=4$ analog of the well-known GKO coset construction [21] widely used in the case of bosonic affine algebras. It is also of importance to perform a more detailed analysis of the enveloping algebra of $N=2$ $\widehat{sl(3)}$ with the aim to list all irreducible composite $N=4$ supermultiplets and to study possible $N=4$ extended W type algebras generated by these composite supercurrents. At last, it still remains to classify all possible $N=2$ affine superalgebras admitting the hidden $N=4$ structure, i.e. $N=4$ affine superagebras. As is clear from the two available examples ($\widehat{sl(2)\oplus u(1)}$ and $\widehat{sl(3)}$) a sufficient condition of the existence of such a structure on the given affine superalgebra is the possiblity to define $N=4$ SCA on it via the corresponding "long" Sugawara construction, with the full $N=2$ stress-tensor included.

Acknowledgments

F.T. wishes to express his gratitude to the JINR-Bogoliubov Laboratory of Theoretical

Physics, where this work has been completed, for the kind hospitality. E.I. and S.K. acknowledge a support from the grants RFBR-99-02-18417, INTAS-96-0308 and INTAS-96-538.

Appendix: the second flow of the "long" $\widehat{sl(3)}$ $N=4$ mKdV

For completeness we present here the evolution equations for the second flow of the "long" $\widehat{sl(3)}$ mKdV hierarchy (it is the first non-trivial flow). We have [3]

$$
\begin{aligned}
\dot{H} \;=\;& -2\partial^2 H - 2\alpha(2HD\partial\overline{H} + \partial HD\overline{H} - SD\partial\overline{S} - \partial SD\overline{S} - RD\partial\overline{R} - \partial RD\overline{R}) - \\
& -4\overline{\alpha}\partial H\overline{D}H + 2\alpha(\overline{F}S\partial R + \overline{F}\partial SR - HS\partial\overline{S} - D\overline{F}S\overline{D}R + D\overline{FD}SR) - \\
& -2\overline{\alpha}HR\partial\overline{R} - 2(1+\alpha)(H\partial S\overline{S} + \partial HS\overline{S}) - 2(1-\overline{\alpha})(H\partial R\overline{R} + \partial HR\overline{R}) + \\
& +2(2H\overline{D}FD\overline{F} + H\overline{D}RD\overline{R} + H\overline{D}SD\overline{S} - 2\overline{D}HFD\overline{F} - \overline{D}HRD\overline{R} - \overline{D}HSD\overline{S} - 2H\partial F\overline{F} - \\
& -2\partial HF\overline{F}) + 2\alpha(2H\overline{H}FD\overline{F} + 2HD\overline{H}F\overline{F} + S\overline{S}RD\overline{R} + SD\overline{S}R\overline{R}) - \\
& -2\overline{\alpha}(H\overline{H}RD\overline{R} + HD\overline{H}R\overline{R} + \overline{H}D\overline{F}SR - D\overline{H}\overline{F}SR) + \\
& +2(2HF\overline{S}D\overline{R} - 2HFD\overline{S}\overline{R} - H\overline{F}S\overline{D}R + H\overline{F}\overline{D}SR + H\overline{H}SD\overline{S} + HD\overline{H}S\overline{S} + \overline{D}H\overline{F}SR) - \\
& -2\alpha H\overline{H}\overline{F}SR - 2HS\overline{S}R\overline{R} \\
\dot{S} \;=\;& 2\overline{\alpha}(D\overline{H}\partial S - \overline{D}H\partial S + D\partial\overline{H}S - \partial H\overline{D}S) - 2\overline{D}\partial HS - 2\partial FD\overline{R} - \\
& -2\alpha(2H\partial\overline{H}S + SR\partial\overline{R} + S\partial R\overline{R} + \partial H\overline{H}S) - \\
& -2\overline{\alpha}(H\overline{D}FD\overline{R} - \overline{D}HFD\overline{R} + \partial SS\overline{S}) - \\
& -2(F\overline{F}\partial S + FD\overline{FD}S + H\overline{H}\partial S + HD\overline{HD}S - H\partial F\overline{R} - S\overline{D}RD\overline{R} + S\overline{D}SD\overline{S} + 2\overline{D}SRD\overline{R} + \\
& +\partial SR\overline{R}) + 2\alpha(FSD\overline{S}R - FS\overline{S}D\overline{R}) - 2\overline{\alpha}(HF\overline{FD}S + H\overline{D}HF\overline{R} + D\overline{H}F\overline{F}S + \\
& +\overline{H}FD\overline{F}S) + 2(1+\alpha)(\overline{F}S\overline{D}SR + H\overline{D}SR\overline{R}) - 2(2HS\overline{D}R\overline{R} + 2HS\overline{D}S\overline{S} - 2H\overline{D}F\overline{F}S - \\
& -H\overline{D}H\overline{H}S + \overline{D}HF\overline{F}S) + 2\alpha H\overline{H}F\overline{F}S - 2\overline{\alpha}H\overline{H}SR\overline{R} + 2HFS\overline{S}R \\
\dot{R} \;=\;& 2\alpha(\overline{D}\partial HR - D\overline{H}\partial R) - 2\overline{\alpha}(\overline{D}H\partial R + \partial H\overline{D}R) - 2(D\partial\overline{H}R - \partial FD\overline{S}) + \\
& +2\alpha(H\partial F\overline{S} - \partial RR\overline{R}) + 2\overline{\alpha}(H\overline{D}FD\overline{S} - 2H\partial\overline{H}R - S\partial\overline{S}R - \overline{D}HFD\overline{S} - \\
& -\partial H\overline{H}R - \partial S\overline{S}R) - 2(F\overline{F}\partial R + FD\overline{FD}R + H\overline{H}\partial R + HD\overline{HD}R + R\overline{D}RD\overline{R} + S\overline{S}\partial R + \\
& +2SD\overline{SD}R - \overline{D}SD\overline{S}R) + 2\alpha(2HR\overline{D}R\overline{R} - 2H\overline{D}F\overline{F}R - H\overline{D}H\overline{H}R - 2H\overline{D}S\overline{S}R + \\
& +\overline{D}HF\overline{F}R) + 2\overline{\alpha}(F\overline{S}RD\overline{R} + FD\overline{S}R\overline{R} - HF\overline{FD}R) + 2(1+\overline{\alpha})(\overline{F}SR\overline{D}R) - \\
& -2(1+\alpha)HS\overline{SD}R - 2(H\overline{D}HF\overline{S} - \overline{H}FD\overline{F}R - D\overline{H}F\overline{F}R) + \\
& +2\alpha(HF\overline{S}R\overline{R} - H\overline{H}S\overline{S}R) + 2\overline{\alpha}H\overline{H}F\overline{F}R \\
\dot{F} \;=\;& 2\partial^2 F - 4\alpha D\overline{H}\partial F - 4\overline{\alpha}(\overline{D}H\partial F + \overline{D}\partial HF) + 2(\overline{D}S\partial R - S\overline{D}\partial R + \overline{D}\partial SR - \\
& -\partial S\overline{D}R) - 2\overline{\alpha}(4HF\overline{D}F\overline{F} - 2H\overline{D}H\overline{H}F + \overline{H}FRD\overline{R} - D\overline{H}FR\overline{R}) - \\
& -2\alpha\overline{D}HFS\overline{S} - 2(1+\overline{\alpha})(H\overline{D}FS\overline{S} + HF\overline{D}S\overline{S}) + 2i\sqrt{3}(HF\overline{D}R\overline{R} + H\overline{D}FR\overline{R}) + \\
& +2(2F\overline{F}S\overline{D}R - 2F\overline{FD}SR + H\overline{H}S\overline{D}R - H\overline{HD}SR + \overline{H}FSD\overline{S} + 2SR\overline{D}R\overline{R} - 2S\overline{D}S\overline{S}R +
\end{aligned}
$$

[3] In order to save space and to avoid an unnecessary duplication we present the equations only for the non-linear chiral sector.

$$+2\overline{D}F\overline{F}SR+\overline{D}HFR\overline{R}+\overline{D}H\overline{H}SR-D\overline{H}FS\overline{S})+$$
$$+2\alpha(D\overline{H}S\overline{D}R-D\overline{H}\overline{D}SR)-2\overline{\alpha}\partial\overline{H}SR-2(FR\partial\overline{R}+FS\partial\overline{S}+2F\overline{D}FD\overline{F}+$$
$$+F\overline{D}RD\overline{R}+F\overline{D}SD\overline{S}+2H\overline{H}\partial F+2HD\overline{H}\overline{D}F+2H\partial\overline{H}F+\overline{D}FRD\overline{R}+\overline{D}FSD\overline{S}+$$
$$+2\partial FF\overline{F}+2\partial FR\overline{R}+2\partial FS\overline{S})+2(1+\alpha)H\overline{H}FR\overline{R}+2(1+\overline{\alpha})H\overline{H}FS\overline{S}\ .$$

The parameters $\alpha, \bar{\alpha}$ have been defined in eq. (9).

References

[1] P. Di Francesco, P. Ginsparg, J. Zinn-Justin, Phys. Reports **254** (1995) 1.

[2] N. Seiberg, E. Witten, Nucl. Phys. **B 426** (1994) 19;
A. Gorskii, I. Krichever, A. Marshakov, A. Mironov, A. Morozov, Phys. Lett. **B 355** (1995) 466.

[3] V. Drinfeld, V. Sokolov, J. Sov. Math. **30** (1984) 1975.

[4] Yu. I. Manin, A.O. Radul, Commun. Math. Phys. **98** (1985) 65;
P.P. Kulish, Lett. Math. Phys **10** (1985) 87.

[5] P. Mathieu, Phys. Lett. **B 203** (1988) 287;
C.A. Laberge, P. Mathieu, Phys. Lett. **B 215** (1988) 718;
P. Labelle, P. Mathieu, J. Math. Phys. **89** (1991) 923.

[6] T. Inami, H. Kanno, Commun. Math. Phys. **136** (1991) 519; Int. J. Mod. Phys. **A 7** Suppl. 1A (1992) 419.

[7] L. Bonora, S. Krivonos, A. Sorin, Nucl. Phys. **B 477** (1996) 835; Lett. Math. Phys. **45** (1998) 63; Phys. Lett. **A 240** (1998) 201;
L. Bonora, S. Krivonos, Mod. Phys. Lett. **A 12** (1997) 3037.

[8] Z. Popowicz, Phys. Lett. **A 194** (1994) 375; J. Phys. **A 29** (1996) 1281; *ibid* **30** (1997) 7935; Phys. Lett. **B 459** (1999) 150.

[9] J.C. Brunelli, A. Das, J. Math. Phys. **36** (1995) 268.

[10] F. Toppan, Int. J. Mod. Phys. **A 10** (1995) 895.

[11] E. Ivanov, S. Krivonos, F. Toppan, Phys. Lett. **B 405** (1997) 85.

[12] F. Delduc, E. Ivanov, Phys. Lett. **B 309** (1993) 312;
F. Delduc, E. Ivanov, S. Krivonos, J. Math. Phys. **37** (1996) 1356.

[13] E. Ivanov, S. Krivonos, Phys. Lett. **A 231** (1997) 75.

[14] Ph. Spindel, A. Sevrin, W. Troost, A. Van Proeyen, Phys. Lett. **B 206** (1988) 71.

[15] F. Delduc, L. Gallot, E. Ivanov, Phys. Lett. **B 396** (1997) 122.

[16] S. Krivonos, K. Thielemans, Class. Quant. Grav. **13** (1996) 2899.

[17] C.M. Hull, B. Spence, Phys. Lett. **B 241** (1990) 357.

[18] C. Ahn, E. Ivanov, A. Sorin, Commun. Math. Phys. **183** (1997) 205.

[19] E. Ivanov, Mod. Phys. Lett. **A 13** (1998) 2855.

[20] M. Roček, C. Ahn, K. Schoutens, A. Sevrin, Superspace WZW Models and Black Holes, in: Workshop on Superstrings and Related Topics, Trieste. Aug. 191,IASSNS-HEP-91/69, ITP-SB-91-49, LBL-31325, UCB-PTH-91/50.

[21] P. Goddard, A. Kent, D. Olive, Phys. Lett. **B 152** (1985) 88.